1 Probability space

1a Axiomatic approach

The probability space is left unspecified. Only its general properties, stipulated by the following definition, are used.

1a1 Definition. A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ consists of:

$$
\left.\begin{array}{l}
\Omega \neq \emptyset \\
\mathcal{F} \subset 2^{\Omega} \\
\left.\begin{array}{l}
\left.\begin{array}{l}
\emptyset, \Omega \in \mathcal{F} \\
A \in \mathcal{F} \quad \Longrightarrow \quad \Omega \setminus A \in \mathcal{F} \\
A, B \in \mathcal{F} \quad \Longrightarrow \quad A \cap B, A \cup B \in \mathcal{F}
\end{array}\right\} \begin{array}{c} \mathcal{F} \text{ is a field} \\ \text{(algebra)} \end{array} \\
A_1, A_2, \dots \in \mathcal{F} \quad \Longrightarrow \quad (A_1 \cup A_2 \cup \dots) \in \mathcal{F}
\end{array}\right\} \begin{array}{c} \mathcal{F} \text{ is a } \sigma\text{-field} \\ (\sigma\text{-algebra}) \end{array} \\
\left.\begin{array}{l}
\left.\begin{array}{l}
P : \mathcal{F} \to [0,1] \\
P(\Omega) = 1 \\
P(A \uplus B) = P(A) + P(B)
\end{array}\right\} \begin{array}{c} P \text{ is a finitely ad-} \\ \text{ditive probability} \end{array} \\
P(A_1 \uplus A_2 \uplus \dots) = P(A_1) + P(A_2) + \dots
\end{array}\right\} \begin{array}{c} P \text{ is a proba-} \\ \text{bility measure} \end{array}
\end{array}\right\} \begin{array}{c} (\Omega, \mathcal{F}, P) \text{ is a} \\ \text{probability space} \end{array}
$$

1a2 Definition. A *random variable* is a function $X : \Omega \to \mathbb{R}$ that is $\mathcal{F}$-measurable, which means

$$\forall x \in \mathbb{R} \quad \{\omega : X(\omega) \le x\} \in \mathcal{F} \,.$$

Strangely enough, there is no simple example of a continuous probability space. We'll return to the point later (in Sect. 5).

1b Analytic approach

The probability space used is simply the interval $(0, 1)$ with the uniform probability distribution:

$$
\begin{aligned}
&\Omega = (0,1), \\
&\mathbb{P}\big([a,b]\big) = b - a \qquad \text{for } 0 < a < b < 1.
\end{aligned}
$$

(Subsets of $(0, 1)$ more complicated than intervals and their finite unions will be discussed later.) This approach allows us to use the notion of integral, well-known from the analysis. We will see that the integral is nothing but the expectation!

For the first time, only monotone (or piecewise monotone) functions on $(0, 1)$ will be considered as random variables. This is why complicated subsets will not appear.

1b1 Example. The function

$$
X(\omega) = \begin{cases} 0 & \text{for } \omega \in (0,\, 1/2), \\ 1 & \text{for } \omega \in (1/2,\, 1) \end{cases}
$$

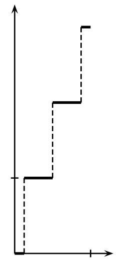
is a random variable taking two values $0, 1$ with probabilities $1/2$, $1/2$. The value $X(1/2)$ does not matter. The function $1 - X$ is another random variable with the same distribution. There are infinitely many of them:

1b2 Example. The function

$$X(\omega) = \begin{cases} 0 & \text{for } \omega \in (0,\, 1/8), \\ 1 & \text{for } \omega \in (1/8,\, 1/2), \\ 2 & \text{for } \omega \in (1/2,\, 7/8), \\ 3 & \text{for } \omega \in (7/8,\, 1) \end{cases}$$

is a random variable with the binomial distribution $B(3,\, 1/2)$:

$$\begin{aligned} \mathbb{P}\big(\{\omega : X(\omega) = 0\}\big) = \mathbb{P}\big(X = 0\big) = p_X(0) &= 1/8, \\ p_X(1) &= 3/8, \\ p_X(2) &= 3/8, \\ p_X(3) &= 1/8. \end{aligned}$$



1b3 Exercise. For the given function

$$X(\omega) = \begin{cases} 1 & \text{for } \omega \in (0,\, 1/4), \\ 0 & \text{for } \omega \in (1/4,\, 3/4), \\ 3 & \text{for } \omega \in (3/4,\, 1) \end{cases}$$

find an increasing function Y having the same distribution as X.

1b4 Example. The function

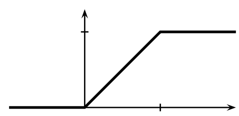
$$X(\omega) = \omega$$

is a random variable such that

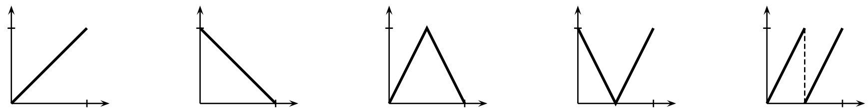
$$p_X(x) = \mathbb{P}\{\omega : X(\omega) = x\} = 0 \quad \text{for any } x\,.$$

Such a distribution cannot be described by p_X, but still can be described by its *cdf* — cumulative distribution function F_X:

$$F_X(x) = \mathbb{P}\{\omega : X(\omega) \le x\} = \mathbb{P}\big((0, x]\big) = x \quad \text{for } x \in [0, 1)\,.$$



Some other functions with the same distribution (that is, the same cdf):



Later on it will be shown that for any X there is an *increasing* Y with the same distribution as X.

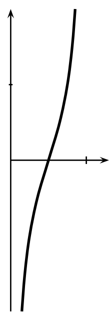
Till the end of the section we restrict ourselves to increasing functions X.

The following case is especially simple and important; it will be called the *smooth case:*

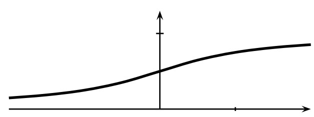
(1b5)
$$\left.\begin{array}{l}\forall\omega\in(0,1)\quad X'(\omega)\text{ exists, is continuous, and }X'(\omega)>0;\\ X(0+)=-\infty,\qquad X(1-)=+\infty.\end{array}\right\}\text{ smooth case}$$

It follows that the inverse function X^{-1} exists, $X^{-1}:\mathbb{R}\to(0,1)$.

1b6 Example.



$X(\omega)=\tan\pi\big(\omega-\frac12\big)$;



$X^{-1}(x)=\frac12+\frac1\pi\arctan x$.

Clearly, $\{\omega : X(\omega)\le x\}=\big(0,X^{-1}(x)\big]$. Hence

(1b7)
$$F_X(x)=X^{-1}(x);\qquad X(\omega)=F_X^{-1}(\omega)\,.$$

1c Density

The inverse function has its derivative:

$$F_X'(x)=\frac{1}{X'(p)}\bigg|_{p=F_X(x)}\,,$$

which may be also written as

$$\frac{dp}{dx}\cdot\frac{dx}{dp}=1\,.$$

1c1 Example. $X(\omega)=\tan\pi\big(\omega-\frac12\big)$, $F_X(x)=\frac12+\frac1\pi\arctan x$,

$$X'(\omega)=\frac{\pi}{\cos^2\pi\big(\omega-\frac12\big)},\qquad F_X'(x)=\frac1\pi\,\frac{1}{1+x^2}\,,$$

$$\frac{\pi}{\cos^2\pi\big(\omega-\frac12\big)}\cdot\frac1\pi\,\frac{1}{1+x^2}=1\quad\text{when }x=\tan\pi\big(\omega-\tfrac12\big)\,.$$

Clearly,

$$\mathbb{P}\left(X \in [a,b]\right) = F_X(b) - F_X(a) = \int_a^b F'_X(x)\,dx\,,$$

that is,

$$\mathbb{P}\left(X \in [a,b]\right) = \int_a^b f_X(x)\,dx \quad \text{whenever} \quad -\infty < a < b < \infty\,, \tag{1c2}$$

where

$$f_X(x) = F'_X(x)\,. \tag{1c3}$$

A function f_X satisfying (1c2) is called the *density* (of the distribution) of the random variable X. The formula (1c3) gives the density under the smoothness assumptions (1b5). The general case will be considered later. The question of uniqueness of the density also will be addressed later.

Let a continuous strictly positive function $f : \mathbb{R} \to (0,\infty)$ be given such that

$$\int_{-\infty}^{+\infty} f(x)\,dx = 1\,. \tag{1c4}$$

Then the function

$$F(x) = \int_{-\infty}^{x} f(z)\,dz \tag{1c5}$$

is a continuous strictly increasing function $F : \mathbb{R} \to (0,1)$, $F(-\infty) = 0$, $F(+\infty) = 1$. It has an inverse function $F^{-1} : (0,1) \to \mathbb{R}$. Denote

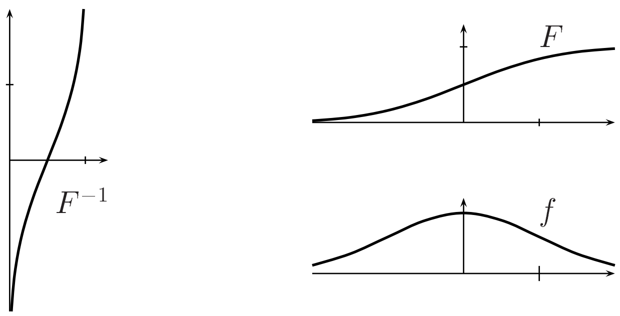
$$X = F^{-1}\,,$$

then X satisfies the smoothness conditions (1b5), and

$$F = F_X\,.$$

The given function f is the density of the constructed random variable X.

1c6 Example. $f(x) = \dfrac{1}{\sqrt{2\pi}} \exp\bigl(-x^2/2\bigr)$. It may be shown that (1c4) is satisfied. Unfortunately, the integral (1c5) is not an elementary function.[1] Nevertheless the function F exists, and is well-known as the standard normal cdf. Its values are available by tables and computer programs.



[1] That is, not a finite combination of exponential, logarithmic, trigonometric, and inverse trigonometric functions.