8 Miscellany

8a Beyond the one-dimensional time

Scaling limits of models driven by *two-dimensional* arrays of random signs are evidently important. They appear in percolation theory. Also the Brownian web is an example and, after all, it may be treated as an oriented percolation.

In such cases, independent sub-σ-fields should correspond to disjoint regions of $\mathbb{R}^2$, not only of the form $(s,t)\times\mathbb{R}$. In fact, a rudimentary use of these can be found in Sect. 7 (recall 'cells' in 7c). In general it is unclear, what kind of regions can be used; at least, regions with piecewise smooth boundary should fit.

For the *general* theory of stability, spectral measures, decomposable processes etc., dimension of the underlying space is of little importance. Basically, regions must form a Boolean algebra. Such a general approach is used in [11], [10]. A classical noise is generated by decomposable processes, and a decomposable process can be extended naturally from an algebra of sets ('regions') to a σ-field; thus, the noise can be extended. The property appears to be characteristic; no nonclassical noise can be extended (under natural assumptions) to a σ-field [10].

Nonclassical factorizations appear already in zero-dimensional 'time', be it a Cantor set, or even a convergent sequence with limit point. For Cantor sets, see [11, Sect. 4]; some interesting models of combinatorial nature, with large symmetry groups (instead of 'time shifts' of a noise) are examined there. For a convergent sequence with limit point, see Chapter 1 here (namely, 1b1), and [10, Appendix].

8b The 'wave noise' approach

A completely different way of constructing noises is sketched here.

Consider the linear wave equation in dimension $1+1$,

$$\text{(8b1)} \qquad \Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \Big) u(x,t) = 0 \,,$$

with initial conditions $u(x,0)=0$, $u_t(x,0)=f(x)$. Its solution is well-known:

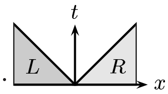
$$u(x,t) = \frac12 \int_{x-t}^{x+t} f(y)\,dy = \frac12 F(x+t) - \frac12 F(x-t) \,,$$

where F is defined by $F'(x)=f(x)$. The formula holds in a generalized sense for nonsmooth F, which covers the following case: $F(x)=B(x)=$ Brownian motion (combined out of two independent branches, on $[0,+\infty)$ and on $(-\infty,0]$); $f(x)=B'(x)=$ white noise. The random field on $(-\infty,\infty)\times[0,\infty)$,

$$u(x,t) = \frac12 B(x+t) - \frac12 B(x-t) \,, \qquad B = \text{ Brownian motion,}$$

is continuous, stationary in x, scaling invariant (for any c the random field $u(cx,ct)/\sqrt{c}$ has the same distribution as $u(x,t)$), satisfies the wave equation (8b1) and the following independence condition:

(8b2) $$u\big|_L \text{ and } u\big|_R \text{ are independent, where } L = \{(x,t) : x < -t < 0\}, \quad R = \{(x,t) : x > t > 0\}\,.$$



The independence is a manifestation of:

- independence of the white noise: its integrals over disjoint segments are independent;
- hyperbolicity of the wave equation: propagation speed does not exceed 1.

A solution with such properties is essentially unique. That is, if $u(x,t)$ is a continuous random field on $(-\infty,\infty)\times(0,\infty)$, stationary in x, satisfying the wave equation (8b1) and the independence condition (8b2), then necessarily $u(x,t) = \mu_0 + \mu_1 t + \sigma\big(B(x+t) - B(x-t)\big)$ for a Brownian motion B. Scaling invariance forces $\mu_0 = \mu_1 = 0$.

It is instructive that a wave equation may be used in a non-traditional way. Traditionally, a solution is determined by its initial values. In contrast, the independence condition (8b2), combined with some more conditions, determines a random solution with no help of initial conditions! Not an individual sample function is determined, of course, but its distribution (a probability measure on the space of solutions of the wave equation).

A man with no preexisting idea of white noise or Brownian motion can, in principle, use the above approach. Observing that $u(x,0) = 0$ but $u_t(x,0)$ does not exist (in the classical sense), he may investigate $u(x,t)/t$ for $t \to 0$ as a way toward the white noise.

8b3 Question. Can we construct a nonclassical (especially, black) noise, using a nonlinear hyperbolic equation?

I tried once such a nonlinear wave equation:

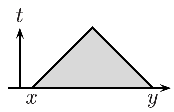
(8b4) $$\Big(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\Big) u(x,t) = \varepsilon t^{-(3-\varepsilon)/2} \sin\big(t^{-(1+\varepsilon)/2} u(x,t)\big)\,,$$

ε being a small positive parameter. The equation is scaling-invariant: if $u(x,t)$ is a solution, then $u(cx,ct)/c^{(1+\varepsilon)/2}$ is also a solution. We search for a random field $u(t,x)$, continuous, stationary in x, scaling invariant, satisfying (8b4) and the independence condition (8b2). Its behavior for $t \to 0$ should give us a new noise. The following questions arise.

- Does such a random field exist?
- Is it unique (in distribution)?

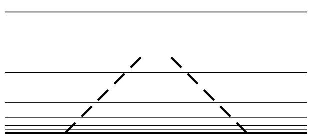
If the answers are positive, then we get a noise,

$$\mathcal{F}_{x,y} \text{ is the } \sigma\text{-field generated by } \{u(z,t) : x+t < z < y-t\}\,,$$



and maybe it is black. However, I did not succeed with it.

A modified 'waive noise' approach was used successfully in [11, Sect. 5], proving in the first time existence of a black noise. The modification is, to keep the auxiliary dimension, but make it discrete rather than continuous:



More specifically, consider a sequence of stationary random processes $u_k(\cdot)$ on $\mathbb{R}$ such that

- u_k is $2\varepsilon_k$-dependent (for some $\varepsilon_k \to 0$); it means that $u_k\big|_{(-\infty,-\varepsilon_k]}$ and $u_k\big|_{[\varepsilon_k,+\infty)}$ are independent;
- $u_{k-1}(x)$ is uniquely determined by $u_k\big|_{[x-(\varepsilon_{k-1}-\varepsilon_k),\,x+(\varepsilon_{k-1}-\varepsilon_k)]}$.

Such a sequence (u_k) determines a noise; namely, $\mathcal{F}_{x,y}$ is generated by all $u_k(z)$ such that $x+\varepsilon_k \le z \le y-\varepsilon_k$. White noise can be obtained by a linear system of Gaussian processes:

$$u_{k-1}(x) = \int_{x-(\varepsilon_{k-1}-\varepsilon_k)}^{x+(\varepsilon_{k-1}-\varepsilon_k)} V_k(y-x)u_k(y)\,dy$$

where kernels V_k, concentrated on $[-(\varepsilon_{k-1}-\varepsilon_k),(\varepsilon_{k-1}-\varepsilon_k)]$, are chosen appropriately. A nonlinear system (of quite non-Gaussian processes) of the form

$$u_{k-1}(x) = \varphi\biggl(\frac{\text{const}}{\varepsilon_{k-1}-\varepsilon_k}\int_{x-(\varepsilon_{k-1}-\varepsilon_k)}^{x+(\varepsilon_{k-1}-\varepsilon_k)} u_k(y)\,dy\biggr)$$

was used for constructing a black noise. Though, it is not really a *construction* of a specific noise. Existence of (u_k) is proven, but uniqueness (in distribution) is not. True, every such (u_k) determines a black noise. However, no one of them is singled out.

8c Groups, semigroups, kernels

A Brownian motion X in a topological group G is defined as a continuous G-valued random process with stationary independent increments, starting from the unit of G. For example, if G is the additive group of reals, then the general form of a Brownian motion in G is $X(t) = \sigma B(t) + vt$, where $B(\cdot)$ is the standard Brownian motion, $\sigma \in [0,\infty)$ and $v \in \mathbb{R}$ are parameters. If G is a Lie group, then Brownian motions X in G correspond to Brownian motions Y in the tangent space of G (at the unit) via the stochastic differential equation $(dX)\cdot X^{-1} = dY$ (in the sense of Stratonovich).

A noise corresponds to every Brownian motion in a topological group, just as the white noise corresponds to $B(\cdot)$. If the noise is classical, it is the white noise of some dimension $(0,1,2,\dots$ or $\infty)$. If it is the case for all Brownian motions in G, we'll call G a *white group.* Thus, $\mathbb{R}$ is white, and every Lie group is white. Every commutative topological group is white (see [7, Th. 1.8]). The group of all unitary operators in l_2 (equipped with the strong operator topology) is white (see [7, Th. 1.6]). Many other groups are white since they are embeddable to a group known to be white; for example, the group of diffeomorphisms is white (an old result of Baxendale).

8c1 Question. The group of all homeomorphisms of (say) $[0,1]$, is it white?

In a topological group, Brownian motions X and continuous stochastic flows ξ are basically the same:

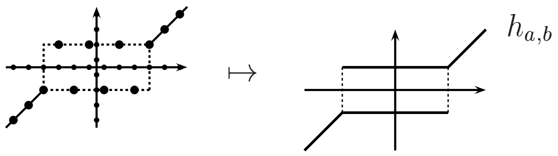
$$X(t) = \xi_{0,t}\,; \qquad \xi_{s,t} = X^{-1}(s)X(t)\,.$$

In a semigroup, however, a noise corresponds to a flow, not to a Brownian motion (see also 4c4).

A nonclassical noise (of stickiness) was constructed in Sect. 4 out of an abstract flow in a 3-dimensional semigroup G_3; however, G_3 is not a topological semigroup, since composition is discontinuous.

8c2 Question. Can a nonclassical noise arise from an abstract stochastic flow in a finite-dimensional topological semigroup?

The continuous (but not topological) semigroup G_3 emerged in Sect. 4 from the discrete semigroup G_3^{discrete} via scaling limit. Or rather, a flow in G_3 emerged from a flow in G_3^{discrete} via scaling limit. A similar approach to the discrete model of 1e1 gives something unexpected. The continuous semigroup emerged is G_2, the two-dimensional topological semigroup described in (4d1). However, its representation is not single-valued:



You see, $h_{a,b}(x)$ for $x \in (-b, b)$ is $\pm(a+b)$, that is, either $a+b$ or $-(a+b)$ with probabilities $0.5, 0.5$. Such h is not a function, of course. Rather, it is a *kernel,* that is, a measurable map from $\mathbb{R}$ into the space of probability measures on $\mathbb{R}$. Composition of kernels is well-defined, thus, a representation (of a semigroup) by kernels (rather than functions) is also well-defined.

The stochastic flow in G_2, resulting from 1e1 via scaling limit, is identical to the flow $(\xi_{s,t}^{(2)})$ of 4g. Its noise is the usual (one-dimensional) white noise. The representation of G_2 by kernels turns the abstract flow into a *stochastic flow of kernels* as defined by Le Jan and Raimond [4, Def. 1.1.3]. However, a kernel (unlike a function) introduces an additional level of randomness. You see, when the kernel says that $h_{a,b}(x) = \pm(a+b)$, someone must choose at random one of the two possibilities. The additional randomness does not contribute to the stable σ-field, which leads to a nonclassical noise. See [4, Sect. 4.4].

The general theory of Le Jan and Raimond deals with convolution semigroups in semigroups of kernels, in spite of the fact that the semigroup of kernels is usually not a standard Borel space, and the composition (of kernels) is not a Borel measurable operation.

8c3 Question. Can we reformulate the theory of Le Jan and Raimond in terms of standard Borel spaces and Borel operations?

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