MATH 253 - SEC 104 - W2011T1

1. Let

$$f(x,y) = \frac{xy^3 - x^3y}{x^2+y^2}\,.$$

   (a) What is the domain of definition of f?

   (b) Calculate f_x, f_y, f_{xy}, f_{yx} and verify that $f_{xy} = f_{yx}$.

   (c) Calculate $\lim_{x\to 0} f_{xy}(x,0)$ and $\lim_{y\to 0} f_{xy}(0,y)$. Does $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ exist?

   Solution:

   (a) $(x,y) \neq (0,0)$

   (b) $f_x(x,y) = \frac{y^5 - x^4y - 4x^2y^3}{(x^2+y^2)^2}$

   $f_y(x,y) = \frac{-x^5 + xy^4 + 4x^3y^2}{(x^2+y^2)^2}$

   $f_{xy}(x,y) = f_{yx}(x,y) = \frac{-x^6 - 9x^4y^2 + 9x^2y^4 + y^6}{(x^2+y^2)^3}$

   (c) For any $x \neq 0$, $f_{xy}(x,0) = \frac{-x^6}{x^6} = -1$ so $\lim_{x\to 0} f_{xy}(x,0) = -1$. Similarly, $\lim_{y\to 0} f_{xy}(0,y) = \lim_{y\to 0} \frac{y^6}{y^6} = 1$. Since $1 \neq -1$ we see that $\lim_{(x,y)\to(0,0)} f_{xy}(x,y)$ does not exist.

2. Let

$$f(x, y) = x \arctan(x^2 - y) \, .$$

(a) What is the domain of definition of f?

(b) Calculate f_x and f_y and find the only point where the tangent plane is horizontal.

(c) Find the second order Taylor approximation at the point from (b). Can you say whether this point is a local minimum, a local maximum or neither?

Solution:

(a) f is defined on the entire plane.

(b) $f_x(x, y) = \arctan(x^2 - y) + \frac{2x^2}{1+(x^2-y)^2}$.

$f_y(x, y) = \frac{-x}{1+(x^2-y)^2}$.

The tangent plane is horizontal when both $f_x(x, y)$ and $f_y(x, y)$ are zero, i.e. when (x, y) is a critical point for f. The equation $\frac{-x}{1+(x^2-y)^2} = 0$ yield that $x = 0$. Putting this back into $\arctan(x^2 - y) + \frac{2x^2}{1+(x^2-y)^2} = 0$ we get $\arctan(-y) = 0$ and so $y = 0$ too. So our point is $(0, 0)$.

(c) $f_{xx}(x, y) = \frac{2x^5+4x^3y-6x-6xy^2}{(1+(x^2-y)^2)^2}$ so $f_{xx}(0, 0) = 0$.

$f_{xy}(x, y) = \frac{1-3x^4+2x^2y+y^2}{(1+(x^2-y)^2)^2}$ so $f_{xy}(0, 0) = 1$.

$f_{yy}(x, y) = \frac{2x^3-2xy}{(1+(x^2-y)^2)^2}$ so $f_{yy}(0, 0) = 0$.

$f(0, 0) = 0$

Hence, the 2nd order Taylor polynomial of f around $(0, 0)$ is $T_2(x, y) = xy$. This function describes a hyperbolic paraboloid and so $(0, 0)$ is a saddle point for f.

3. Let

$$f(x, y) = \sqrt{1 - x^2 - y^2}\,.$$

(a) What is the domain of definition of f?

(b) Write the equation of the tangent plane at the point $(a, b, f(a, b))$ in terms of a and b.

(c) Find a and b such that this tangent plane passes through the points $(1, 1, 1)$ and $(1, -2, 4)$.

Solution:

(a) $x^2 + y^2 \leq 1$ - the disc of radius 1 around the origin.

(b) $f_x(x, y) = \frac{-x}{\sqrt{1-x^2-y^2}}$

$f_y(x, y) = \frac{-y}{\sqrt{1-x^2-y^2}}$

The tangent plane at $(a, b, f(a, b))$ is described by

$$z = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(x - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(y - b)\,.$$

(c) We want to solve the following equations for a and b:

$$1 = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(1 - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(1 - b)\,.$$

$$4 = \sqrt{1 - a^2 - b^2} - \frac{a}{\sqrt{1 - a^2 - b^2}}(1 - a) - \frac{b}{\sqrt{1 - a^2 - b^2}}(-2 - b)\,.$$

Multiplying by $\sqrt{1 - a^2 - b^2}$ and simplifying yields

$$\sqrt{1 - a^2 - b^2} = 1 - a - b \tag{1}$$

$$4\sqrt{1 - a^2 - b^2} = 1 - a + 2b$$

This implies that

$$4(1 - a - b) = 1 - a + 2b$$

so

$$a = 1 - 2b\,.$$

Putting this back into equation 1 gives

$$\sqrt{1 - (1 - 2b)^2 - b^2} = 1 - (1 - 2b) - b\,.$$

Squaring yields

$$1 - (1 - 4b + 4b^2) - b^2 = b^2$$

so

$$4b - 6b^2 = 0$$

which means that either $b = 0$ (and $a = 1$) or $b = \frac{2}{3}$ (and $a = 1 - 2\frac{2}{3} = -\frac{1}{3}$).

Note that the first solution $(a, b) = (0, 1)$ is on the boundary of our domain and the partial derivatives are not defined there. The second solution $(a, b) = (\frac{2}{3}, -\frac{1}{3})$ is the only solution.