Some results and problems on quantum Bell-type inequalities

B. S. Tsirelson

School of Mathematics, Tel Aviv University, Tel Aviv 69978, Israel

4 old and 6 new theorems (without proofs), and 11 problems are presented in this review on single-time quantum Bell-type inequalities.*

1. Quantum-free prelude

Behaviors

Suppose that two correlated, but non-interacting subsystems of a physical system are given. Consider some finite set of (generally, incompatible) measurements over the first subsystem, each measurement having a finite set of possible outcomes. Denote by M_k the set of possible outcomes of the k-th measurement; we treat sets $M_1, \dots, M_K$ as disjoint and put $M = M_1 \cup \ldots \cup M_K$. A probability p_m corresponds to each $m \in M$, that is, the probability of obtaining the result m from the corresponding measurement k ($m \in M_k$); so,

$$\forall k = 1, \dots, K \quad \sum_{m \in M_k} p_m = 1. \tag{1.1}$$

This p_m is in fact a transition probability $k \to m$, but we prefer** one-index notation for it, exploiting the fact that k is uniquely determined by m. Similarly, for the second subsystem we introduce $N = N_1 \cup \ldots \cup N_L$, and

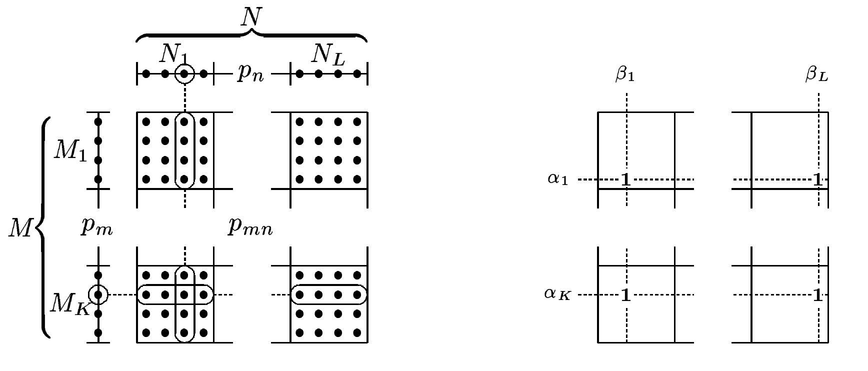
$$\forall l = 1, \dots, L \quad \sum_{n \in N_l} p_n = 1. \tag{1.2}$$

We treat N as being disjoint of M; this allows us to use the same letter p both for p_m and for p_n.

For examining correlations, introduce joint probabilities: p_{mn} is the probability of obtaining the combination (m, n) of results from a pair (k, l) of measurements ($m \in M_k$, $n \in N_l$). So,

$$\forall k, n \quad \sum_{m \in M_k} p_{mn} = p_n; \qquad \forall l, m \quad \sum_{n \in N_l} p_{mn} = p_m. \tag{1.3}$$

Each family $\{p_{mn}\}_{m \in M, n \in N}$ of nonnegative numbers satisfying*** (1.1–1.3) is called a *behavior* over the given *behavior scheme* $(M_1, \dots, M_K; N_1, \dots, N_L)$.



A behavior

A deterministic behavior

Fig. 1

* For a more embracing, but older and more concise review see [KT92, Sect. 1].

** Matrix notation, as p_{kl} or $p(k \to l)$, obscures the symmetry of the situation: points of M_1 may be rearranged independently of M_2, and so on.

*** More exactly: such that (1.3) is fulfilled with some $\{p_m\}_{m \in M}$, $\{p_n\}_{n \in N}$ satisfying (1.1), (1.2).

A behavior $\{p_{mn}\}$ is called *deterministic,* if each p_{mn} is either 1 or 0. Clearly, a deterministic behavior may be determined by $\alpha_1 \in M_1, \dots, \alpha_K \in M_K$ and $\beta_1 \in N_1, \dots, \beta_L \in N_L$:

$$p_{\alpha_k \beta_l} = 1, \qquad \text{other } p_{mn} = 0.$$

Hence, the set of deterministic behaviors may be identified with $M_1 \times \dots \times M_K \times N_1 \times \dots \times N_L$.

The set of all behaviors X_{B} is a convex polytope of dimension

$$d = (|M| - K + 1)(|N| - L + 1) - 1; \tag{1.4}$$

here $|M| = |M_1| + \dots + |M_K|$ means the number of elements in M. Each deterministic behavior is a vertex of the polytope:

$$X_{\mathrm{DB}} \subset \mathrm{ex}(X_{\mathrm{B}});$$

here X_{DB} is the set of deterministic behaviors (it is finite, $|X_{\mathrm{DB}}| = |M_1| \cdot \ldots \cdot |M_K| \cdot |N_1| \cdot \ldots \cdot |N_L|$), and $\mathrm{ex}(X_{\mathrm{B}})$ means the set of extremal points (vertices) of X_{B}.

It is vital for the very existence of any Bell-type inequality (classical or quantum), that in general

$$X_{\mathrm{DB}} \neq \mathrm{ex}(X_{\mathrm{B}}),$$

or, what is the same,

$$\mathrm{co}(X_{\mathrm{DB}}) \neq X_{\mathrm{B}};$$

here $\mathrm{co}(X_{\mathrm{DB}})$ means the convex hull of X_{DB}. It is another convex polytope $X_{\mathrm{HDB}} = \mathrm{co}(X_{\mathrm{DB}})$ of the same dimension d; and

$$X_{\mathrm{DB}} = \mathrm{ex}(X_{\mathrm{HDB}}).$$

Behaviors belonging to X_{HDB} are called *hidden deterministic,* because they (and only they) can be described in the framework of a local hidden variables theory. So, it is vital that in general

$$X_{\mathrm{HDB}} \neq X_{\mathrm{B}}.$$

Inequalities

A linear function of a behavior may be written as

$$\sum_{m,n} \lambda_{mn} p_{mn},$$

or, what is the same,

$$\sum_{k,l} \sum_{\substack{m \in M_k \\ n \in N_l}} f_{kl}(m,n) p_{mn} \tag{1.5}$$

with arbitrary real-valued functions $f_{kl} : M_k \times N_l \to \mathbb{R}$. The value of the function on a deterministic behavior $(\alpha_1, \dots, \alpha_K; \beta_1, \dots, \beta_L)$ is

$$f(\alpha_1, \dots, \alpha_K; \beta_1, \dots, \beta_L) = \sum_{k,l} f_{kl}(\alpha_k, \beta_l). \tag{1.6}$$

Clearly it is a special kind of function on $M_1 \times \dots \times M_K \times N_1 \times \dots \times N_L$. The space of all such functions is $(d+1)$-dimensional (d being defined by (1.4)) and may be identified with the space of all linear functions of behaviors (the additional dimension resulting from constant functions).

Positive* functions of the form (1.6) constitute a polyhedral convex cone in the above $(d+1)$-dimensional space. Being the cone dual to X_{HDB}, it may be denoted by X°_{HDB}. So,

$$f \in X^{\circ}_{\mathrm{HDB}} \iff \forall x \in X_{\mathrm{HDB}} \quad f(x) \geq 0; \tag{1.7a}$$

$$x \in X_{\mathrm{HDB}} \iff \forall f \in X^{\circ}_{\mathrm{HDB}} \quad f(x) \geq 0. \tag{1.7b}$$

The larger polytope X_{B} generates a smaller cone:

$$X_B \supset X_{\mathrm{HDB}},\ X_B \neq X_{\mathrm{HDB}} \quad \Longrightarrow \quad X^{\circ}_B \subset X^{\circ}_{\mathrm{HDB}},\ X^{\circ}_B \neq X^{\circ}_{\mathrm{HDB}}.$$

Each element f of X°_{HDB}, not belonging to X°_{B}, determines a *classical Bell-type inequality* $f(x) \geq 0$. But a finite number of them is of special interest; these are extremal rays.

* Not strictly; that is, $f(\alpha_1, \dots, \alpha_K; \beta_1, \dots, \beta_L) \geq 0$ for all combinations of variables.

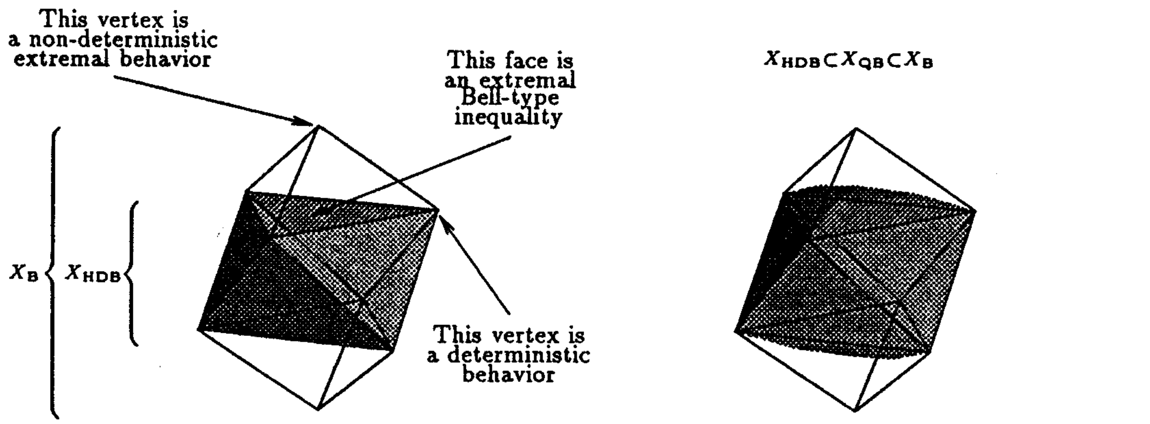
A classical Bell-type inequality $f(x) \ge 0$ is called *extremal*, if f lies on an extremal ray of the cone,

$$f \in \mathrm{exr}(X^{\circ}_{\mathrm{HDB}}) \setminus X^{\circ}_{\mathrm{B}}.$$

Considering all extremal classical Bell-type inequalities $f_1(x) \ge 0, \dots, f_\nu(x) \ge 0$, we obtain

$$\forall x \in X_{\mathrm{B}} \quad (\, x \in X_{\mathrm{HDB}} \quad \Longleftrightarrow \quad f_1(x) \ge 0, \dots, f_\nu(x) \ge 0\,). \tag{1.8}$$

So, these inequalities form a full and non-redundant set of consequences of local realism for a given behavior scheme.



A three-dimensional caricature to X_{B} and X_{HDB} **The set X_{QB} of quantum behaviors (see Sect. 2)**

Fig. 2

The simplest scheme

The simplest non-trivial behavior scheme is $(2+2) \times (2+2)$:

$$\left(\begin{array}{cc|cc} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ \hline p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{array}\right) \qquad \begin{array}{c} p_{m1} + p_{m2} = p_{m3} + p_{m4}, \\ p_{1n} + p_{2n} = p_{3n} + p_{4n}. \end{array} \tag{1.9}$$

The scheme has numerous symmetries; 8 "horizontal" symmetries

$$\begin{array}{cccc} (\,p_1\,p_2\,|\,p_3\,p_4\,), & (\,p_1\,p_2\,|\,p_4\,p_3\,), & (\,p_2\,p_1\,|\,p_3\,p_4\,), & (\,p_2\,p_1\,|\,p_4\,p_3\,), \\ (\,p_3\,p_4\,|\,p_1\,p_2\,), & (\,p_3\,p_4\,|\,p_2\,p_1\,), & (\,p_4\,p_3\,|\,p_1\,p_2\,), & (\,p_4\,p_3\,|\,p_2\,p_1\,) \end{array}$$

together with 8 similar "vertical" symmetries lead to a symmetry group of 64 elements, allowing us to give a concise description of X_{B} and X_{HDB}. Both sets are 8-dimensional ($d = (4-2+1)(4-2+1) - 1 = 8$) convex polytopes; X_{B} has 24 vertices and 16 faces, X_{HDB} has 16 vertices and 24 faces. Vertices of X_{HDB} are exactly the deterministic behaviors; one of them follows, with the others being symmetric to it:

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \tag{1.10}$$

They are all vertices of X_{B}, as well; 8 other vertices of X_{B}, *non-deterministic extremal behaviors,* are symmetric to the following one:

$$\frac{1}{2} \cdot \left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right). \tag{1.11}$$

The dual space may be identified with the 9-dimensional space of functions f of four binary variables $\alpha_1, \alpha_2, \beta_1, \beta_2$, taking two values ± 1 each, of the form

$$a_1\alpha_1 + a_2\alpha_2 + b_1\beta_1 + b_2\beta_2 + c_{11}\alpha_1\beta_1 + c_{12}\alpha_1\beta_2 + c_{21}\alpha_2\beta_1 + c_{22}\alpha_2\beta_2 + d \tag{1.12}$$

with 9 coefficients a_k, b_l, c_{kl}, d. Such a function is identified with the following linear function of a behavior:

$$a_1\alpha_1 + a_2\alpha_2 + b_1\beta_1 + b_2\beta_2 + c_{11}\gamma_{11} + c_{12}\gamma_{12} + c_{21}\gamma_{21} + c_{22}\gamma_{22} + d \tag{1.13}$$

where

$$\begin{aligned} &\alpha_1 = (p_{11} + p_{12}) - (p_{21} + p_{22}) = (p_{13} + p_{14}) - (p_{23} + p_{24}), \\ &\gamma_{11} = p_{11} - p_{12} - p_{21} + p_{22}, \end{aligned} \tag{1.14}$$

and so on. Now, one face of X_{B} is

$$\alpha_1 + \beta_1 - \gamma_{11} \le 1, \tag{1.15}$$

with the others being symmetric. They are all faces of X_{HDB}, as well; 8 other faces of X_{HDB}, *extremal classical Bell-type inequalities*, are symmetric to the following one:

$$\gamma_{11} + \gamma_{12} + \gamma_{21} - \gamma_{22} \le 2. \tag{1.16}$$

Polytopes X_{B} and X_{HDB} are dual to one another: there exists a symmetric non-degenerated bilinear form b on the 8-dimensional space such that

$$\begin{aligned} x \in X_{\mathrm{B}} \quad &\Longleftrightarrow \quad \forall y \in X_{\mathrm{HDB}} \quad b(x,y) \le 1; \\ y \in X_{\mathrm{HDB}} \quad &\Longleftrightarrow \quad \forall x \in X_{\mathrm{B}} \quad b(x,y) \le 1. \end{aligned} \tag{1.17}$$

The matrix of the form b in the basis $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22})$ follows:

$$\begin{pmatrix} -1/2 & -1/2 & 0 & 0 & & & & \\ -1/2 & +1/2 & 0 & 0 & & 0 & & \\ 0 & 0 & -1/2 & -1/2 & & & & \\ 0 & 0 & -1/2 & +1/2 & & & & \\ & & & & -1/4 & -1/4 & -1/4 & -1/4 \\ & 0 & & & -1/4 & +1/4 & -1/4 & +1/4 \\ & & & & -1/4 & -1/4 & +1/4 & +1/4 \\ & & & & -1/4 & +1/4 & +1/4 & -1/4 \end{pmatrix}. \tag{1.18}$$

Remarks

A clear and general understanding of the geometric meaning of classical Bell-type inequalities was reached by M. Froissart [Fr81]. He pointed out that:

(1) Any "logical configuration" (which is treated more extensively by him than "behavior scheme" by myself) determines a polytope, consisting of all probabilities (here "behaviors") compatible with local realism.
(2) Faces of the polytope are exactly classical Bell-type inequalities.
(3) The faces may be found algorithmically, in a finite number of steps.
(4) Several examples were solved by a computer, giving new inequalities.

Strangely enough, I failed to find even a single reference to [Fr81], except for mine. Accordingly, a recent paper [Le89] is cited as giving the most general set of inequalities following from local realism [HS91, p.46]; the very idea of reducing the continuum of inequalities to a finite number does not appear in [Le89].

Having the algorithm [Fr81], we still hope for an analytical method of finding the extremal classical Bell-type inequalities. Some progress was made by Fine [Fi82] and Garg, Mermin [GM84], but the problem remains unsolved.

The class of all behaviors X_{B} appeared in [KT85], [Ra85]. Each behavior is a convex combination (a probabilistic mix) of extremal behaviors. However, extremal behaviors are identified for the $(2+2)\times(2+2)$ scheme only. An algorithm is again available, but no analytical method has been proposed.

The duality of X_{B} and X_{HDB} for the $(2+2)\times(2+2)$ scheme seems to be new. Its origin and meaning remain vague.

Another general approach to classical Bell-type inequalities was presented by Pitowsky [Pi86, 89, 91a]. His "correlation polytopes" have much in common with polytopes X_{HDB}. Pitowsky proved the high algorithmical complexity* of several natural tasks related to his polytopes [Pi89, 91a]. A general inequality including known classical Bell-type inequalities was pointed out [Pi91, eq. (2.5)] with a challenge to find an inequality not contained in the given one.

Generalization of X_{B} and X_{HDB} to any finite number of subsystems is straightforward. Much more broad generalization, including continuous variables instead of k, l, m, n; continuous space-time instead of a

* Modulo some well-known problems of the theory of algorithmical complexity, see [Pi89, 91a].

finite collection of space-separated subsystems; and possible non-local observables, was introduced in [KT85]. A discrete multi-time case was considered in [KT92] and, in explicit connection with algebraic field theory, in [VT92].

2. Quantum restrictions

Behaviors

We continue to consider two correlated but non-interacting subsystems of a physical system, however the system is now supposed to be a quantum system. Hence, the joint probabilities p_{mn}, introduced in Sect. 1, may be expressed as follows:

$$p_{mn} = \mathrm{Tr}(F_m F_n W), \tag{2.1}$$

where W is a density matrix,

$$W \geq 0, \qquad \mathrm{Tr}(W) = 1, \tag{2.2}$$

and F_m, F_n are operators, satisfying

$$\forall m \in M \quad F_m \geq 0, \qquad \forall n \in N \quad F_n \geq 0, \tag{2.3a}$$

$$\forall k = 1, \dots, K \quad \sum_{m \in M_k} F_m = 1, \qquad \forall l = 1, \dots, L \quad \sum_{n \in N_l} F_n = 1, \tag{2.3b}$$

$$\forall m \in M \ \forall n \in N \quad F_m F_n = F_n F_m. \tag{2.3c}$$

These are minimal quantum requirements, while maximal ones follow:

$$p_{mn} = \langle \Psi | P_m \otimes P_n | \Psi \rangle, \tag{2.4}$$

where Ψ is an "entangled" state vector,

$$\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad \langle \Psi | \Psi \rangle = 1, \tag{2.5}$$

and P_m, P_n are projection operators in $\mathcal{H}_1$, $\mathcal{H}_2$ respectively:

$$\begin{aligned} &\forall m \in M \quad P_m : \mathcal{H}_1 \to \mathcal{H}_1, \quad P_m^* = P_m, \quad P_m^2 = P_m, \\ &\forall n \in N \quad P_n : \mathcal{H}_2 \to \mathcal{H}_2, \quad P_n^* = P_n, \quad P_n^2 = P_n, \end{aligned} \tag{2.6a}$$

$$\forall k = 1, \dots, K \quad \sum_{m \in M_k} P_m = 1, \qquad \forall l = 1, \dots, L \quad \sum_{n \in N_l} P_n = 1. \tag{2.6b}$$

It follows from (2.6b) that $P_{m_1} P_{m_2} = 0$ for $m_1, m_2 \in M_k$, $m_1 \neq m_2$. So, each $\{P_m\}_{m \in M_k}$ is a "projection measure." Without loss of generality we may suppose that each m is a real number, that is, $M \subset \mathbb{R}$, and also $N \subset \mathbb{R}$. Forming Hermitian operators

$$A_k = \sum_{m \in M_k} m P_m, \qquad B_l = \sum_{n \in N_l} n P_n, \tag{2.7}$$

we see that $\{p_{mn}\}_{m \in M_k, n \in N_l}$ is nothing but the joint distribution of two observables A_k, B_l (acting on $\mathcal{H}_1, \mathcal{H}_2$ correspondingly). Note that in general

$$A_{k_1} A_{k_2} \neq A_{k_2} A_{k_1}, \qquad B_{l_1} B_{l_2} \neq B_{l_2} B_{l_1}.$$

The setting (2.1–2.3) is more general than (2.4–2.6) in the following:

(1) The quantum state may be mixed.
(2) Measurements may be non-ideal.
(3) Two observed subsystems may be correlated with other (unobserved) subsystems.
(4) The von Neumann algebra for the system is not necessarily a tensor product, because of superselection, non-type-I factors, or other reasons.

Nevertheless:

> The class of behaviors generated by the setting (2.1–2.3) coincides with the class of behaviors generated by the setting (2.4–2.6).

A behavior* $\{p_{mn}\}$ admitting a representation of the form (2.1–2.3), and hence also (2.4–2.6), is called a *quantum behavior.*

The set of all quantum behaviors (over a given behavior scheme $(M_1, \dots, M_K; N_1, \dots, N_L)$) is a d-dimensional convex compact body X_{QB} (d being defined by (1.4));

$$X_{\mathrm{HDB}} \subset X_{\mathrm{QB}} \subset X_{\mathrm{B}}, \tag{2.8}$$

and in general

$$X_{\mathrm{HDB}} \neq X_{\mathrm{QB}} \neq X_{\mathrm{B}}; \tag{2.9}$$

see Fig. 2. The noncoincidence $X_{\mathrm{HDB}} \neq X_{\mathrm{QB}}$ is equivalent to the existence of classical Bell-type inequalities, while the noncoincidence $X_{\mathrm{QB}} \neq X_{\mathrm{B}}$ is equivalent to the existence of quantum Bell-type inequalities .

Inequalities

The class X_{HDB} may be defined in terms of (2.1–2.3) or (2.4–2.6), and very simply: by demanding *all* used operators to commute. From the quantum point of view, it means that:

> Classical Bell-type inequalities are inequalities for commuting observables, while quantum Bell-type inequalities are inequalities for observables commuting only when related to different subsystems.

In contrast to X_{B} and X_{HDB}, the convex body X_{QB} is in general not a polytope. Hence, it cannot be described by a finite number of linear inequalities. It seems plausible that its boundary ∂X_{QB} is a piecewise smooth surface, but this has not been proved.

2.10. Problem. Does the set of quantum behaviors admit a description by a finite number of analytic inequalities? Or even — polynomial inequalities?

Not any boundary point of X_{QB} is an extremal point, since ∂X_{QB} contains some flat regions (see Fig. 2), and some flat pieces of smaller dimension. An extremal point of X_{QB}, — an *extremal quantum behavior,* is a quantum behavior that cannot be decomposed into a probabilistic mix of other quantum behaviors. It seems plausible that the set $\mathrm{ex}(X_{\mathrm{QB}})$ of all extremal quantum behaviors consists of a finite number of analytical pieces of various dimensions, but this has not been proved.

2.11. Problem. What is the dimension of $\mathrm{ex}(X_{\mathrm{QB}})$, that is, of its most multi-dimensional piece?

X_{QB}, being a convex compact, may be described by an infinite system of linear inequalities. We know a general form of linear function of a behavior,

$$f(x) = \sum_{k,l} \sum_{\substack{m \in M_k \\ n \in N_l}} f_{kl}(m,n) p_{mn},$$

see (1.5); it is easy to see that

$$\max_{x \in X_{\mathrm{QB}}} f(x) = \max \left(\max \ \mathrm{spec} \sum_{k,l} f_{k,l}(A_k, B_l) \right) ; \tag{2.12}$$

the left-hand side is the *quantum bound* for f; on the right-hand side $f_{k,l}(A_k, B_l)$ is the operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$, resulting from applying the scalar function f_{kl} to the pair of commuting operators A_k, B_l introduced in (2.7); "max spec" means the maximal number belonging to the spectrum of the written operator;** and the outer maximum is taken over all collections $(A_1, \dots, A_K; B_1, \dots, B_L)$ of operators on $\mathcal{H}_1, \mathcal{H}_2$ respectively, with

$$\forall k = 1, \dots, K \quad \mathrm{spec}(A_k) \subset M_k, \qquad \forall l = 1, \dots, L \quad \mathrm{spec}(B_l) \subset N_l. \tag{2.13}$$

So, to find a quantum bound for a linear function, we have to find an "optimal" collection of operators. Its existence is guaranteed by the fact that X_{QB} is compact.

2.14. Problem. What are algebraic properties characterizing "optimal" collections of operators?

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* It is easy to see that numbers p_{mn} defined by (2.1) or (2.4) under imposed conditions form a behavior in the sense of Sect. 1. Quantum theory does not predict faster-than-light communication!

** The decomposition of f into the sum of f_{kl} is not unique, but nevertheless the written sum of operators is determined uniquely.

Correlation matrices

If we restrict ourselves to bilinear functions $f_{k,l}(A_k, B_l) = c_{kl}A_kB_l$ in (2.12), we reach the following notion.

A matrix $\gamma = \{\gamma_{kl}\}$ is called a *quantum correlation matrix,* if it admits a representation

$$\gamma_{kl} = \mathrm{Tr}(A_kB_lW) \tag{2.15}$$

with some density matrix W and some Hermitian operators $A_1, \dots, A_K, B_1, \dots, B_L$, satisfying

$$\forall k \quad \|A_k\| \le 1, \qquad \forall l \quad \|B_l\| \le 1, \tag{2.16a}$$

$$\forall k, l \quad A_kB_l = B_lA_k. \tag{2.16b}$$

Here $\|A_k\|$ is the operator norm of A_k; so, $\|A_k\| \le 1$ if and only if $\mathrm{spec}(A_k) \subset [-1, +1]$. The above definition follows the style of (2.1–2.3), but it may clearly be reformulated in the style of (2.4–2.6).

The set of all quantum correlation matrices of a given size $K \times L$ is a convex compact body M_{QB} in the KL-dimensional space of matrices.

Any matrix from M_{QB} can be represented in the form (2.15–2.16) with the additional restriction

$$\forall k \quad A_k^2 = 1, \qquad \forall l \quad B_l^2 = 1, \tag{2.17}$$

that is, $\mathrm{spec}(A_k)$ and $\mathrm{spec}(A_l)$ contain ± 1 only.

For any matrix $c = \{c_{kl}\}$ the corresponding quantum bound is

$$\max_{\gamma \in M_{\mathrm{QB}}} \sum_{kl} c_{kl}\gamma_{kl} = \max \| \sum_{k,l} c_{kl}A_kB_l \|, \tag{2.18}$$

the maximum on the right-hand side being taken over all collections of operators satisfying (2.16) or, equivalently, (2.16b, 2.17).

2.19. Theorem. A matrix $\{\gamma_{kl}\}$ is a quantum correlation matrix if and only if it admits a representation

$$\gamma_{kl} = \langle x_k, y_l \rangle \tag{2.20}$$

with some unit vectors x_k, y_l in a Euclidean space.

Extremal quantum correlation matrices $\gamma \in \mathrm{ex}(M_{\mathrm{QB}})$ are of special interest. A necessary condition close to a sufficient one follows.

2.21. Theorem. Let $\gamma \in \mathrm{ex}(M_{\mathrm{QB}})$; let $x_1, \dots, x_K, y_1, \dots, y_L$ be unit vectors in $\mathbb{R}^r$ such that each vector of $\mathbb{R}^r$ is a linear combination of $x_1, \dots, x_K, y_1, \dots, y_L$; and let $\gamma_{kl} = \langle x_k, y_l \rangle$ for all k, l. Then

(a) each vector of $\mathbb{R}^r$ is both a linear combination of $x_1, \dots, x_K$ and a linear combination of $y_1, \dots, y_L$;

(b) any quadratic form Q on $\mathbb{R}^r$ satisfying the equations $Q(x_k) = 0$, $Q(y_l) = 0$ for all k, l is identically zero;

(c) there exist numbers $\lambda_1, \dots, \lambda_K, \mu_1, \dots, \mu_L$ such that

$$\forall k \quad \lambda_k \ge 0; \qquad \sum_k \lambda_k = 1; \qquad \forall l \quad \mu_l \ge 0; \qquad \sum_l \mu_l = 1;$$

$$\text{and } \sum_k \lambda_k Q(x_k) = \sum_l \mu_l Q(y_l) \text{ for any quadratic form } Q \text{ on } \mathbb{R}^r.$$

2.22. Theorem. Let unit vectors $x_1, \dots, x_K, y_1, \dots, y_L \in \mathbb{R}^r$ have the properties (a–c) of Theorem 2.21, and in addition let $\lambda_k > 0$, $\mu_l > 0$ for all k, l. Then $\gamma \in \mathrm{ex}(M_{\mathrm{QB}})$, where $\gamma_{kl} = \langle x_k, y_l \rangle$.

The dimension r of the Euclidean space $\mathbb{R}^r$ is uniquely determined by $\gamma \in \mathrm{ex}(M_{\mathrm{QB}})$; we call r the *rank* of γ. It follows from (b) and (c) that

$$\frac{r(r+1)}{2} \le K + L - 1. \tag{2.23}$$

Vectors $x_1, \dots, x_K, y_1, \dots, y_L$ are determined by γ up to an isometry.

A matrix $\gamma = \{\gamma_{kl}\}$ is called a *classical correlation matrix,* if it admits a representation

$$\gamma_{kl} = \langle A_kB_l \rangle = \int_\Omega A_k(\omega)\mathrm{B}_l(\omega)\mathbb{P}(d\omega), \tag{2.24}$$

$$\forall k, l \;\; \forall \omega \in \Omega \quad |A_k(\omega)| \le 1, \; |B_l(\omega)| \le 1; \tag{2.25}$$

here A_k, B_l are random variables, that is, measurable functions on a probability space $(\Omega, \mathbb{P})$. Once again, an additional condition $A_k^2 = 1$, $B_l^2 = 1$ may be imposed without changing the class of all classical correlation matrices. These form a convex polytope M_{HDB} in the KL-dimensional space of matrices,

$$M_{\text{HDB}} \subset M_{\text{QB}}.$$

There is a natural connection between M_{QB} and M_{HDB} on the one hand, and $X_{\text{QB}}, X_{\text{HDB}}$ for the $(2 + \ldots + 2) \times (2 + \ldots + 2) = (2K) \times (2L)$ behavior scheme, on the other. Indeed, a behavior in such a scheme may be described by means of parameters

$$\alpha_k = \langle A_k \rangle, \quad \beta_l = \langle B_l \rangle, \quad \gamma_{kl} = \langle A_k B_l \rangle \tag{2.26}$$

as in (1.14). The KL-dimensional space of matrices γ may now be considered as a subspace of the d-dimensional space of triples (α, β, γ); here $d = (|M| - K + 1)(|N| - L + 1) - 1 = (2K - K + 1)(2L - L + 1) - 1 = K + L + KL$, see (1.4). The subspace is determined by equations $\alpha_1 = \ldots = \alpha_K = 0$, $\beta_1 = \ldots = \beta_L = 0$. It is easy to see that the intersection of the subspace with X_{HDB} is M_{HDB}, and the intersection with X_{QB} is M_{QB}. A natural projection of the d-dimensional space onto the KL-dimensional subspace emerges by discarding all α_k, β_l. The projection maps X_{HDB} onto M_{HDB}, and X_{QB} onto M_{QB}.

2.27 Theorem. If $\gamma = \{\gamma_{kl}\} \in M_{\text{QB}}$, then $\gamma' = \{\gamma'_{kl}\} \in M_{\text{HDB}}$, where

$$\gamma'_{kl} = \frac{2}{\pi} \arcsin \gamma_{kl}, \qquad \gamma_{kl} = \sin \frac{\pi}{2} \gamma'_{kl}. \tag{2.28}$$

The converse is false. An example follows of γ, γ' satisfying (2.28) such that $\gamma' \in M_{\text{HDB}}$, but $\gamma \notin M_{\text{QB}}$:

$$\gamma = \frac{1}{2} \begin{pmatrix} 2 & 1 & -1 & -2 & -1 \\ 1 & 2 & 1 & -1 & 1 \\ -1 & 1 & 2 & 1 & -1 \\ -2 & -1 & 1 & 2 & 1 \\ -1 & 1 & -1 & 1 & 2 \end{pmatrix}, \qquad \gamma' = \frac{1}{3} \begin{pmatrix} 3 & 1 & -1 & -3 & -1 \\ 1 & 3 & 1 & -1 & 1 \\ -1 & 1 & 3 & 1 & -1 \\ -3 & -1 & 1 & 3 & 1 \\ -1 & 1 & -1 & 1 & 3 \end{pmatrix}. \tag{2.29}$$

2.30. Theorem. Let $\gamma = \{\gamma_{kl}\}$, $\gamma' = \{\gamma'_{kl}\}$, $\gamma'' = \{\gamma''_{kl}\}$, and $\forall k,l \;\; \gamma_{kl} = \gamma'_{kl}\gamma''_{kl}$. Then

$$\begin{aligned} \gamma', \gamma'' \in M_{\text{HDB}} \quad &\Longrightarrow \quad \gamma \in M_{\text{HDB}}; \\ \gamma', \gamma'' \in M_{\text{QB}} \quad &\Longrightarrow \quad \gamma \in M_{\text{QB}}. \end{aligned}$$

2.31. Corollary. Let $\gamma = \{\gamma_{kl}\}$ and $\gamma' = \{\gamma'_{kl}\}$ be connected by the relation

$$\gamma'_{kl} = f(\gamma_{kl}), \qquad f(t) = \sum_{i=1}^{\infty} c_i t^i, \qquad \sum_{i=1}^{\infty} |c_i| \le 1.$$

Then

$$\gamma \in M_{\text{HDB}} \quad \Longrightarrow \quad \gamma' \in M_{\text{HDB}}; \qquad \gamma \in M_{\text{QB}} \quad \Longrightarrow \quad \gamma' \in M_{\text{QB}}.$$

Applying Theorem 2.31 with

$$f(t) = \left(\sinh \frac{\pi}{2}\right)^{-1} \sin \frac{\pi}{2} t$$

together with Theorem 2.27, we obtain

$$\gamma \in M_{\text{QB}} \quad \Longrightarrow \quad \left(\sinh \frac{\pi}{2}\right)^{-1} \gamma \in M_{\text{HDB}}.$$

The best constant, however, is the well-known* Grothendieck's constant K_{G}:

$$\gamma \in M_{\text{QB}} \quad \Longrightarrow \quad (K_{\text{G}})^{-1} \gamma \in M_{\text{HDB}}. \tag{2.32}$$

The Grothendieck's constant has been studied by mathematicians since 1956, but as yet it is only known that $K_{\text{G}} \approx 1.73 \pm 0.06$. This enigmatic constant is an exact constant for (2.32), when matrices of any size $K \times L$ are considered. For 2×2 matrices** the exact constant is $\sqrt{2}$.

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* In mathematics, but not yet in physics!

** And even for $3 \times L$ matrices with any L.

The role that Grothendieck's constant plays in correlation matrices of any size is the same role that $\sqrt{2}$ plays in 2×2 correlation matrices.

It appears to be unexpectedly difficult to give a low-dimensional example of $\gamma \in M_{\mathrm{QB}}$ such that $(1/\sqrt{2})\gamma \notin M_{\mathrm{HDB}}$. The best result is now a 20×20 matrix giving the ratio $\approx 1.428 > \sqrt{2}$ [FR93].

The simplest scheme

For the $(2+2) \times (2+2)$ behavior scheme we deal with four operators A_1, A_2, B_1, B_2, see (2.26), such that $A_k^2 = 1$, $B_l^2 = 1$, see (2.17). Fortunately, all the operators necessarily commute with $A_1A_2 + A_2A_1$ and $B_1B_2 + B_2B_1$. This good fortune (available for the $(2+2) \times (2+2)$ scheme exclusively!) allows us to reduce the general case to the well-studied pair of spin-1/2 particles. So, an explicit description of X_{QB} is available [Ts80], but it is too cumbersome to be reproduced here. In contrast, M_{QB} is simple enough: the necessary condition 2.27 appears to be sufficient for the $(2+2) \times (2+2)$ scheme. So, $\gamma \in M_{\mathrm{QB}}$ if and only if $\gamma' \in M_{\mathrm{HDB}}$ (see 2.28). But γ' belongs to M_{HDB} if and only if it satisfies 8 extremal Bell-type inequalities, see (1.8) and (1.16), that is,

$$\gamma' \in M_{\mathrm{HDB}} \iff \forall k,l \quad |\gamma'_{11} + \gamma'_{12} + \gamma'_{21} + \gamma'_{22} - 2\gamma'_{kl}| \le 2. \tag{2.33}$$

Hence

$$\gamma \in M_{\mathrm{QB}} \iff \forall k,l \quad |\arcsin\gamma_{11} + \arcsin\gamma_{12} + \arcsin\gamma_{21} + \arcsin\gamma_{22} - 2\arcsin\gamma_{kl}| \le \pi. \tag{2.34}$$

Trigonometric functions may be eliminated; an explicit algebraic formula was given [La88]:

$$|\gamma_{11}\gamma_{12} - \gamma_{21}\gamma_{22}| \le \sqrt{1-\gamma_{11}^2}\sqrt{1-\gamma_{12}^2} + \sqrt{1-\gamma_{21}^2}\sqrt{1-\gamma_{22}^2}, \tag{2.35}$$

and an explicit polynomial formula (of degree 6) was given [Ts85].

Remarks

Investigation of quantum restrictions was started in [Ts80]. Theorems 2.19 and 2.21(a–b) were proved in [Ts85]; 2.21(c) and 2.22 are new. Theorems 2.27, 2.30, and Corollaries 2.31, 2.32 are due to Grothendieck [Gr56], but of course for X_{QB} defined by (2.20) rather than (2.15–2.16); Bell-type version (2.32) of the corresponding Grothendieck's result was given in [Ts85], while 2.27, 2.30, and 2.31 are presented for the first time. Grothendieck's bounds for K_{G} were: $1.571 \approx \pi/2 \le K_{\mathrm{G}} \le \sinh(\pi/2) \approx 2.301$. A better upper bound $K_{\mathrm{G}} \le \pi/2\log(1+\sqrt{2}) \approx 1.782$ was given by Krivine [Kr79]. A lower bound better than $\pi/2$ was recently found by Reeds [Re93] in connection with a work [FR93] encouraged by [Ts85]. The main result of Fishburn and Reeds [FR93] states that the constant $\sqrt{2}$ is not suitable for 20×20 matrices. For sizes $4, 5, \dots, 19$ the question remains open!

Another approach was proposed [Pi86, 89] with (2.1) substituted by $p_{mn} = \mathrm{Tr}\big((P_m \wedge P_n)W\big)$ with noncommuting projections P_m, P_n; here $P_m \wedge P_n = \lim_{\nu\to\infty}(P_mP_n)^\nu$ is the projection onto the intersection $P_m(H) \cap P_n(H)$. Waiving locality, this approach missed crucial points of the theory presented here.

A. M. Vershik repeatedly asked me about the asymptotical ratio (in some sense) between X_{QB} and X_{HDB}, as the scheme grows (in some sense); but I am unable to reply.

3. Related properties of observables and states

Behaviors

The further from mathematics and closer to physics, the more detailed the description required for observables and states implementing quantum behaviors of interest. However, limitations peculiar to present-day technologies are beyond the scope of this article; see [FMS90, HS91, Sa91] for limitations, and [TWC91, Zu91, Ha91, YS93] for new technologies.

Any quantum behavior $p = \{p_{mn}\} \in X_{\mathrm{QB}}$ may be given by Hermitian operators $A_k : \mathcal{H}_1 \to \mathcal{H}_1$, $\mathrm{B}_l : \mathcal{H}_2 \to \mathcal{H}_2$ and a density matrix W on $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$p_{mn} = \mathrm{Tr}\big((E_m(A_k) \otimes E_n(B_l))W\big) \quad \text{for } m \in M_k,\ n \in N_L; \tag{3.1}$$

here $E_m(A_k)$ is the spectral projection operator, corresponding to $m \in M_k$, $M_k = \mathrm{spec}(A_k)$.

3.2. Problem. Does any $p \in X_{\mathrm{QB}}$ admit a representation (3.1) with finite-dimensional $\mathcal{H}_1$, $\mathcal{H}_2$?

If all A_k commute (that is, $A_{k_1}A_{k_2} = A_{k_2}A_{k_1}$ for all k_1, k_2), then $p \in X_{\mathrm{HDB}}$. If all B_l commute, then again $p \in X_{\mathrm{HDB}}$. Conversely, if $p \in X_{\mathrm{HDB}}$ for all W, then all A_k commute or/and all B_l commute [La87, p. 117].

If $W = \sum c_\nu W_\nu^{(1)} \otimes W_\nu^{(2)}$ with some $c_\nu \ge 0$ and some density matrices $W_\nu^{(1)}$ on $\mathcal{H}_1$ and $W_\nu^{(2)}$ on $\mathcal{H}_2$ (such W are called *classically correlated* or *decomposable*), then $p \in X_{\mathrm{HDB}}$. The converse is wrong: R.Werner [We89] discovered the existence of a density matrix W that is not classically correlated, but nevertheless $p \in X_{\mathrm{HDB}}$ for any choice of A_k, B_l. However, if $W = |\Psi\rangle\langle\Psi|$ for a vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$, and $p \in X_{\mathrm{HDB}}$ for any choice of A_k, B_l, then necessarily $\Psi = \Psi_1 \otimes \Psi_2$; see [HS91], [Gi91], [GP92], [MNR92].

Correlation matrices

The notion of quantum correlation matrix (see 2.15–2.16) was defined by means of arbitrary operators. Surprisingly, it appears to be closely related to anticommuting operators.

Suppose r is an even natural number, and Hermitian operators $X_1, \dots, X_r : \mathcal{H} \to \mathcal{H}$ satisfy

$$\forall i \ne j \quad X_i X_j = -X_j X_i; \qquad \forall i \quad X_i^2 = 1. \tag{3.3}$$

Then $\mathcal{H}$ may be identified with a tensor product $\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}'$ such that $\dim \mathcal{H}_r = 2^{r/2}$ and each X_i acts in fact on $\mathcal{H}_r$, that is, $X_i = X_i^{(r)} \otimes 1$, $X_i^{(r)} : \mathcal{H}_r \to \mathcal{H}_r$, $1 : \mathcal{H}' \to \mathcal{H}'$. The collection $(\mathcal{H}_r; X_1^{(r)}, \dots, X_r^{(r)})$ satisfying (3.3) exists for each $r = 2, 4, 6, \dots$ and is unique up to unitary equivalence. Let us call it *Clifford representation* of order r.*

There exists one and only one (up to a phase factor) entangled unit vector $\Psi \in \mathcal{H}_r \otimes \mathcal{H}_r$ satisfying

$$\forall i \quad \langle \Psi | X_i^{(r)} \otimes X_i^{(r)} | \Psi \rangle = 1. \tag{3.4}$$

Let us call it the *Clifford singlet state vector* of rank r.**

3.5. Theorem. Any quantum correlation matrix $\gamma \in M_{\mathrm{QB}}$ may be written as

$$\gamma_{kl} = \langle \Psi | A_k \otimes B_l | \Psi \rangle \tag{3.6}$$

where Ψ is the Clifford singlet state vector of some order r, all A_k, B_l being some linear combinations of $X_1^{(r)}, \dots, X_r^{(r)}$.

It is a luck! Even the existence of finite-dimensional implementation is not evident (cf. 3.2). However, the proof is simple: represent γ_{kl} as $\langle x_k, y_l \rangle$ following 2.20, and take

$$A_k = \sum_i x_k^{(i)} X_i^{(r)}, \qquad B_l = \sum_j y_l^{(j)} X_j^{(r)}; \tag{3.7}$$

here $x_k^{(1)}, \dots, x_k^{(r)}$ are coordinates of the vector x_k.

So, arbitrary operators may be replaced with Clifford operators (that is, linear combinations of $X_i^{(r)}$). The following theorem shows that Clifford operators are irreplaceable for an extremal case.

3.8. Theorem. Let (2.15–2.16) be fulfilled for some $\gamma \in \mathrm{ex}(M_{\mathrm{QB}})$, and $r = \mathrm{rank}(\gamma)$ be even. Then the Hilbert space $\mathcal{H}$, on which the operators A_k, B_l, W act, admits a decomposition

$$\mathcal{H} = \mathcal{H}_r \otimes \mathcal{H}_r \otimes \mathcal{H}' \oplus \mathcal{H}'' \tag{3.9}$$

into a pair of Clifford representation spaces and additional spaces $\mathcal{H}', \mathcal{H}''$ (of dimensions — zero, finite, or infinite) satisfying the following two conditions. First,

$$W = (\,|\Psi\rangle\langle\Psi|\,) \otimes W' \oplus 0 \tag{3.10}$$

with the Clifford singlet state vector $\Psi \in \mathcal{H}_r \otimes \mathcal{H}_r$ and some density matrix W' on $\mathcal{H}'$. Second, the subspace $\mathcal{H}_r \otimes \mathcal{H}_r \otimes \mathcal{H}'$ is invariant for all operators A_k, B_l, their restrictions onto $\mathcal{H}_r \otimes \mathcal{H}_r \otimes \mathcal{H}'$ being of the form

$$A_k|_{\mathcal{H}_r \otimes \mathcal{H}_r \otimes \mathcal{H}'} = A_k^{(r)} \otimes 1 \otimes 1, \qquad B_l|_{\mathcal{H}_r \otimes \mathcal{H}_r \otimes \mathcal{H}'} = 1 \otimes B_l^{(r)} \otimes 1 \tag{3.11}$$

with some Clifford operators $A_k^{(r)}, B_l^{(r)}$.

---

* For the simplest case $r = 2$ operators $X_1^{(2)}, X_2^{(2)}$ may be identified with well-known Pauli spin matrices σ_x, σ_y.

** For $r = 2$ it may be identified with the well-known singlet state of a pair of spin-1/2 particles, but with one particle rotated 180° around the z-axis.

Implementation of an extremal quantum correlation matrix (of even rank) is unique up to irrelevant tensor factor, irrelevant direct summand, and unitary equivalence. The single Clifford singlet state implements all matrices of a given rank.

The case of odd $r = \text{rank}(\gamma)$ is similar, but more involved; see [Ts85].

Schmidt coefficients

Any unit vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition

$$\Psi = \sum_{i=0}^{\infty} \lambda_i \varphi_i \otimes \theta_i \tag{3.12}$$

with some orthogonal unit vectors $\varphi_i \in \mathcal{H}_1$, $\theta_i \in \mathcal{H}_2$ and some $\lambda_0 \ge \lambda_1 \ge \dots \ge 0$, $\sum \lambda_i^2 = 1$. The sequence $\{\lambda_i\}$ — the *spectrum* of Ψ — is the sole invariant of an entangled vector Ψ, when no additional structure on $\mathcal{H}_1, \mathcal{H}_2$ is available.

The singlet state of a pair of spin-j particles has $2j+1$ equal Schmidt coefficients [Me80]:

$$\lambda_i = (2j+1)^{-1/2} \quad \text{for } 0 \le i < 2j+1, \qquad \lambda_i = 0 \quad \text{for } i \ge 2j+1. \tag{3.13}$$

The Clifford singlet state vector (3.4) of an even rank r has $2^{r/2}$ equal Schmidt coefficients. So, the Clifford singlet state of an even rank r may be identified with the singlet state for the spin j such that $2j+1 = 2^{r/2}$, if all operators are considered feasible observables.

3.14. Theorem. Let $\gamma \in \text{ex}(M_{\text{QB}})$, and $r = \text{rank}(\gamma)$ be even. Then for any unit vector $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ the following two conditions are equivalent.

(a) There exist Hermitian operators $A_k : \mathcal{H}_1 \to \mathcal{H}_1$ and $B_l : \mathcal{H}_2 \to \mathcal{H}_2$ such that for all k, l

$$\|A_k\| \le 1, \quad \|B_l\| \le 1, \quad \text{and} \quad \langle \Psi | A_k \otimes B_l | \Psi \rangle = \gamma_{kl}.$$

(b) The spectrum of Ψ has multiplicity $2^{r/2}$, that is, the sequence of Schmidt coefficients contains each number $2^{r/2}$ times.

3.15. Corollary.

It is impossible to implement all quantum correlation matrices (of all sizes $K \times L$) with a single state vector.

The simplest scheme

Applying Theorem 3.8 to a single but famous extremal quantum correlation matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} +1 & +1 \\ +1 & -1 \end{pmatrix}, \tag{3.16}$$

(the only matrix maximally violating the Bell-CHSH inequality $\gamma_{11} + \gamma_{12} + \gamma_{21} - \gamma_{22} \le 2$), and taking into account that the Clifford singlet for $r = 2$ is the usual singlet for spin-1/2 particles, we see that [SW87a, p.2442], [PR92]:

Each state maximally violating the Bell-CHSH inequality is essentially the same as the singlet state for a pair of spin-1/2 particles.

Nevertheless, such states in general are mixed [BMR92], since an arbitrary (irrelevant!) density matrix W' appears in Theorem 3.8 in addition to the (relevant) pure state $|\Psi\rangle\langle\Psi|$.

It is clear from (3.13) and Theorem 3.14 that the maximal violation of Bell-CHSH inequality can be implemented with the singlet state of a pair of spin-j particles for any half-integer j [GP92], but for no integer j [PR92].

Several estimations are known for the maximal violation R of Bell-CHSH inequality, implementable with a given entangled vector Ψ with known Schmidt coefficients λ_i:

$R \ge 2(1 + 4\lambda_0^2\lambda_1^2)^{1/2}$; [Gi91] (3.17$a$)

(in fact, $2(1 + 4\lambda_0\lambda_1)^{-1/2}$ was written instead, — an obvious mistake)

$$R \geq 2\bigl(1 + 4(\lambda_0\lambda_1 + \lambda_2\lambda_3 + \ldots)^2\bigr)^{1/2}; \qquad \text{[GP92]} \quad (3.17b)$$

$$R \geq 2 + 2(\lambda_0^2 + \lambda_1^2)\left(\sqrt{1+c^2} - 1\right) \quad \text{with } c = 2\lambda_0\lambda_1/(\lambda_0^2 + \lambda_1^2); \qquad \text{[MNR92]} \quad (3.17c)$$

$$R \geq 2\lambda_0^2 + 2\sqrt{2}(1 - \lambda_0^2); \qquad (3.17d)$$

(Tsirelson; announced [KT92, p. 894], proved [PT93]).

Only (3.17b) gives an exact result $2\sqrt{2}$ when $\{\lambda_i\}$ has multiplicity 2: $\lambda_0 = \lambda_1 \geq \lambda_2 = \lambda_3 \geq \ldots$. Only (3.17d) shows that $R \to 2\sqrt{2}$ when $\lambda_0 \to 0$.

Implementing all quantum behaviors with a single state

Let $\Psi \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a unit vector, and p a quantum behavior. A collection $\{P_m\}$, $\{P_n\}$, satisfying (2.6), such that $\forall m, n \ p_{mn} = \langle\Psi|P_m \otimes P_n|\Psi\rangle$, may be called an *implementation* of p with Ψ. Corollary 3.15 shows that for each Ψ there exists p, admitting no implementation with Ψ. This is why we introduce the following definition.

Let $\Psi_1, \Psi_2, \ldots \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be a sequence of unit vectors, and p a quantum behavior. A collection $\{P_m\}$, $\{P_n\}$, satisfying (2.6), is called an implementation of p with $\{\Psi_i\}$, if

$$\forall m, n \quad p_{mn} = \lim_{i\to\infty} \langle\Psi_i|P_m \otimes P_n|\Psi_i\rangle \qquad (3.18)$$

(that is, the limit exists and is equal to p_{mn}).

3.19. Theorem. There exists a sequence $\Psi_1, \Psi_2, \ldots \in \mathcal{H}_1 \otimes \mathcal{H}_2$ such that any quantum behavior (over any behavior scheme) admits an implementation with $\{\Psi_i\}$.

The general theory of states on C^*-algebras gives us a state ρ such that $\rho(A) = \lim\langle\Psi_i|A|\Psi_i\rangle$ for each A such that the limit exists. So, any quantum behavior admits an implementation with ρ: $p_{mn} = \rho(P_m \otimes P_n)$.

> A single state can implement all quantum behaviors (over all schemes), and all quantum correlation matrices (of all sizes), and maximally violate all Bell-type inequalities.

However, the existence of such ρ is highly nonconstructive; no concrete example of ρ can be given. This is why I prefer a sequence.

We saw a connection between Bohm's version of the EPR thought experiment and implementation of all quantum 2×2 correlation matrices. Interestingly, there is a connection between the original EPR thought experiment [EPR35] and implementation of all quantum behaviors!

Consider a pair of one-dimensional spinless particles with coordinate operators Q_1, Q_2 and momentum operators P_1, P_2. A sequence $\{\Psi_i\}$ of entangled state vectors of the pair will be called an *EPR-sequence,* if

$$\bigl\langle\Psi_i|(Q_1 - Q_2)^2|\Psi_i\bigr\rangle \to 0 \qquad \text{and} \qquad \bigl\langle\Psi_i|(P_1 + P_2)^2|\Psi_i\bigr\rangle \to 0 \qquad \text{for } i \to \infty. \qquad (3.20)$$

3.21. Theorem. There exists an EPR-sequence $\{\Psi_i\}$ implementing all quantum behaviors.

Clearly, for any EPR sequence

$$\bigl\langle\Psi_i|(Q_1 + Q_2)^2|\Psi_i\bigr\rangle \to \infty \qquad \text{and} \qquad \bigl\langle\Psi_i|(P_1 - P_2)^2|\Psi_i\bigr\rangle \to \infty \qquad \text{for } i \to \infty. \qquad (3.22)$$

Indeed, the uncertainty relation gives

$$\Delta(Q_1 - Q_2) \cdot \Delta(P_1 - P_2) \geq h, \qquad \Delta(Q_1 + Q_2) \cdot \Delta(P_1 + P_2) \geq h. \qquad (3.23)$$

Equalities hold for coherent states that give us the most natural example of an EPR sequence. However, my proof of Theorem 3.21 gives $\{\Psi_i\}$ such that the quantity

$$S = \frac{1}{h^2} \lim_{i\to\infty} \bigl(\Delta_i(Q_1 - Q_2) \cdot \Delta_i(P_1 - P_2) \cdot \Delta_i(Q_1 + Q_2) \cdot \Delta_i(P_1 + P_2)\bigr) \qquad (3.24)$$

is equal to ∞.

3.25. Problem. Is there an EPR sequence $\{\Psi_i\}$ with $S < \infty$, or even with $S = 1$, implementing all quantum behaviors?

3.26. Problem. Is there a Bell-type inequality that holds for all coherent states, but not for arbitrary states?

As was shown by Summers and Werner [SW87b], the vacuum state of the free boson field can simulate the EPR state with respect to some observables localized in spacelike separated regions of special kind (complementary wedge regions). They conclude that the Bell-CHSH inequality can be maximally violated in the vacuum state. Is it true for higher Bell-type inequalities?

Combining two pairs of particles, each having an entangled state vector $\Psi = \sum \lambda_i \varphi_i \otimes \theta_i,\ \lambda_0 < 1$, we obtain

$$\Psi^2 = \Psi \otimes \Psi = \sum_{i,j} \lambda_i \lambda_j (\varphi_i \otimes \varphi_j) \otimes (\theta_i \otimes \theta_j),$$

and clearly $\lambda_0(\Psi^2) = \big(\lambda_0(\Psi)\big)^2 = \lambda_0^2$. Similarly, $\lambda_0\big(\Psi^N\big) = \lambda_0^N$. Hence, $\lambda_0\big(\Psi^N\big) \to 0$ for $N \to \infty$. Applying (3.17d), we obtain [PT93]:

> Bell-CHSH inequality can be maximally violated with an infinite collection of independently and identically prepared correlated pairs.

3.27. Problem. Is it true for any Bell-type inequality?

Remarks

Theorem 3.5 is taken from [Ts85]; Theorem 3.8 is essentially a reformulation of a theorem from [Ts85]. Theorem 3.14 follows easily from 3.8 (and 3.5). All results of the last subsection ("implementing ... with a single state") are new.

4. Generalizations

The case of three and more correlated subsystems is attracting increasing attention [Zu91, Ha91, YS93] after recognizing the following surprising distinction between two- and three-point correlations: for the famous CHSH linear function of a behavior,*

$$F_{\mathrm{CHSH}}(p) = \gamma_{11} + \gamma_{12} + \gamma_{21} - \gamma_{22} = \langle A_1 B_1 + A_1 B_2 + A_2 B_1 - A_2 B_2 \rangle \tag{4.1}$$

we have three different bounds

$$\max_{p \in X_{\mathrm{HDB}}} F_{\mathrm{CHSH}}(p) = 2, \qquad \max_{p \in X_{\mathrm{QB}}} F_{\mathrm{CHSH}}(p) = 2\sqrt{2}, \qquad \max_{p \in X_{\mathrm{B}}} F_{\mathrm{CHSH}}(p) = 4, \tag{4.2}$$

while for a similar three-point expression

$$F_3(p) = \gamma_{211} + \gamma_{121} + \gamma_{112} - \gamma_{222} = \langle A_2 B_1 C_1 + A_1 B_2 C_1 + A_1 B_1 C_2 - A_2 B_2 C_2 \rangle \tag{4.3}$$

two of them coincide:

$$\max_{p \in X_{\mathrm{HDB}}} F_3(p) = 2, \qquad \max_{p \in X_{\mathrm{QB}}} F_3(p) = 4, \qquad \max_{p \in X_{\mathrm{B}}} F_3(p) = 4. \tag{4.4}$$

This fact was discovered by Greenberger, Horne, and Zeilinger (see [GHSZ90] and [Me90b, eq. (6)]) and, simultaneously, by Palatnik (see [KT92, eq. (1.7)]; appeared in preprint version of [KT92] in 1990). From the geometric point of view it means that X_{B} contains a q-face (of some dimension q) intersecting X_{QB}, but not X_{HDB}. It leads to "Bell's theorem without inequalities" [GHSZ90].

All general notions used in previous sections for two subsystems — behavior schemes, behaviors of various kinds (X_{B}, X_{DB}, X_{HDB}, X_{QB}), correlation matrices of corresponding kinds (M_{B}, M_{DB}, M_{HDB}, M_{QB}), quantum bounds and implementations — can readily be generalized to three and more subsystems. However, no generalization is known for the Schmidt decomposition (3.12)** and the Clifford singlet state (3.4). Theorem 2.19 cannot be generalized, because quantum bounds for three-point correlations, unlike two-point ones, are not of quadratic nature [Ts85, Prop. 5.2 with Remark].

4.5. Problem. Is there an absolute constant $K_3 < \infty$ such that

$$\gamma \in M_{\mathrm{QB}}^{(3)} \quad \Longrightarrow \quad (K_3)^{-1} \gamma \in M_{\mathrm{HDB}}^{(3)}$$

for any three-point behavior scheme? Here $M_{\mathrm{QB}}^{(3)}$ and $M_{\mathrm{HDB}}^{(3)}$ are three-point counterparts of M_{QB} and M_{HDB}; that is, $\gamma \in M_{\mathrm{QB}}^{(3)}$ when $\gamma_{klm} = \mathrm{Tr}(A_k B_l C_m W)$, see (2.15–2.16), and $\gamma \in M_{\mathrm{HDB}}^{(3)}$ when $\gamma_{klm} =$

* For notation see (2.18).

** We may write $\Psi = \sum \lambda_i \varphi_i \otimes \theta_i \otimes \zeta_i$, but it is not a general form for a vector of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

$\int A_k(\omega)B_l(\omega)C_m(\omega)\,\mathbb{P}(d\omega)$, see (2.24–2.25). If such K_3 exists, then its exact (minimal) value may be called the *triple Grothendieck-type constant.*

4.6. Problem. Find a generalization of Theorem 3.5 for triple correlations.

4.7. Problem. Find a generalization of Theorem 3.8 for triple correlations.

4.8. Problem. Is there a sequence $\{\Psi_i\}$ of vectors from $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ such that any quantum behavior (over any three-point behavior scheme) admits an implementation with $\{\Psi_i\}$? (Cf. Theorem 3.19).

As to the vacuum state of a quantum field: Landau [La87] showed that *some* non-hidden-deterministic behavior can be implemented with the vacuum state by an appropriate choice of observables localized in three given space-like separated domains (even if the domains are small and distant).

A study of a large number of subsystems was pioneered by Mermin [Me90a]. He gave the following generalization of (4.3) for the behavior scheme $(2+2)\times\ldots\times(2+2)=(2+2)^\nu$ with ν subsystems:

$$F_\nu = \langle \mathrm{Im}\big(\underbrace{(A_1+iA_2)(B_1+iB_2)\ldots}_{\nu\text{ factors}}\big)\rangle \tag{4.9}$$

(Im denotes the imaginary part) and found that (in our terms)

$$\max_{p\in X_{\mathrm{HDB}}} F_\nu(p) \le \begin{cases} 2^{\nu/2}, & \text{for even } \nu,\\ 2^{(\nu-1)/2}, & \text{for odd } \nu; \end{cases} \qquad \max_{p\in X_{\mathrm{QB}}} F_\nu(p) = 2^{\nu-1}; \qquad \max_{p\in X_{\mathrm{B}}} F_\nu(p) = 2^{\nu-1}. \tag{4.10}$$

So, the quantum/classical ratio grows exponentially with the number of subsystems, in contrast to the case of many observables but two subsystems (2.32).

A method to prepare an entangled state of ν spin-1/2 particles (in fact, atoms) can be found in [Ha91].

A definition of classical and quantum behaviors for multi-time behavior schemes was discussed [Ts80, KT85, KT92, VT92] on the basis of the standard description of local measurements [HK64, Sch68, HK70] (see also [Di91]). Predictions of local observables can be in contradiction [Pi91b]. Another kind of problem arises from nonlocal measurements, see [AA80]; Schmidt coefficients appear in this connection [AAV86]. Unexpectedly, measurements over non-correlated particles can also lead to nontrivial problems [PW91]. We have no reason to doubt that only quantum behaviors are feasible. Nevertheless, some mathematical mechanisms producing non-quantum behaviors (but respecting locality) were considered [La92, VT92].

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