MATH 253 - SEC 104 - W2011T1

1. Determine whether the center of the sphere $x^2 + 2x + y^2 + z^2 = 6y + 14z + 5$ lies on the line $(2, 0, 1) + t(-1, 1, 2)$.

   Rewriting the sphere equation $x^2 + 2x + y^2 + z^2 = 6y + 14z + 5$ as $(x + 1)^2 + (y - 3)^2 + (z - 7)^2 = 64$ we see that it is centered around $(-1, 3, 7)$.

   Hence we need to find whether there is a t satisfying the equation

   $$(2, 0, 1) + t(-1, 1, 2) = (-1, 3, 7)\,.$$

   This is equivalent to $t(-1, 1, 2) = (-3, 3, 6)$ and we see that indeed $t = 3$ satisfies the equation and hence the center of the sphere is on the line.

2. What is the angle between the line $x = y = z$ and the following planes?

   (a) $x + y + z = 7$

   (b) $x + y = 3$

   (c) $2x - y - z = 0$

   (d) $x - y = 2$

First, we need to find a vector in the direction of the line. one way of doing this is finding two points on the line and taking the difference between them. By substituting $x = 0$ we get the point $(0, 0, 0)$ and by substituting $x = 1$ we get $(1, 1, 1)$. Hence, the vector $v = (1, 1, 1) - (0, 0, 0) = (1, 1, 1)$ is a vector in the direction of the line.

Next, we calculate α, the angle with the normal to the plane using the dot product and then β, angle with the plane is $\beta = \frac{\pi}{2} - \alpha$.

(a) $n = (1, 1, 1)$: $\cos\alpha = \frac{n \cdot v}{|n||v|} = \frac{3}{\sqrt{3}\sqrt{3}} = 1$ hence $\alpha = 0$ and $\beta = \frac{\pi}{2}$. Indeed, in this case the normal is in the same direction of v.

(b) $n = (1, 1, 0)$: $\cos\alpha = \frac{2}{\sqrt{2}\sqrt{3}} = 0.816$ hence $\alpha = 0.615$ and $\beta = \frac{\pi}{2} - 0.615 = 0.955$.

(c) $n = (2, -1, -1)$: $\cos\alpha = \frac{0}{\sqrt{6}\sqrt{3}} = 0$ hence $\alpha = \frac{\pi}{2}$ and $\beta = 0$. The line is in the direction of the plane. We choose an arbitrary point on the line $(0, 0, 0)$ and see that it is on the plane, hence the line is lying on the plane.

(d) $n = (1, -1, 0)$: $\cos\alpha = \frac{0}{\sqrt{2}\sqrt{3}} = 0$ hence $\alpha = \frac{\pi}{2}$ and $\beta = 0$. The line is in the direction of the plane. Since the point $(0, 0, 0)$ lie on the line but not the plane, the line and the plane here are parallel, but do not intersect, so there is actually no angle between them at all.

3. Find the line which passes through the point $(7, 1, 2)$ and is orthogonal to the plane $x = y$. Present both the vector equation and the symmetric equations of the line.

   Rewriting the plane equation as $x - y = 0$ we see that $(1, -1, 0)$ is a normal to this plane. Hence $u = (7, 1, 2) + t(1, -1, 0)$ is a vector equation of the required line. Converting to scalar equations we get $x = 7 + t$, $y = 1 - t$ and $z = 2$. Solving for t yields $t = x - 7 = 1 - y$ (which is equivalent to $x + y - 8 = 0$) and $z = 2$.

4. Determine whether the following two planes intersect or are parallel. If they intersect, find the equation of the line of intersection, if they are parallel, find the distance between them. Plane A: $x + y + z = 1$; Plane B: $(0, 0, 7) + s(1, -1, 0) + t(0, 1, -1)$.

   First we find a normal to plane A directly from the plane equation $n_A = (1, 1, 1)$.

   Then we find a normal to plane B by taking the cross product of $(1, -1, 0)$ and $(0, 1, -1)$. This yields

$$n_B = (1, -1, 0) \times (0, 1, -1) = (1, 1, 1)\,.$$

   Since the normals are in the same direction (in fact they're identical, but this is coincidental) the planes are parallel. The distance between them can be found by calculating the distance from one of them to any point of the other. Specifically, we see that $u = (0, 0, 7)$ belongs to plane B and that $u_0 = (1, 0, 0)$ belongs to plane A, so the distance between u and plane A is

$$\frac{|(u - u_0) \cdot n_A|}{|n_A|} = \frac{6}{\sqrt{3}} = 3.464\,.$$

5. Find the equations of the two planes which contain the points $(1, 1, 1)$ and $(1, -2, 4)$ and are tangent to the sphere of radius 1 around the origin.

Let the plane(s) equation be $Ax + By + Cz + D = 0$. Then A, B, C, D has to satisfy the equations

$$A + B + C + D = 0 \tag{1}$$

and

$$A - 2B + 4C + D = 0\,. \tag{2}$$

A plane is tangent to a sphere of radius R if and only if its distance to the center of the sphere is R. Hence we need our plane to be of distance 1 from $(0, 0, 0)$. Using the distance formula we get that

$$\frac{|D|}{\sqrt{A^2 + B^C + C^2}} = 1$$

which is equivalent to

$$D^2 = A^2 + B^2 + C^2\,. \tag{3}$$

Taking the difference between equations 1 and 2 we get $3B - 3C = 0$ and therefore $C = B$. Substituting back in equation 1 yields $A + 2B + D = 0$ so $D = -(A + 2B)$.

Putting these in equation 3 we get

$$(A + 2B)^2 = A^2 + 2B^2$$

which is equivalent to

$$A^2 + 4AB + 4B^2 = A^2 + 2B^2$$

which is equivalent to

$$4AB + 2B^2 = 0$$

which is equivalent to

$$2B(2A + B) = 0\,.$$

So, either $B = C = 0$ and $D = -A$, or $B = C = -(2A)$ and $D = 3A$. Taking $A = 1$ to get concrete equations we get the following solutions:

Plane 1: $x = 1$.

Plane 2: $x - 2y - 2z + 3 = 0$.