3 Scaling limit and independence

3a Product of coarse probability spaces

Having two coarse probability spaces $\big((\Omega_1[i],\mathcal{F}_1[i],P_1[i])_{i=1}^\infty,\mathcal{A}_1\big)$ and $\big((\Omega_2[i],\mathcal{F}_2[i],P_2[i])_{i=1}^\infty,\mathcal{A}_2\big)$, we define their product as the coarse probability space $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty,\mathcal{A}\big)$ where for each i,

$$(\Omega[i],\mathcal{F}[i],P[i]) = (\Omega_1[i],\mathcal{F}_1[i],P_1[i]) \times (\Omega_2[i],\mathcal{F}_2[i],P_2[i])$$

is the usual product of probability spaces, and $\mathcal{A}$ is the smallest coarse σ-field that contains $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$, where $A_1 \times A_2 \subset \Omega[\text{all}]$ is defined by $\forall i \quad (A_1 \times A_2)[i] = A_1[i] \times A_2[i]$. Existence of such $\mathcal{A}$ is ensured by Lemma 2b3. We write $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$.

3a1 Lemma. The refinement of the product of two coarse probability spaces is (canonically isomorphic to) the product of their refinements.

Proof. Denote these refinements by $(\Omega_1,\mathcal{F}_1,P_1)$, $(\Omega_2,\mathcal{F}_2,P_2)$ and $(\Omega,\mathcal{F},P)$. Both $\text{MALG}(\Omega_1,\mathcal{F}_1,P_1)$ and $\text{MALG}(\Omega_2,\mathcal{F}_2,P_2)$ are naturally embedded into $\text{MALG}(\Omega,\mathcal{F},P)$ as *independent* subalgebras. They generate $\text{MALG}(\Omega,\mathcal{F},P)$ due to Lemma 2c6. □

Given an arbitrary coarse σ-field $\mathcal{A}$ on the product coarse sample space $\big((\Omega_1[i],\mathcal{F}_1[i],P_1[i]) \times (\Omega_2[i],\mathcal{F}_2[i],P_2[i])\big)_{i=1}^\infty$, we may ask, whether $\mathcal{A}$ is a product, that is, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$ for some $\mathcal{A}_1,\mathcal{A}_2$, or not. No need to check all $\mathcal{A}_1,\mathcal{A}_2$. Rather, we have to check

$$\mathcal{A}_1 = \{A_1 : A_1 \times \Omega_2 \in \mathcal{A}\}\,, \quad \mathcal{A}_2 = \{A_2 : \Omega_1 \times A_2 \in \mathcal{A}\}\,;$$

of course, $A_1 \times \Omega_2 \subset \Omega[\text{all}]$ is defined by $\forall i \ (A_1 \times \Omega_2)[i] = A_1[i] \times \Omega_2[i]$. If $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$ generates $\mathcal{A}$, then $\mathcal{A}$ is a product; otherwise, it is not.

The refinement $\mathcal{F}$ of $\mathcal{A}$ contains two sub-σ-fields $\mathcal{F}_1 = \{(A_1 \times \Omega_2)[\infty] : A_1 \in \mathcal{A}_1\}$, $\mathcal{F}_2 = \{(\Omega_1 \times A_2)[\infty] : A_2 \in \mathcal{A}_2\}$. They are independent:

$$P(A \cap B) = P(A)P(B) \quad \text{for } A \in \mathcal{F}_1,\ B \in \mathcal{F}_2\,.$$

3a2 Lemma. $\mathcal{A}$ is a product if and only if $\mathcal{F}_1,\mathcal{F}_2$ generate $\mathcal{F}$.

Proof. We apply Lemma 2c6 to $\{A_1 \times A_2 : A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$. □

3a3 Note. It is well-known that a generating pair of independent sub-σ-fields means that $(\Omega,\mathcal{F},P)$ is (isomorphic to) the product of two probability spaces. So, a coarse probability space is a product if and only if its refinement is a product. (Assuming, of course, that the coarse *sample* space is a product.)

Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. Consider Hilbert spaces $H_1[i] = L_2(\Omega_1[i],\mathcal{F}_1[i],P_1[i])$, $H_2[i] = L_2(\Omega_2[i],\mathcal{F}_2[i],P_2[i])$, $H[i] = L_2(\Omega[i],\mathcal{F}[i],P[i])$. For each i, the space $H[i]$ is (canonically isomorphic to) $H_1[i] \otimes H_2[i]$. Indeed, for $x_1 \in H_1[i]$, $x_2 \in H_2[i]$ we define $x_1 \otimes x_2 \in H[i]$ by $(x_1 \otimes x_2)(\omega_1,\omega_2) = x_1(\omega_1)x_2(\omega_2)$, then $\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle = \langle x_1,y_1\rangle\langle x_2,y_2\rangle$, and factorizable vectors (of the form $x_1 \otimes x_2$) span the space $H[i]$. We know (see Lemma 2c7) that the refinement $H[\infty]$ of $\big((H[i])_{i=1}^\infty, L_2(\mathcal{A})\big)$ is $L_2(\Omega,\mathcal{F},P)$. Also, $H_1[\infty] = L_2(\Omega_1,\mathcal{F}_1,P_1)$ and $H_2[\infty] = L_2(\Omega_2,\mathcal{F}_2,P_2)$. Using Lemma 3a1 we get $H[\infty] = H_1[\infty]\otimes H_2[\infty]$. In that sense,

$$\operatorname{Lim}\big(H_1[i]\otimes H_2[i]\big) = \big(\operatorname{Lim} H_1[i]\big)\otimes\big(\operatorname{Lim} H_2[i]\big)\,.$$

If $x\in L_2(\mathcal{A}_1)$, $y\in L_2(\mathcal{A}_2)$, we define $x\otimes y$ by $(x\otimes y)[i] = x[i]\otimes y[i]$ for all i, and we get $x\otimes y\in L_2(\mathcal{A})$ and $(x\otimes y)[\infty] = x[\infty]\otimes y[\infty]$, that is,

$$\operatorname{Lim}\big(x[i]\otimes y[i]\big) = \big(\operatorname{Lim} x[i]\big)\otimes\big(\operatorname{Lim} y[i]\big)\,. \tag{3a4}$$

Indeed, it holds for (linear combinations of) indicators of coarse events. Note also that linear combinations of factorizable vectors are dense in $L_2(\mathcal{A})$.

Assume that $R_1[i] : H_1[i]\to H_1[i]$, $R_2[i] : H_2[i]\to H_2[i]$ are linear operators, possessing limits $R_1[\infty] = \operatorname{Lim} R_1[i]$, $R_2[\infty] = \operatorname{Lim} R_2[i]$. Consider linear operators $R_1[i]\otimes R_2[i] = R[i] : H[i]\to H[i]$. (It means that $R[i]x[i] = R_1[i]x_1[i]\otimes R_2[i]x_2[i]$ whenever $x[i] = x_1[i]\otimes x_2[i]$.) If $\sup_i\|R_1[i]\|<\infty$, $\sup_i\|R_2[i]\|<\infty$, then $\operatorname{Lim} R[i] = R_1[\infty]\otimes R_2[\infty]$, that is,

$$\operatorname{Lim}\big(R_1[i]\otimes R_2[i]\big) = \big(\operatorname{Lim} R_1[i]\big)\otimes\big(\operatorname{Lim} R_2[i]\big)\,. \tag{3a5}$$

Indeed, we have to check that

$$\operatorname{Lim}\big(R_1[i]\otimes R_2[i]\big)x[i] = \big(\operatorname{Lim} R_1[i]\otimes\operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x[i]\big)$$

for all $x\in L_2(\mathcal{A})$. We may assume that x is factorizable, $x = x_1\otimes x_2$; then

$$\begin{aligned}
&\operatorname{Lim}\big(R_1[i]\otimes R_2[i]\big)\big(x_1[i]\otimes x_2[i]\big) = \\
&\qquad = \operatorname{Lim}\big(R_1[i]x_1[i]\otimes R_2[i]x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i]x_1[i]\big)\otimes\big(\operatorname{Lim} R_2[i]x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i]\big)\big(\operatorname{Lim} x_1[i]\big)\otimes\big(\operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x_2[i]\big) = \\
&\qquad = \big(\operatorname{Lim} R_1[i]\otimes\operatorname{Lim} R_2[i]\big)\big(\operatorname{Lim} x_1[i]\otimes\operatorname{Lim} x_2[i]\big)\,.
\end{aligned}$$

Especially, let $R_2[i]$ be the orthogonal projection to the one-dimensional subspace of constants (basically, the expectation), and $R_1[i]$ be the unit (identity) operator, then $\big(R_1[i]\otimes R_2[i]\big)\big(x[i]\big) = \mathbb{E}\,\big(\,x[i]\,\big|\,\mathcal{F}_1[i]\,\big)$; indeed, it holds for factorizable vectors. Further, $R_2[\infty] = \operatorname{Lim} R_2[i]$ is the expectation on $(\Omega_2,\mathcal{F}_2,P_2)$, since convergence of vectors implies convergence of one-dimensional projections, and constant functions on $\Omega_2[\text{all}]$ belong to $L_2(\mathcal{A})$. So,

$$\operatorname{Lim}\mathbb{E}\,\big(\,x[i]\,\big|\,\mathcal{F}_1[i]\,\big) = \mathbb{E}\,\big(\,\operatorname{Lim} x[i]\,\big|\,\mathcal{F}_1\,\big) \tag{3a6}$$

for all $x\in L_2(\mathcal{A})$.

All the same holds for the product of any finite number of spaces (not just two).

3b Dyadic case

Let $(\Omega[i],\mathcal{F}[i],P[i])$ be the space of all maps $\frac{1}{i}\mathbb{Z}\to\{-1,+1\}$ with the usual product measure. That is, we have independent random signs $\tau_{k/i}$ for all integer k;[14] each random sign takes on

[14]Rigorously, I should denote it by $\tau_k[i]$, but $\tau_{k/i}$ is more expressive. Though, $\tau_{2/6}$ is not the same as $\tau_{1/3}$, but hopefully it does not harm.

two values ± 1 with probabilities 50%, 50%. The coarse sample space $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$ will be called the *dyadic coarse sample space.*[15] Let $\mathcal{A}$ be a coarse σ-field on the dyadic coarse sample space. What about decomposing it, say, into the past and the future w.r.t. a given instant?

Let us define a *coarse instant* as a sequence $t = \big(t[i]\big)_{i=1}^\infty$ such that $t[i] \in \frac1i \mathbb{Z}$ (that is, $it[i] \in \mathbb{Z}$) for all i, and there exists $t[\infty] \in \mathbb{R}$ (call it the refinement of the coarse instant) such that $t[i] \to t[\infty]$ for $i \to \infty$. A *coarse time interval* is a pair (s,t) of coarse instants s,t such that $s \le t$ in the sense that $s[i] \le t[i]$ for all i.

For every coarse time interval (s,t) we define the coarse probability space $\big((\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i])_{i=1}^\infty, \mathcal{A}\big)$ as follows. First, $\Omega_{s,t}[i]$ is the space of all maps $\big(\frac1i \mathbb{Z} \cap [s[i], t[i])\big) \to \{-1,+1\}$.[16] Second, $\mathcal{F}_{s,t}[i]$ and $P_{s,t}[i]$ are defined naturally, and we have the canonical measure preserving map $(\Omega[i], \mathcal{F}[i], P[i]) \to (\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i])$. Third, each $A \subset \Omega_{s,t}[\text{all}]$ has its inverse image in $\Omega[\text{all}]$; if the inverse image of A belongs to $\mathcal{A}$ then (and only then) A belongs to $\mathcal{A}_{s,t}$, which is the definition of $\mathcal{A}_{s,t}$. It is easy to see that $\mathcal{A}_{s,t}$ is a coarse σ-field.

Given coarse time intervals (r,s) and (s,t), we have

$$\big(\Omega_{r,t}[i], \mathcal{F}_{r,t}[i], P_{r,t}[i]\big) = \big(\Omega_{r,s}[i], \mathcal{F}_{r,s}[i], P_{r,s}[i]\big) \times \big(\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i]\big)\,,$$

and we may ask, whether $\mathcal{A}_{r,t}$ is a product, that is, $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$, or not.

3b1 Definition. A *dyadic coarse factorization* is a coarse probability space $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ such that $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$ is the dyadic coarse sample space, and

$$\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$$

whenever r,s,t are coarse instants such that $r[i] \le s[i] \le t[i]$ for all i; and

$$\mathcal{A} \text{ is generated by } \bigcup_{(s,t)} \mathcal{A}_{s,t}\,,$$

where the union is taken over all coarse time intervals (s,t).

Every family $(\mathcal{A}_{s,t})_{s \le t}$ of coarse σ-fields $\mathcal{A}_{s,t}$ on coarse sample spaces $\big(\Omega_{s,t}[i], \mathcal{F}_{s,t}[i], P_{s,t}[i]\big)_{i=1}^\infty$, indexed by all coarse time intervals (s,t) and satisfying $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$ whenever $r \le s \le t$, corresponds to a dyadic coarse factorization.

3b2 Example. Given a coarse time interval (s,t), we consider $f_{s,t} : \Omega[\text{all}] \to \mathbb{R}$,

$$f_{s,t}(\omega) = \frac{1}{\sqrt{i}} \sum_{k: s[i] \le k/i < t[i]} \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

Only $s[\infty], t[\infty]$ matter, in the sense that

$$\int_{\Omega[i]} \frac{|\tilde f[i] - f[i]|}{1 + |\tilde f[i] - f[i]|}\, dP[i] \xrightarrow[i\to\infty]{} 0 \tag{3b3}$$

[15] Sometimes a subsequence is used; say, $i \in \{2,4,8,16,\dots\}$ only; or equivalently, $\Omega[i]$ is the space of maps $2^{-i}\mathbb{Z} \to \{-1,+1\}$; see 3b6, 3b7.

[16] It may happen that $s[i] = t[i]$, then $\Omega_{s,t}[i]$ contains a single point.

if $f = f_{s,t}$ and $\tilde f = f_{\tilde s,\tilde t}$ is such a function built for a different coarse time interval $(\tilde s, \tilde t)$ satisfying $\tilde s[\infty] = s[\infty]$, $\tilde t[\infty] = t[\infty]$. Moreover, $\|\tilde f[i] - f[i]\|_{L_2[i]} \to 0$ for $i \to \infty$. We choose a sequence of coarse time intervals, $(s_n, t_n)_{n=1}^\infty$, such that the sequence of their refinements, $(s_n[\infty], t_n[\infty])$ is dense among all (usual, not coarse) intervals, then the sequence $\big(f_{s_n,t_n}\big)_{n=1}^\infty$ satisfies the condition of Lemma 2c10 and therefore it generates a coarse σ-field $\mathcal{A}$. It is easy to see that $\mathcal{A}$ does not depend on the choice of (s_n, t_n). Clearly, the refinement of $f_{s,t}$ is the increment $B(t[\infty]) - B(s[\infty])$ of the usual Brownian motion $B(\cdot)$.

Given three coarse instants $r \le s \le t$, we have

$$f_{r,t} = f_{r,s} + f_{s,t}\,.$$

It shows that $f_{r,t}$ is coarsely measurable w.r.t. the product of two coarse σ-fields $\mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$, which means $\mathcal{A}_{r,t} = \mathcal{A}_{r,s} \otimes \mathcal{A}_{s,t}$. So, we have a dyadic coarse factorization. We may call it the Brownian coarse factorization.

3b4 Example. Let $f_{s,t}(\omega)$ be the same as in 3b2 and in addition,

$$g_{s,t}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} (-1)^k \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

In the scaling limit we get two independent Brownian motions B_1, B_2; the refinement of $f_{s,t}$ is $B_1(t[\infty]) - B_1(s[\infty])$, the refinement of $g_{s,t}$ is $B_2(t[\infty]) - B_2(s[\infty])$. By the way, $(-1)^k$ cannot be replaced with $(-1)^{k-s[i]}$; it would violate the condition of 2c10.

We may also consider

$$f_{s,t}^{(n)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \exp\Big(2\pi\sqrt{-1}\cdot\frac{k}{n}\Big) \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]$$

for $n = 1, 2, 3, \dots$ In the scaling limit we get two real-valued Brownian motions B_1, B_2 and infinitely many complex-valued Brownian motion $B_3, B_4, \dots$ All B_n are independent.

Another construction of that kind:

$$f_{s,t}^{(\lambda)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \exp\Big(2\pi\sqrt{-1}\cdot\lambda\frac{k}{\sqrt{i}}\Big) \tau_{k/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

In the scaling limit, each $\lambda \in (0, \infty)$ gives a complex-valued Brownian motion B_λ. Any finite or countable set of numbers λ may be used, and leads to independent Brownian motions. Note that we cannot use more than a countable set of λ, since separability is stipulated by the definition of a coarse probability space.

3b5 Example. For $n = 1, 2, \dots$ we introduce

$$f_{s,t}^{(n)}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \prod_{m=1}^{n} \tau_{(k+m)/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,.$$

In the scaling limit we get independent Brownian motions B_n.

Another construction of that kind:

$$f^{(\lambda)}_{s,t}(\omega) = \frac{1}{\sqrt{i}} \sum_{k:s[i]\le k/i<t[i]} \prod_{m=1}^{\operatorname{entier}(\lambda\sqrt{i})} \tau_{(k+m)/i}(\omega) \quad \text{for } \omega \in \Omega[i]\,;$$

any finite or countable set of numbers $\lambda \in (0,\infty)$ may be used, and leads to independent Brownian motions B_λ.

Note that we cannot take the product over $m = 1,\dots,\operatorname{entier}(\lambda i)$; that would destroy factorizability.

3b6 Example. Here we restrict ourselves to $i \in \{2,4,8,16,\dots\}$, thus violating a little our framework. We let

$$g_{s,t}(\omega) = \sum_{k:s[i]\le k/i<(k+n-1)/i<t[i]} \frac{1+\tau_{k/i}(\omega)}{2} \prod_{m=1}^{n-1} \frac{1-\tau_{(k+m)/i}(\omega)}{2} \quad \text{for } \omega \in \Omega[i],\ i = 2^n\,.$$

That is, $g_{s,t} : \Omega[\mathrm{all}] \to \{0,1,2,\dots\}$ counts combinations '$+\,-\,\dots-$' of one plus sign and $(n-1)$ minus signs in succession. In the scaling limit we get the Poisson process.

3b7 Example. Let $f_{s,t}$ be as in Example 3b2 (Brownian), while $g_{s,t}$ be as in Example 3b6 (Poisson). Taken together, they generate a coarse σ-field. The corresponding scaling limit consists of two *independent* processes, Brownian and Poisson.

Let $\bigl((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty,\mathcal{A}\bigr)$ be a dyadic coarse factorization. Being a coarse probability space, it has a refinement $(\Omega,\mathcal{F},P)$. For every coarse time interval (s,t) we have a coarse sub-σ-field $\mathcal{A}_{s,t} \subset \mathcal{A}$ and its refinement, a sub-σ-field $\mathcal{F}_{s,t} \subset \mathcal{F}$. By Lemma 3a1,

$$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} \quad \text{whenever } r \le s \le t\,.$$

3b8 Lemma. If $s[\infty] = t[\infty]$ then $\mathcal{F}_{s,t}$ is degenerate (that is, contains sets of probability 0 or 1 only).

Proof. Consider the coarse instant r,

$$r[i] = \begin{cases} s[i] & \text{for } i \text{ even,} \\ t[i] & \text{for } i \text{ odd.} \end{cases}$$

For every $A \in \mathcal{A}_{s,r}$,

$$P(A) = \lim_{i\to\infty} P[i]\bigl(A[i]\bigr) = \lim_{i\to\infty} P[2i]\bigl(A[2i]\bigr) \in \{0,1\}\,,$$

since $\mathcal{F}_{s,r}[2i]$ is degenerate. So, $\mathcal{A}_{s,r}$ is degenerate. Similarly, $\mathcal{A}_{r,t}$ is degenerate. However, $\mathcal{A}_{s,t} = \mathcal{A}_{s,r} \otimes \mathcal{A}_{r,t}$. $\square$

3b9 Lemma. $\mathcal{F}_{s,t}$ depends only on $s[\infty], t[\infty]$.

Proof. Let (u, v) be another coarse time interval such that $u[\infty] = s[\infty]$ and $v[\infty] = t[\infty]$; we have to prove that $\mathcal{F}_{s,t} = \mathcal{F}_{u,v}$. Assume that $s[\infty] < t[\infty]$ (otherwise both $\mathcal{F}_{s,t}$ and $\mathcal{F}_{u,v}$ are degenerate). Assume also that $s[i] \le v[i]$ and $u[i] \le t[i]$ for all i (otherwise we correct them on a finite set of indices i).

Further, we may assume that $s \le u \le v \le t$; otherwise we turn to $s \wedge u \le s \vee u \le t \wedge v \le t \vee v$, where $(s \wedge u)[i] = s[i] \wedge u[i] = \min\bigl(s[i], u[i]\bigr)$, etc; both $\mathcal{F}_{s,t}$ and $\mathcal{F}_{u,v}$ are sandwiched between $\mathcal{F}_{s\wedge u,t\vee v}$ and $\mathcal{F}_{s\vee u,t\wedge v}$.

Finally, $\mathcal{F}_{s,t} = \mathcal{F}_{s,u} \otimes \mathcal{F}_{u,v} \otimes \mathcal{F}_{v,t} = \mathcal{F}_{u,v}$, since $\mathcal{F}_{s,u}$ and $\mathcal{F}_{v,t}$ are degenerate by Lemma 3b8. ☐

So, a sub-σ-field $\mathcal{F}_{s,t} \subset \mathcal{F}$ is well-defined for every interval $(s, t) \subset \mathbb{R}$ (rather than a coarse time interval), and

$$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} \quad \text{whenever } -\infty < r \le s \le t < +\infty\,.$$

3b10 Lemma. The union of sub-σ-fields $\mathcal{F}_{s+\varepsilon,t-\varepsilon}$ over $\varepsilon > 0$ generates $\mathcal{F}_{s,t}$.

Proof. Consider $\mathcal{F}_{\varepsilon,1}$. We have to prove that $\mathbb{E}\bigl(\,x \,\big|\, \mathcal{F}_{\varepsilon,1}\,\bigr)$ converges to x (in $L_2(\Omega)$, for $\varepsilon \to 0+$) for every $x \in L_2(\mathcal{F}_{0,1})$, or for $x[\infty]$ where $x \in L_2(\mathcal{A}_{0,1})$. Assume the contrary, then

$$\|\mathbb{E}\bigl(\,x[\infty] \,\big|\, \mathcal{F}_{\varepsilon,1}\,\bigr)\| < c < \|x[\infty]\|$$

for all ε small enough, and some constant c. We know that

$$\mathbb{E}\bigl(\,x[\infty] \,\big|\, \mathcal{F}_{\varepsilon,1}\,\bigr) = \operatorname{Lim} \mathbb{E}\bigl(\,x[i] \,\big|\, \mathcal{F}_{\varepsilon,1}[i]\,\bigr)$$

for each ε.[17] Therefore

$$\|\mathbb{E}\bigl(\,x[i] \,\big|\, \mathcal{F}_{\varepsilon,1}[i]\,\bigr)\| \xrightarrow[i\to\infty]{} \|\mathbb{E}\bigl(\,x[\infty] \,\big|\, \mathcal{F}_{\varepsilon,1}\,\bigr)\| < c\,.$$

We choose a sequence $\varepsilon[i] \xrightarrow[i\to\infty]{} 0$ such that $\|\mathbb{E}\bigl(\,x[i] \,\big|\, \mathcal{F}_{\varepsilon[i],1}[i]\,\bigr)\| < c$ for all i large enough. However, $\operatorname{Lim} \mathbb{E}\bigl(\,x[i] \,\big|\, \mathcal{F}_{\varepsilon[i],1}[i]\,\bigr) = \mathbb{E}\bigl(\,x[\infty] \,\big|\, \mathcal{F}_{\varepsilon[\infty],1}\,\bigr) = \mathbb{E}\bigl(\,x[\infty] \,\big|\, \mathcal{F}_{0,1}\,\bigr) = x[\infty]$; a contradiction. ☐

3c Scaling limit of Fourier-Walsh coefficients

We still consider a dyadic coarse factorization. The Hilbert space $L_2[i] = L_2\bigl(\Omega[i], \mathcal{F}[i], P[i]\bigr)$ consists of all functions of random signs τ_m, $m \in \frac{1}{i}\mathbb{Z}$. The well-known Fourier-Walsh (orthonormal) basis of $L_2[i]$ consists of products

$$\tau_M = \prod_{m\in M} \tau_m\,, \quad M \in \mathcal{C}[i] = \{M \subset \tfrac{1}{i}\mathbb{Z} : M \text{ is finite}\}\,.$$

Every $f \in L_2[i]$ is of the form

$$f = \sum_M \hat{f}_M \tau_M = \hat{f}_\emptyset + \sum_{m\in\frac{1}{i}\mathbb{Z}} \hat{f}_{\{m\}}\tau_m + \sum_{m_1,m_2\in\frac{1}{i}\mathbb{Z},m_1<m_2} \hat{f}_{\{m_1,m_2\}}\tau_{m_1}\tau_{m_2} + \dots\,;$$

[17] Or rather, an appropriate coarse instant is meant in $\mathcal{F}_{\varepsilon,1}[i]$.

coefficients $\hat{f}_M$ are called Fourier-Walsh coefficients of f. We define the *spectral measure* μ_f on the countable set $\mathcal{C}[i]$ by

$$\mu_f(\mathcal{M}) = \sum_{M \in \mathcal{M}} |\hat{f}_M|^2 \quad \text{for } \mathcal{M} \subset \mathcal{C}[i]\,;$$

it is a finite positive measure;

$$\mu_f(\mathcal{C}[i]) = \|f\|^2\,; \quad \mu_f(\{\emptyset\}) = (\mathbb{E}\, f)^2\,; \quad \mu_f(\mathcal{C}[i] \setminus \{\emptyset\}) = \operatorname{Var}(f)\,.$$

Let (s,t) be a coarse time interval. We have

$$\mathbb{E}\left(\, \tau_M \,\middle|\, \mathcal{F}_{s,t}[i] \,\right) = \begin{cases} \tau_M & \text{if } M \subset [s[i], t[i]), \\ 0 & \text{otherwise;} \end{cases}$$
$$\|\mathbb{E}\left(\, f \,\middle|\, \mathcal{F}_{s,t}[i] \,\right)\|^2 = \mu_f\big(\{M \in \mathcal{C}[i] : M \subset [s[i], t[i])\}\big)\,.$$

We apply it to $f = x[i]$ for an arbitrary $x \in L_2(\mathcal{A})$ and arbitrary i; μ_f becomes $\mu_{x[i]}$ or $\mu_x[i]$;

$$\mu_x[i]\big(\{M \in \mathcal{C}[i] : M \subset [s[i], t[i])\}\big) = \|\mathbb{E}\left(\, x[i] \,\middle|\, \mathcal{F}_{s,t}[i] \,\right)\|^2 \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(\, x[\infty] \,\middle|\, \mathcal{F}_{s,t}[\infty] \,\right)\|^2$$

by (3a6). For every $\varepsilon > 0$ we can choose s,t so that $\|x[\infty]\|^2 - \|\mathbb{E}\left(\, x[\infty] \,\middle|\, \mathcal{F}_{s,t}[\infty] \,\right)\|^2 \le \varepsilon$, and moreover,

(3c1) $$\mu_x[i]\big(\{M \in \mathcal{C}[i] : M \subset [s[i], t[i])\}\big) \le \varepsilon \quad \text{for all } i\,.$$

We consider each $\mu_x[i]$ as a measure on the space $\mathcal{C}[\infty]$ of all compact subsets of $\mathbb{R}$, equipped with the Hausdorff metric

(3c2) $$\operatorname{dist}(M_1, M_2) = \sup_{x\in\mathbb{R}} \left| \min_{y\in M_1} |x-y| - \min_{y \in M_2} |x-y| \right|$$

for nonempty M_1, M_2; and $\operatorname{dist}(\emptyset, M) = 1$ for $M \ne \emptyset$. Clearly, $\mathcal{C}[i] \subset \mathcal{C}[\infty]$ for each i; thus, a measure on $\mathcal{C}[i]$ is also a measure on $\mathcal{C}[\infty]$. The set $\{M \in \mathcal{C}[\infty] : M \subset [u,v]\}$ is well-known to be compact, for every $[u,v] \subset \mathbb{R}$. Thus, (3c1) shows that the sequence of measures $\mu_x[i]$ on $\mathcal{C}[\infty]$ is tight.

Let (s_1,t_1) and (s_2,t_2) be two coarse time intervals, $s_1 \le t_1 \le s_2 \le t_2$. Sub-$\sigma$-fields $\mathcal{F}_{s_1,t_1}[i]$ and $\mathcal{F}_{s_2,t_2}[i]$ are independent; they generate a sub-σ-field that may be denoted by

$$\mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] = \mathcal{F}_{s_1,t_1}[i] \otimes \mathcal{F}_{s_2,t_2}[i]\,.$$

We have

$$\mathbb{E}\left(\, \tau_M \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \,\right) = \begin{cases} \tau_M & \text{if } M \subset [s_1[i], t_1[i]) \cup [s_2[i], t_2[i]), \\ 0 & \text{otherwise;} \end{cases}$$
$$\|\mathbb{E}\left(\, f \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \,\right)\|^2 = \mu_f\big(\{M \in \mathcal{C}[i] : M \subset [s_1[i], t_1[i]) \cup [s_2[i], t_2[i])\}\big)\,;$$
$$\mu_x[i]\big(\{M \in \mathcal{C}[i] : M \subset [s_1[i], t_1[i]) \cup [s_2[i], t_2[i])\}\big) =$$
$$= \|\mathbb{E}\left(\, x[i] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[i] \,\right)\|^2 \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(\, x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)} \,\right)[\infty]\|^2\,,$$

where $\mathcal{F}_{(s_1,t_1)\cup(s_2,t_2)}[\infty] = \mathcal{F}_{s_1,t_1}[\infty] \otimes \mathcal{F}_{s_2,t_2}[\infty] = \mathcal{F}_{s_1[\infty],t_1[\infty]} \otimes \mathcal{F}_{s_2[\infty],t_2[\infty]}$. A generalization of (3a6) to the product of more than two spaces was used here.

The same holds for more than two coarse time intervals:

$$(3c3)\quad \mu_x[i]\big(\{M \in \mathcal{C}[i] : M \subset [s_1[i],t_1[i]) \cup \dots \cup [s_n[i],t_n[i])\}\big) \xrightarrow[i\to\infty]{} \\ \xrightarrow[i\to\infty]{} \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup\dots\cup(s_n,t_n)}[\infty] \right)\|^2 \,.$$

We have convergence of spectral measures on a special class of subsets of $\mathcal{C}[\infty]$. Note that the intersection of two such subsets is again such a subset. Therefore, the convergence holds on the algebra of subsets generated by the class. Here is a generic element of the algebra:

$$(3c4)\qquad \{M \in \mathcal{C}[\infty] : M \subset \cup_{k=1}^n [s_k[i],t_k[i]) \text{ and } M \cap [s_k[i],t_k[i]) \ne \emptyset \text{ for } k=1,\dots,n\}\,.$$

Its diameter (recall the metric (3c2)) does not exceed $\max_k\big(t_k[i]-s_k[i]\big)$. Thus, we get weak convergence of measures, which proves the following result.

3c5 Theorem. For every dyadic coarse factorization $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty,\mathcal{A}\big)$ and every $x \in L_2(\mathcal{A})$, the sequence $\big(\mu_x[i]\big)_{i=1}^\infty$ of spectral measures converges weakly to a (finite, positive) measure $\mu_x[\infty]$ on the Polish space $\mathcal{C}[\infty]$.

Convergence of measures $\mu_x[i]$ on a 'cell' of the form (3c3) (or (3c4)) does not ensure that the limit is $\mu_x[\infty]$ on the 'cell'.[18] Rather, the limit lies between $\mu_x[\infty]$-measures of the interior and the closure of the cell,

$$(3c6)\quad \mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \subset (s_1[i],t_1[i]) \cup \dots \cup (s_n[i],t_n[i])\}\big) \le \\ \le \|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup\dots\cup(s_n,t_n)}[\infty] \right)\|^2 \le \\ \mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \subset [s_1[i],t_1[i]] \cup \dots \cup [s_n[i],t_n[i]]\}\big)\,.$$

3c7 Lemma. For every $t \in \mathbb{R}$,

$$\mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \ni x\}\big) = 0\,.$$

Proof. Lemma 3b10 gives us

$$\|\mathbb{E}\left(x[\infty] \,\middle|\, \mathcal{F}_{(-\infty,-\varepsilon)\cup(\varepsilon,+\infty)}[\infty] \right)\|^2 \xrightarrow[\varepsilon\to 0]{} \|x[\infty]\|^2\,,$$

therefore

$$\mu_x[\infty]\big(\{M \in \mathcal{C}[\infty] : M \subset (-\infty,\varepsilon] \cup [\varepsilon,+\infty)\}\big) \xrightarrow[\varepsilon\to 0]{} \mu_x[\infty]\big(\mathcal{C}[\infty]\big)\,.$$

□

Now we see that the boundary of a 'cell' is negligible (of measure 0); inequalities (3c6) are, in fact, equalities.

Applying Fubini theorem we see that $\mu_x[\infty]$ is concentrated on compact sets M of Lebesgue measure 0 (therefore, nowhere dense).

[18] Think for example about an atom at the point $\frac{1}{n}$ of $\mathbb{R}$, and 'cells' of the form $(x,y]$.

3d The limiting object

3d1 Definition. A *continuous factorization* (of probability spaces) (over $\mathbb{R}$) consists of a probability space $(\Omega, \mathcal{F}, P)$ and a two-parameter family $(\mathcal{F}_{s,t})_{s\le t}$ of sub-σ-fields $\mathcal{F}_{s,t} \subset \mathcal{F}$ such that

$$\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t} \quad \text{whenever } r \le s \le t$$

(that is, $\mathcal{F}_{r,s}$ and $\mathcal{F}_{s,t}$ are independent, and together generate $\mathcal{F}_{r,t}$), and

$$\bigcup_{\varepsilon>0} \mathcal{F}_{s+\varepsilon,t-\varepsilon} \text{ generates } \mathcal{F}_{s,t} \text{ whenever } s < t,$$

and

$$\bigcup_{n=1}^{\infty} \mathcal{F}_{-n,n} \text{ generates } \mathcal{F}\,.$$

The refinement of any dyadic coarse factorization is a continuous factorization (as was shown in Sect. 3b). Also, every continuous factorization is (isomorphic to) the refinement of some dyadic coarse factorization. (I omit the proof.)

Given a continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ and $x \in L_2(\Omega, \mathcal{F}, P)$, we may define the spectral measure μ_x of x as the (finite, positive) measure on the space $\mathcal{C} = \mathcal{C}[\infty]$ of compact subsets of $\mathbb{R}$ such that

$$\mu_x\big(\{M \in \mathcal{C}[\infty] : M \subset (s_1,t_1) \cup \ldots \cup (s_n,t_n)\}\big) = \|\mathbb{E}\left(x \,\middle|\, \mathcal{F}_{(s_1,t_1)\cup\ldots\cup(s_n,t_n)} \right)\|^2$$

whenever $s_1 \le t_1 \le s_2 \le \cdots \le t_{n-1} \le s_n \le t_n$; here $\mathcal{F}_{(s_1,t_1)\cup\ldots\cup(s_n,t_n)}$ stands for the sub-σ-field generated by $\mathcal{F}_{s_1,t_1}, \ldots, \mathcal{F}_{s_n,t_n}$.

The spectral measure is concentrated on (the set of all) nowhere dense compact sets, and

$$\mu_x\big(\{M \in \mathcal{C} : M \ni t\}\big) = 0 \quad \text{for each } t \in \mathbb{R}\,.$$

3d2 Example. The refinement of the Brownian coarse factorization (see 3b2) is the Brownian continuous factorization,

$$\mathcal{F}_{s,t} \text{ is generated by } \{B(v) - B(u) : s \le u \le v \le t\}\,,$$

where $B(\cdot)$ is the usual Brownian motion. Every $x \in L_2$ admits Itô's decomposition into multiple stochastic integrals,

$$x = \hat{x}(\emptyset) + \int \hat{x}(\{t_1\})\, dB(t_1) + \iint\limits_{t_1<t_2} \hat{x}(\{t_1,t_2\})\, dB(t_1)dB(t_2) + \cdots =$$

$$= \sum_{n=0}^{\infty} \int\!\!\cdots\!\!\int\limits_{t_1<\cdots<t_n} \hat{x}(\{t_1,\ldots,t_n\})\, dB(t_1)\ldots dB(t_n)\,,$$

where $\hat{x} \in L_2(\mathcal{C}_{\text{finite}})$, $\mathcal{C}_{\text{finite}}$ being the space of all finite subsets of $\mathbb{R}$, equipped with the natural (Lebesgue) measure, making the transform $x \leftrightarrow \hat{x}$ unitary, according to the formula

$$\mathbb{E}\,|x|^2 = |\hat{x}(\emptyset)|^2 + \int |\hat{x}(\{t_1\})|^2 \, dt_1 + \iint\limits_{t_1<t_2} |\hat{x}(\{t_1,t_2\})|^2 \, dt_1 dt_2 + \dots = \\ = \sum_{n=0}^{\infty} \int \dots \int\limits_{t_1<\dots<t_n} |\hat{x}(\{t_1,\dots,t_n\})|^2 \, dt_1 \dots dt_n \, .$$

The spectral measure μ_x of x is

$$\mu_x(A) = \sum_{n=0}^{\infty} \int \dots \int\limits_{t_1<\dots<t_n, \{t_1,\dots,t_n\}\in A} |\hat{x}(\{t_1,\dots,t_n\})|^2 \, dt_1 \dots dt_n \, .$$

That is an important property of the Brownian continuous factorization: the spectral measure (of any random variable) is concentrated on the subset $\mathcal{C}_{\text{finite}} \subset \mathcal{C}$, and absolutely continuous w.r.t. Lebesgue measure on $\mathcal{C}_{\text{finite}}$.

In particular, for $x = \exp\bigl(i\sqrt{\lambda}B(t)\bigr)$ the measure μ_x is just the distribution of the Poisson process of rate λ on $(0,t)$. Indeed,

$$\exp\bigl(i\sqrt{\lambda}B(t)\bigr) = e^{-\lambda t/2} \sum_{n=0}^{\infty} \lambda^{n/2} \int \dots \int\limits_{0<t_1<\dots<t_n<t} dB(t_1) \dots dB(t_n) \, .$$

3d3 Example. Recall the process Y_ε of 1b3;

$$Y_\varepsilon(t) = \exp\bigl(iB(\ln t) - iB(\ln \varepsilon)\bigr) \, .$$

We define $\mathcal{F}_{s,t}$ as the σ-field generated by 'multiplicative increments' $\frac{Y_\varepsilon(v)}{Y_\varepsilon(u)}$ for all $(u,v) \subset (s,t)$, that is, by (usual) Brownian increments on $(\ln s, \ln t)$. The spectral measure $\mu_{Y_\varepsilon(t)}$ is the distribution of a non-homogeneous Poisson process on (ε,t), the image of the usual Poisson process (of rate 1) on $(\ln\varepsilon, \ln t)$ under the map $u \mapsto e^u$. The rate of the non-homogeneous Poisson process is $\lambda(s) = 1/s$.

The limiting process Y was discussed in 1b3. It may be treated as the refinement of Y_ε for $\varepsilon \to 0$ (I leave detail to the reader). The spectral measure $\mu_{Y(t)}$ should be the distribution of a non-homogeneous Poisson process on $(0,t)$, of the rate $\lambda(s) = 1/s$. Random points accumulate to 0; we add 0 to the random set, making it compact. However, the equality $\mu(\{M : M \ni 0\}) = 1$ does not conform to Lemma 3c7! It happens because the limiting object is not a *continuous* factorization. Denote by $\mathcal{F}_{0+,1}$ the σ-field generated by $\cup_{\varepsilon>0}\mathcal{F}_{\varepsilon,1}$. Every $Y(1)/Y(t)$ for $t>0$ is $\mathcal{F}_{0+,1}$-measurable, but $Y(1)$ is not. The global phase is missing. Of course, for every $t>0$ there exists an independent complement of $\mathcal{F}_{0+,t}$ in $\mathcal{F}_{-\infty,t}$ (for example, the σ-field generated by $Y(t)$). However, we cannot choose a single complement (to be denoted by $\mathcal{F}_{-\infty,0+}$) for all $t>0$, since the tail σ-field $\cap_{t>0}\mathcal{F}_{-\infty,t}$ is degenerate.

3d4 Lemma. For every continuous factorization $\bigl((\Omega,\mathcal{F},P),(\mathcal{F}_{s,t})_{s\le t}\bigr)$ and every $s \le t$,

$$\mathcal{F}_{s,t} = \bigcap_{\varepsilon>0} \mathcal{F}_{s-\varepsilon,t+\varepsilon} \, .$$

Proof. The σ-field $\cap_{\varepsilon>0}\mathcal{F}_{0,\varepsilon}$ is degenerate by Kolmogorov's zero-one law applied to $\mathcal{F}_{1,\infty}, \mathcal{F}_{1/2,1}, \mathcal{F}_{1/3,1/2}, \dots$ Further, $\mathcal{F}_{-\infty,\varepsilon} = \mathcal{F}_{-\infty,0} \otimes \mathcal{F}_{0,\varepsilon} \xrightarrow[\varepsilon\to 0]{} \mathcal{F}_{-\infty,0}$. Though, the equality $\lim(\mathcal{A} \vee \mathcal{B}_n) = \mathcal{A} \vee (\lim \mathcal{B}_n)$ does not hold in general; but for independent $\mathcal{A}$ and $\mathcal{B}_1$ the equality holds, which is a rather trivial part of Weizsäcker's criteria [16]. The rest of the proof is left to the reader.

$\square$

3e Time shift; noise

Let $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$ be a dyadic coarse factorization. For each i the lattice $\frac{1}{i}\mathbb{Z}$ acts on $\Omega[i]$ by measure preserving transformations $\alpha_t : \Omega[i] \to \Omega[i]$ (time shift),

$$\alpha_t(\omega)(s) = \omega(s-t) \quad \text{for all } s \in \frac{1}{i}\mathbb{Z}\,.$$

For each coarse instant $t = (t[i])_{i=1}^\infty$ we have a map $\alpha_t : \Omega[\text{all}] \to \Omega[\text{all}]$,

$$\alpha_t(\omega)[i](s) = \omega[i](s - t[i]) \quad \text{for all } s \in \frac{1}{i}\mathbb{Z}\,.$$

Such α_t is an automorphism of the dyadic coarse sample space, but the coarse σ-field $\mathcal{A}$ need not be invariant under α_t. We consider such a condition:

(3e1) $\qquad \mathcal{A}$ is invariant under α_t for every coarse instant t.

Dyadic coarse factorizations of Examples 3b2, 3b5, 3b6, 3b7 satisfy (3e1), but 3b4 does not.

If (3e1) is satisfied, then the refinement $\alpha_t[\infty] = \lim_{i\to\infty,\mathcal{A}} \alpha_t[i]$ is an automorphism of the refinement $(\Omega, \mathcal{F}, P)$ of the dyadic coarse factorization. Existence of the limit for *every* converging sequence $t = (t[i])$ implies that $\alpha_t[\infty]$ depends on $t[\infty]$ only, and we get a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of $(\Omega, \mathcal{F}, P)$. The group is continuous,

$$\mathbb{P}\big(A \triangle \alpha_t(A) \big) \xrightarrow[t\to 0]{} 0 \quad \text{for all } A \in \mathcal{F}\,,$$

which is ensured by (3e1); the proof is left to the reader.

3e2 Definition. A *noise* $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}, (\alpha_t)_{t\in\mathbb{R}}\big)$ consists of a continuous factorization $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s\le t}\big)$ and a one-parameter group of automorphisms α_t of $(\Omega, \mathcal{F}, P)$ such that

$$\alpha_t^{-1}(\mathcal{F}_{r,s}) = \mathcal{F}_{r-t,s-t} \quad \text{for all } r,s,t \in \mathbb{R},\ r \le s\,,$$
$$\mathbb{P}\big(A \triangle \alpha_t^{-1}(A) \big) \xrightarrow[t\to 0]{} 0 \quad \text{for all } A \in \mathcal{F}\,.$$

Unfortunately, the latter assumption (continuity of the group action) is missing in my former publications, which opens the door for pathologies.[19]

[19] Results of these former publications do not depend on the (missing) continuity condition. But anyway, a discontinuous group action is a pathology, no doubt. (In particular, it cannot be Borel measurable.)

3e3 Note. Continuity of the factorization follows from other assumptions, see [7, Lemma 2.1]. For arbitrary factorizations, continuity is restrictive (recall 3d3); waiving it, we get discontinuity points $t \in \mathbb{R}$; they are a finite or countable set. For a noise, however, the set is invariant under time shifts, therefore it is empty.

3e4 Lemma. For every dyadic coarse factorization satisfying (3e1), its refinement is a noise.

The proof is left to the reader.

3e5 Question. Whether every noise is the refinement of some dyadic coarse factorization, or not? I do not know; I guess that the answer is negative. It would be interesting to find some special features of such refinements among all noises. It is also unclear, what happens to the class of such refinements, if subsequences are permitted (like in 3b6).