2 Distribution function and quantile function

2a Distribution function

2a1 Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable (see 1a1, 1a2). The (cumulative) *distribution function* $F_X : \mathbb{R} \to [0,1]$ of the random variable X is defined by

$$F_X(x) = \mathbb{P}\left(\{\omega \in \Omega : X(\omega) \le x\} \right) \quad \text{for all } x \in \mathbb{R}.$$

Note that X is quite arbitrary; it may be discrete (as in 1b1-1b3), or continuous (as in 1b4-1b6), or whatever (we'll return to the point). Accordingly, F_X may be discontinuous.

So, $\mathbb{P}\left(X \le x \right) = F_X(x)$. It follows that $\mathbb{P}\left(a < X \le b \right) = F_X(b) - F_X(a)$, since $(-\infty, a] \uplus (a, b] = (-\infty, b]$ implies $X^{-1}\big((-\infty, a]\big) \uplus X^{-1}\big((a, b]\big) = X^{-1}\big((-\infty, b]\big)$, that is, $\{\omega : X(\omega) \in (-\infty, a]\} \uplus \{\omega : X(\omega) \in (a, b]\} = \{\omega : X(\omega) \in (-\infty, b]\}$. However, $(-\infty, x)$ needs countable additivity:

$$(-\infty, x) = (-\infty, x-1] \uplus (x-1, x-\tfrac{1}{2}] \uplus (x-\tfrac{1}{2}, x-\tfrac{1}{3}] \uplus \dots$$

therefore

$$\begin{aligned} \mathbb{P}\left(X < x \right) &= \mathbb{P}\left(X \le x-1 \right) + \mathbb{P}\left(x-1 < x \le x-\tfrac{1}{2} \right) + \mathbb{P}\left(x-\tfrac{1}{2} < x \le x-\tfrac{1}{3} \right) + \dots = \\ &= F_X(x-1) + \big(F_X(x-\tfrac{1}{2}) - F_X(x-1) \big) + \big(F_X(x-\tfrac{1}{3}) - F_X(x-\tfrac{1}{2}) \big) + \dots \end{aligned}$$

The sum of a series is, by definition, the limit of partial sums:

$$\begin{aligned} &\sum_{k=1}^{\infty} \big(F_X(x-\tfrac{1}{k+1}) - F_X(x-\tfrac{1}{k}) \big) = \lim_{n\to\infty} \sum_{k=1}^{n} \big(F_X(x-\tfrac{1}{k+1}) - F_X(x-\tfrac{1}{k}) \big) = \\ &= \lim_{n\to\infty} \big(F_X(x-\tfrac{1}{n+1}) - F_X(x-1) \big) \end{aligned}$$

(you see, the sum is "telescopic"), therefore

$$\mathbb{P}\left(X < x \right) = \lim_{n\to\infty} F_X(x-\tfrac{1}{n}) \, .$$

The same holds for any sequence $x_n \to x$, $x_n < x$ (not just $x_n = x - \frac{1}{n}$), and so,

$$\mathbb{P}\left(X < x \right) = F_X(x-) \, ,$$

the limit of F at x from the left.

....

The trick with a "telescopic" sum will be used again and again; let us generalize it now for the future use. Instead of events $A_n = X^{-1}\big((-\infty, x_n]\big) = \{\omega : X(\omega) \le x_n\}$ consider arbitrary events $A_n \subset \Omega$, $A_n \in \mathcal{F}$, forming an increasing sequance:

$$A_1 \subset A_2 \subset \dots$$

Their limit is, by definition, their union $A = \cup_{n=1}^{\infty} A_n$, and we write

$$A_n \uparrow A \, .$$

Note that $A \in \mathcal{F}$, since $\mathcal{F}$ is a σ-field. We have

$$
\begin{aligned}
&A = A_1 \uplus (A_2 \setminus A_1) \uplus (A_3 \setminus A_2) \uplus \dots ;\\
&\mathbb{P}\big(A\big) = \mathbb{P}\big(A_1\big) + \mathbb{P}\big(A_2 \setminus A_1\big) + \mathbb{P}\big(A_3 \setminus A_2\big) + \dots =\\
&\quad = \mathbb{P}\big(A_1\big) + \big(\mathbb{P}\big(A_2\big) - \mathbb{P}\big(A_1\big)\big) + \big(\mathbb{P}\big(A_3\big) - \mathbb{P}\big(A_2\big)\big) + \dots =\\
&\quad = \mathbb{P}\big(A_1\big) + \lim_{n\to\infty} \sum_{k=1}^{n} \big(\mathbb{P}\big(A_{k+1}\big) - \mathbb{P}\big(A_k\big)\big) =\\
&\quad = \mathbb{P}\big(A_1\big) + \lim_{n\to\infty} \big(\mathbb{P}\big(A_{n+1}\big) - \mathbb{P}\big(A_1\big)\big) =\\
&\quad = \lim_{n\to\infty} \mathbb{P}\big(A_n\big) .
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
A_n \uparrow A \quad &\Longrightarrow \quad \mathbb{P}\big(A_n\big) \uparrow \mathbb{P}\big(A\big) ,\\
A_n \downarrow A \quad &\Longrightarrow \quad \mathbb{P}\big(A_n\big) \downarrow \mathbb{P}\big(A\big) .
\end{aligned} \tag{2a2}
$$

The latter follows from the former after defining the limit of a decreasing sequence of events $A_1 \supset A_2 \supset \dots$ as the intersection $A = \cap_{n=1}^{\infty} A_n$, and taking complements:

$$
\begin{aligned}
A_n \downarrow A \quad &\Longrightarrow \quad (\Omega \setminus A_n) \uparrow (\Omega \setminus A) \quad \Longrightarrow \quad \mathbb{P}\big(\Omega \setminus A_n\big) \uparrow \mathbb{P}\big(\Omega \setminus A\big) \quad \Longrightarrow\\
&\Longrightarrow \quad 1 - \mathbb{P}\big(A_n\big) \uparrow 1 - \mathbb{P}\big(A\big) \quad \Longrightarrow \quad \mathbb{P}\big(A_n\big) \downarrow \mathbb{P}\big(A\big) .
\end{aligned}
$$

Properties (2a2) are well-known as *continuity* of probability.[1]

. . . .

Now we are in position to calculate any elementary probability in terms of F_X:

$$
\begin{aligned}
\mathbb{P}\big(X \in (-\infty, x]\big) &= F_X(x) ; & \mathbb{P}\big(X \in (-\infty, x)\big) &= F_X(x-) ;\\
\mathbb{P}\big(X \in [x, +\infty)\big) &= 1 - F_X(x-) ; & \mathbb{P}\big(X \in (x, +\infty)\big) &= 1 - F_X(x) ;\\
\mathbb{P}\big(X \in [a, b]\big) &= F_X(b) - F_X(a-) ; & \mathbb{P}\big(X \in (a, b)\big) &= F_X(b-) - F_X(a) ;\\
\mathbb{P}\big(X \in [a, b)\big) &= F_X(b-) - F_X(a-) ; & \mathbb{P}\big(X \in (a, b]\big) &= F_X(b) - F_X(a) ;
\end{aligned} \tag{2a3}
$$
$$
\mathbb{P}\big(X \in [a, b] \cup (c, d)\big) = F_X(d-) - F_X(c) + F_X(b) - F_X(a-) \qquad (a < b < c < d) ,
$$

etc. In that sense, F_X is enough for describing the distribution of X.[2]

2a4 Theorem. For any random variable X on any probability space, its distribution function F_X satisfies the following conditions.

(a) $\forall x \in \mathbb{R} \quad 0 \le F_X(x) \le 1$.

(b) F_X increases, that is, $x_1 \le x_2 \quad \Longrightarrow \quad F_X(x_1) \le F_X(x_2)$.

(c) $F_X(-\infty) = 0$, that is, $\lim_{x\to-\infty} F_X(x) = 0$.

(d) $F_X(+\infty) = 1$, that is, $\lim_{x\to+\infty} F_X(x) = 1$.

(e) F_X is right continuous, that is, $F_X(x+) = F_X(x)$.

[1]Generally, $A_n \to A \quad \Longrightarrow \quad \mathbb{P}\big(A_n\big) \to \mathbb{P}\big(A\big)$ also for non-monotone sequences $A_1, A_2, \dots$ if $\lim A_n$ is defined appropriately. However, we do not need it now.

[2]You may ask about non-elementary probabilities, say, the probability that X is a rational number. We'll return to the point later.

Proof. (a): trivial.
(b): $x_1 \le x_2 \implies (-\infty, x_1] \subset (-\infty, x_2] \implies \mathbb{P}\big(X \in (-\infty, x_1] \big) \le \mathbb{P}\big(X \in (-\infty, x_2] \big) \implies F_X(x_1) \le F_X(x_2)$.
(c) $x_n \downarrow -\infty \implies (-\infty, x_n] \downarrow \emptyset \implies \mathbb{P}\big(X \le x_n \big) \downarrow 0 \implies F_X(x_n) \downarrow 0$.
(d) $x_n \uparrow +\infty \implies (-\infty, x_n] \uparrow \mathbb{R} \implies \mathbb{P}\big(X \le x_n \big) \uparrow 1 \implies F_X(x_n) \uparrow 1$.
(e) $x_n \downarrow x \implies (-\infty, x_n] \downarrow (-\infty, x] \implies \mathbb{P}\big(X \le x_n \big) \downarrow \mathbb{P}\big(X \le x \big) \implies F_X(x_n) \downarrow F_X(x)$. □

We cannot prove left continuity like (e), since $x_n < x$, $x_n \uparrow x \implies (-\infty, x_n] \uparrow (-\infty, x)$ rather than $(-\infty, x]$.

You may ask: why just these 5 properties (a)–(e)? Probably, there are other properties? However, we'll see that (a)–(e) are necessary and sufficient!

2b Quantile function

Probably you are acquainted with a median; that is just an $\frac{1}{2}$-quantile, a special case of the following.

2b1 Definition. Given F satisfying 2a4(a–e), and a number $p \in (0, 1)$, and another number $x \in \mathbb{R}$, we say that x is a *p-quantile* (w.r.t. F), if

$$F(x-) \le p \le F(x) .$$

For now, nothing is claimed about the correspondence between x and p. Does a p-quantile exist for every p? Is it unique? All that will be discussed.

The set of all (x, p) such that x is a p-quantile, may be called *the quantile line.* It is really a continuous line on the (x, p)-plane, however (in general) not a continuous function of x, nor of p. Rather, it appears to be continuous in the variable

$$s = x + p$$

as we'll see soon.[3]

2b2 Lemma. For each $s \in \mathbb{R}$ there exists one and only one pair (x_s, p_s) such that $s_s + p_s = s$ and x_s is a p_s-quantile.

Proof. (x is an $(s-x)-$quantile) $\iff \big(F(x-) \le s - x \le F(x)\big) \iff \big(x + F(x-) \le s \le x + F(x)\big)$. The function $x + F(x)$ is *strictly* increasing,[4] and its limits at $-\infty$ and $+\infty$ are $-\infty$ and $+\infty$, respectively, since $F(x)$ is bounded. We have

$$\mathbb{R} = \{x : x + F(x) < s\} \uplus \{x : x + F(x) \ge s\} ,$$

both sets are non-empty, and each member of the first set is less than every member of the second set. Therefore

$$\sup\{x : x + F(x) < s\} = \inf\{x : x + F(x) \ge s\} ;$$

[3] Of course, $x + p$ is only a convenient trick; if you prefer, say, $x + 2p$, you may use it equally well.
[4] But not continuous, in general. This is why the equation $x + F(x) = s$ need no have a root.

denote this number by x_s. Limit from the right gives $x_s + F(x_s) \ge s$ (since F is right-continuous). Limit from the left gives $x_s + F(x_s-) \le s$. So, $x_s + F(x_s-) \le s \le x_s + F(x_s)$, which means that x_s is a p_s-quantile, where $p_s = s - x_s$. No other x can have the needed property, since

$$\begin{aligned} x > x_s \quad &\Longrightarrow \quad x + F(x-) > x_s + F(x_s) \ge s\,, \\ x < x_s \quad &\Longrightarrow \quad x + F(x) < x_s + F(x_s-) \le s\,. \end{aligned}$$

□

2b3 Lemma. $s_1 \le s_2 \quad \Longrightarrow \quad x_{s_1} \le x_{s_2}$.

Proof. Assume the contrary: $x_{s_1} > x_{s_2}$, then $F(x_{s_1}-) \ge F(x_{s_2})$, therefore $s_1 \ge x_{s_1} + F(x_{s_1}-) > x_{s_2} + F(x_{s_2}) \ge s_2$, which is a contradiction. □

2b4 Lemma. $s_1 \le s_2 \quad \Longrightarrow \quad p_{s_1} \le p_{s_2}$.

Proof. $x_{s_1} \le x_{s_2}$ by 2b3; consider two cases: $x_{s_1} = x_{s_2}$ and $x_{s_1} < x_{s_2}$.

If $x_{s_1} = x_{s_2}$ then $p_{s_1} = s_1 - x_{s_1} < s_2 - x_{s_2} = p_{s_2}$.

If $x_{s_1} < x_{s_2}$ then $F(x_{s_1}) \le F(x_{s_2}-)$, therefore $p_{s_1} \le F(x_{s_1}) \le F(x_{s_2}-) \le p_{s_2}$. □

2b5 Lemma. Both x_s and p_s are continuous functions of s.

Proof. The equality $x_s + p_s = s$ implies

$$(x_{s_2} - x_{s_1}) + (p_{s_2} - p_{s_1}) = s_2 - s_1\,.$$

Assume that $s_1 \le s_2$, then both increments are ≥ 0 by 2b3, 2b4. So,

$$0 \le x_{s_2} - x_{s_1} \le s_2 - s_1\,, \qquad 0 \le p_{s_2} - p_{s_1} \le s_2 - s_1\,;$$

thus $|\Delta x_s| \le |\Delta s|$ and $|\Delta p_s| \le |\Delta s|$, which is more than enough for continuity. □

So, the quantile line is described by two continuous increasing[5] functions x_s and p_s:

$$\{(x,p) : x \text{ is a } p\text{-quantile}\,\} = \{(x_s,p_s) : s \in \mathbb{R}\}\,. \tag{2b6}$$

In other words, x is a p-quantile if and only if $x = x_s$ and $p = p_s$ where $s = x + p$.

2b7 Lemma. For every $p \in (0,1)$ the set of all p-quantiles is a non-empty closed interval.

Proof. First, $\lim_{s\to-\infty} x_s = -\infty$ and $\lim_{s\to+\infty} x_s = +\infty$, since $x_s = s - p_s \in [s-1,s]$. Second, $\lim_{s\to-\infty} p_s = 0$ and $\lim_{s\to+\infty} p_s = 1$ due to 2a4(c,d), since $F(x_s-) \le p_s \le F(x_s)$. Third, the equation $p_s = p$ has (at least one) solution s, since p_s is continuous, and $p_s - p$ changes its sign. So, the set $\{s : p_s = p\}$ is nonempty. The set is an interval, since p_s increases. The interval is closed, since p_s is continuous. Denote the interval as follows:

$$\{s : p_s = p\} = [S_{\min}(p), S_{\max}(p)]\,. \tag{2b8}$$

The set of all p-quantiles is the set of x_s for all s satisfying $p_s = p$, that is, for all $s \in [S_{\min}(p), S_{\max}(p)]$. So,

$$\begin{gathered} \{x : x \text{ is a } p\text{-quantile}\,\} = [X_{\min}(p), X_{\max}(p)]\,, \\ \text{where } X_{\min}(p) = x_{S_{\min}(p)}\,, \quad X_{\max}(p) = x_{S_{\max}(p)}\,, \end{gathered} \tag{2b9}$$

since x_s increases. □

[5] Not strictly increasing, in general.

If $p_1 < p_2$ then $S_{\min}(p_1) \le S_{\max}(p_1) < S_{\min}(p_2) \le S_{\max}(p_2)$, since p_s increases. Therefore functions $S_{\min}(\cdot)$, $S_{\max}(\cdot)$ increase strictly. It follows that functions $X_{\min}(\cdot)$, $X_{\max}(\cdot)$ increase (not strictly, in general).

2b10 Lemma. For every $p \in (0,1)$

$$X_{\min}(p-) = X_{\max}(p-) = X_{\min}(p)\,,$$
$$X_{\min}(p+) = X_{\max}(p+) = X_{\max}(p)\,.$$

Proof. I prove only $X_{\min}(p-) = X_{\min}(p)$; others are similar. Let $p_n < p$, $p_n \uparrow p$; we have to prove that $X_{\min}(p_n) \uparrow X_{\min}(p)$.

The sequence $s_n = S_{\min}(p_n)$ increases and is bounded, therefore it has a limit: $s_n \uparrow s'$. We have $p_{s_n} = p_n \uparrow p$; on the other hand, $p_{s_n} \uparrow p_{s'}$ (since p_s is continuous); therefore $p_{s'} = p$. It follows that $s' \in [S_{\min}(p), S_{\max}(p)]$. However, $s_n < S_{\min}(p)$ and $s_n \uparrow s'$, thus $s' \le S_{\min}(p)$; finally, $s' = S_{\min}(p)$, so,

$$S_{\min}(p_n) \uparrow S_{\min}(p)\,.$$

Continuity of x_s implies $x_{S_{\min}(p_n)} \uparrow x_{S_{\min}(p)}$, that is, $X_{\min}(p_n) \uparrow X_{\min}(p)$. $\square$

2b11 Definition. Given a function F satisfying 2a4(a–e) and another function $X : (0,1) \to \mathbb{R}$, we say that X is a *quantile function* (for F), if

$$X(p) \text{ is a } p\text{-quantile for every } p \in (0,1).$$

It follows from (2b9) that $X(\cdot)$ is a quantile function if and only if

$$X_{\min}(p) \le X(p) \le X_{\max}(p) \quad \text{for all } p \in (0,1).$$

Therefore, a quantile function exists (say, $X_{\min}$, $X_{\max}$, and $(X_{\min} + X_{\max})/2$ are quantile functions). If $X_{\min} = X_{\max}$ everywhere, then there is only one quantile function, otherwise there are infinitely many quantile functions; however, we'll see soon, that they all are equivalent in some sense.

It follows from 2b10 that

$$X(p-) = X_{\min}(p)\,, \qquad X(p+) = X_{\max}(p)$$

for every quantile function X. This is why we may forget about $X_{\min}, X_{\max}$ and use only X.

. . . .

2b12. The following three conditions are equivalent.

- x is a p-quantile;
- $F(x-) \le p \le F(x)$;
- $X(p-) \le x \le X(p+)$.

. . . .

Values $X(p)$ at discontinuity (jump) points p are not determined.[6] However, $X(p-)$ and $X(p+)$ are determined uniquely by the distribution function F.

[6] A quantile function, unlike a distribution function, need not be right-continuous, nor left-continuous.

The statement 2b12 gives us two equivalent descriptions of the quantile line. The line divides the plane in two regions (one being above the line and to the left, the other below and to the right). In terms of $s = x + p$ the formulation is evident:

$$\begin{array}{ll} x < x_s,\ p > p_s & \text{the first region,} \\ x = x_s,\ p = p_s & \text{the quantile line,} \\ x > x_s,\ p < p_s & \text{the second region.} \end{array}$$

The regions may be described similarly to 2b12,

$$\begin{aligned} F(x-) \le p \quad &\iff \quad x \le X(p+)\,, \\ p \le F(x) \quad &\iff \quad X(p-) \le x\,. \end{aligned} \tag{2b13}$$

The former describes the first region plus the quantile line; the latter describes the second region plus the quantile line.

2b14 Lemma. For every $x \in \mathbb{R}$,

$$(0, F(x)) \subset \{p \in (0,1) : X(p) \le x\} \subset (0, F(x)]\,.$$

Proof. $\{p : X(p-) \le x\} = (0, F(x)]$ by (2b13); it follows that $\{p : X(p) \le x\} \subset \{p : X(p-) \le x\} = (0, F(x)]$. It remains to prove that $(0, F(x)) \subset \{p : X(p) \le x\}$.

Let $0 < p < p_1 = F(x)$, then $X(p) \le X(p_1-)$. However, $X(p_1-) \le x$, since $p_1 \in (0, F(x)] = \{p : X(p-) \le x\}$. So, $p < F(x) \implies X(p) \le x$. □

2b15 Theorem. A function F satisfies 2a4(a–e) if and only if $F = F_X$ for some random variable X on some probability space.

Proof. The "if" part: if $F = F_X$ then F satisfies (a–e) by Theorem 2a4.

The "only if" part: assume that F satisfies (a–e). There is a quantile function $X : (0,1) \to \mathbb{R}$ for F. Consider X as a random variable on $(0,1)$, according to the analytic approach (see 1b).[7] For any $x \in \mathbb{R}$ the set $\{\omega \in \Omega : X(\omega) \le x\} = \{p \in (0,1) : X(p) \le x\}$ is either $(0, F(x))$ or $(0, F(x)]$ by 2b14. Anyway, the set is an interval of length $F(x)$, therefore, an event of probability $F(x)$,

$$\mathbb{P}\big(X \le x\big) = F(x)\,,$$

which means that $F_X = F$. □

A function F will be called a *distribution function,* if it satisfies the equivalent conditions of Theorem 2b15. So:

. . . .

2b16. The following three conditions are equivalent:

- F is a distribution function;
- F satisfies 2a4(a–e);
- $F = F_X$ for some X.

[7] A problem persists: we did not show that $(0,1)$ is indeed a probability space. We'll return to the point later.

. . . .

From now on, a quantile function of a random variable X will be denoted by X^*. Note that X is defined on an arbitrary probability space, while X^* is an increasing function on $(0,1)$. According to the analytic approach, X^* is also a random variable. It was shown in the proof of Theorem 2b15, that

(2b17) $$F_{X^*} = F_X \, .$$

There is some arbitrariness in the choice of X^*, however, the arbitrariness does not influence F_{X^*}, and may be ignored.

Random variables[8] X, Y are called *identically distributed,* if

(2b18) $$F_X(u) = F_Y(u) \quad \text{for all } u \in \mathbb{R}$$

or, equivalently,

(2b19) $$X^*(p-) = Y^*(p-) \quad \text{and} \quad X^*(p+) = Y^*(p+) \quad \text{for all } p \in (0,1).$$

Equivalence of (2b18) and (2b19) follows readily from 2b12; both are equivalent to

(2b20) $$\{(x,p) : x \text{ is a } p\text{-quantile for } X\} = \{(x,p) : x \text{ is a } p\text{-quantile for } Y\} \, ,$$

that is, the quantile line of X is identical to the quantile line of Y.

The equality (2b17) means that X and X^* are identically distributed. In fact, a quantile function X^* may be defined as an increasing function on $(0,1)$ such that X^*, X are identically distributed.

The quantile line (for a random variable X) may contain vertical or/and horizontal segments. Consider their probabilistic meaning.

A vertical segment: $F(x-) = p_1 < p_2 = F(x)$; then $\mathbb{P}\big(X = x \big) = p_2 - p_1 > 0$. Such a point x is called an *atom* (of the distribution).

A horizontal segment: $X(p-) = x_1 < x_2 = X(p+)$; then $F(x_1) = p = F(x_2-)$ and $\mathbb{P}\big(x \in (x_1, x_2) \big) = 0$. Such an interval (x_1, x_2) is called a *gap* (of the distribution).

Points not belonging to gaps form the *support* (of the distribution). In other words, a number x belongs to the support of the distribution of X, if and only if

(2b21) $$\mathbb{P}\big(x - \varepsilon < X < x + \varepsilon \big) > 0 \quad \text{for all } \varepsilon > 0 \, .$$

2c Convergence in distribution, or weak convergence of distributions

2c1 Definition. Let $F, F_1, F_2 \dots$ be distribution functions (see 2b16). We say that F_n converges weakly to F,

$$F_n \xrightarrow[n\to\infty]{\text{weak}} F \, ,$$

if $F_n(x) \xrightarrow[n\to\infty]{} F(x)$ for each x satisfying $F(x-) = F(x)$.

[8] Possibly defined on different probability spaces.

2c2 Definition. Let $X, X_1, X_2, \dots$ be random variables.[9] We say that X_n converges to X in distribution,

$$X_n \xrightarrow[n\to\infty]{\text{distr}} X \, ,$$

if $F_{X_n} \xrightarrow[n\to\infty]{\text{weak}} F_X$.

2c3 Example. Let X be a random variable, and $c_1, c_2, \dots$ be real numbers such that $c_n \to 1$. Then $c_n X \xrightarrow[n\to\infty]{\text{distr}} X$. Indeed,

$$F_{X_n}(x) = \mathbb{P}\,\big(X_n \le x \big) = \mathbb{P}\,\big(c_n X \le x \big) = \mathbb{P}\,\big(X \le \tfrac{x}{c_n} \big) = F_X\big(\tfrac{x}{c_n}\big) \, ;$$

if x satisfies $F_X(x-) = F_X(x)$, that is, F_X is continuous at x, then $F_X\big(\frac{x}{c_n}\big) \xrightarrow[n\to\infty]{} F_X(x)$. However, if $F_X(x-) < F_X(x)$ (that is, x is an atom), then $F_X(x/c_n)$ tends to $F(x)$ if $c_n \to 1-$, to $F(x-)$ if $c_n \to 1+$, and has no limit, if $c_n \to 1$ from both sides (oscillating). Note also that $\mathbb{P}\,\big(X_n = x \big)$ need not tend to $\mathbb{P}\,\big(X = x \big)$. This is why the convergence is called weak.

2c4 Example. Let a discrete random variable X_n be distributed uniformly on $\{1, \dots, n\}$, and a continuous random variable X be distributed uniformly on $(0, 1)$. Then $\frac{1}{n} X_n \xrightarrow[n\to\infty]{\text{distr}} X$.

2c5 Example. Let X be distributed uniformly on $[0, 1]$. Then $1 + \frac{1}{n} X \xrightarrow[n\to\infty]{\text{distr}} 1$. Here the number 1 (in the right-hand side) is treated as a degenerate random variable, taking on a single value 1.

2c6 Exercise. Find an example of continuous random variables X_n converging in distribution to a non-degenerate discrete random variable X.

2c7 Example. Let X_n have the binomial distribution $\mathrm{B}(n, \lambda/n)$, and X the Poisson distribution $\mathrm{P}(\lambda)$. Then $X_n \xrightarrow[n\to\infty]{\text{distr}} X$. It follows easily from the following fact known from Introduction to Probability:

$$\binom{n}{k} \Big(\frac{\lambda}{n}\Big)^k \Big(1 - \frac{\lambda}{n}\Big)^{n-k} \xrightarrow[n\to\infty]{} \frac{\lambda^k}{k!} e^{-\lambda} \quad \text{for } k = 0, 1, 2, \dots$$

2c8 Example. Let X_n have the geometric distribution $\mathrm{G}(1/n)$, and X — the exponential distribution

$$F_X(x) = 1 - e^{-x} \quad \text{for } 0 \le x < \infty \, .$$

Then $\frac{1}{n} X_n \xrightarrow[n\to\infty]{\text{distr}} X$.

2c9 Example. Let X_n have the binomial distribution $\mathrm{B}(n, p)$, and X the normal distribution $\mathrm{N}(0, 1)$. Then

$$\frac{1}{\sqrt{np(1-p)}} (X_n - np) \xrightarrow[n\to\infty]{\text{distr}} X \, .$$

That is a limit theorem (De Moivre-Laplace) known (without proof) from Introduction to Probability.

[9] Possibly defined on different probability spaces.

2c10. The following four conditions are equivalent:

- $X_n \xrightarrow[n\to\infty]{\text{distr}} X$;
- $F_{X_n} \xrightarrow[n\to\infty]{\text{weak}} F_X$;
- $F_X(x-) = F_X(x) \quad \Longrightarrow \quad F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x)$.
- $X^*(p-) = X^*(p+) \quad \Longrightarrow \quad X_n^*(p) \xrightarrow[n\to\infty]{} X^*(p)$.

Equivalence of the two last conditions is not evident. Both are equivalent to convergence of quantile lines treated as functions of $s = x + p$. We may define the distance between two distribution functions F_1, F_2 by

$$\operatorname{dist}(F_1, F_2) = \sup_{s\in\mathbb{R}} |x_s^{(1)} - x_s^{(2)}| = \sup_{s\in\mathbb{R}} |p_s^{(1)} - p_s^{(2)}|\,,$$

where $x_s^{(1)}, p_s^{(1)}$ correspond to F_1 and $x_s^{(2)}, p_s^{(2)}$ correspond to F_2 (see (2b6)). It can be shown that

$$F_n \xrightarrow[n\to\infty]{\text{weak}} F \quad \Longleftrightarrow \quad \operatorname{dist}(F_n, F) \xrightarrow[n\to\infty]{} 0\,.$$