7 Conditioning

SUMMARY

7a General theory

	Unconditional	**Conditional**
Quantile function:	$X^*(p)$	$Y^*(p \mid X = x)$
Its properties: (qualitative)	Increases in p.	Increases in p for every x, Borel in (p, x).
(quantitative)	X^* is distributes like X but on $\big((0,1), \mathrm{mes}\big)$.	(X^{**}, Y^{**}) is distributed like (X, Y) but on $\big((0,1)\times(0,1), \mathrm{mes}_2\big)$; here $X^{**}(p_1, p_2) = X^*(p_1)$, $Y^{**}(p_1, p_2) = Y^*\big(p_2 \mid X = X^*(p_1)\big)$.
Distribution function:	$F_X(x)$	$F_{Y\mid X=x}(y)$
Its properties: (qualitative)	Increases in x from 0 to 1 and is right-continuous in x.	Increases in y from 0 to 1 and is right-continuous in y, both for every x; Borel in (x, y).
(quantitative)	$\mathbb{P}\big(X \le x\big) = F_X(x)$.	$\mathbb{P}\big(X \le x \& Y \le y\big) = \int_{(-\infty,x]} F_{Y\mid X=x_1}(y)\, dF_X(x) = \mathbb{E}\big(\mathbf{1}_{(-\infty,x]}(X) F_{Y\mid X}(y)\big)$.
Distribution:	$P_X(B)$	$P_{Y\mid X=x}(B)$
Its properties: (qualitative)	probability measure	probability measure in B for every x, a Borel function in x for every B.
(quantitative)	$\mathbb{P}\big(X \in B\big) = P_X(B)$	$\mathbb{P}\big(X \in A \,\&\, Y \in B\big) = \mathbb{E}\big(\mathbf{1}_A(X) P_{Y\mid X}(B)\big)$.
	$\mathbb{E}\,\varphi(X) = \int \varphi(x) P_X(dx)$	$\mathbb{E}\,\varphi(X, Y) = \int \Big(\int \varphi(x, y) P_{Y\mid X=x}(dy)\Big) P_X(dx)$.

Uniqueness: $F_{Y\mid X}$ is determined uniquely up to a change on a set of the form $A \times \mathbb{R}$ (that is, $x \in A$) with $\mathbb{P}\big(X \in A\big) = 0$.

Existence (without proof): $F_{Y\mid X}$ exists for every two-dimensional distribution.

7b Special cases

Absolutely continuous case: if $P_{X,Y}$ has a density $f_{X,Y}$, then $P_{Y\mid X=x}$ has a density

$$f_{Y\mid X=x}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

for P_X-almost all x.

Discrete X and arbitrary Y:

$$F_{Y|X=x}(y) = \mathbb{P}\left(Y \le y \,\middle|\, X = x \right) = \frac{\mathbb{P}\left(Y \le y \,\&\, X = x \right)}{\mathbb{P}\left(X = x \right)} .$$

Absolutely continuous X and arbitrary Y:

$$F_{Y|X=x}(y) = \frac{f_{X|Y\le y}(x)\mathbb{P}\left(Y \le y \right)}{f_X(x)} .$$

	Total probability formula	**Bayes formula**
General:	$\mathbb{P}\left(Y \in B \right) = \mathbb{E}\mathbb{P}\left(Y \in B \,\middle|\, X \right) =$ $\int_{\mathbb{R}} \left(\int_B dF_{Y|X=x}(y) \right) dF_X(x).$	
Discrete:	$p_Y(y) = \mathbb{E}\, p_{Y|X}(y) =$ $\sum_x \underbrace{p_{Y|X=x}(y)}_{=\frac{p_{X,Y}(x,y)}{p_X(x)}} p_X(x).$	$p_{Y|X=x}(y) p_X(x) =$ $p_{X|Y=y}(x) p_Y(y).$
Absolutely continuous:	$f_Y(y) = \mathbb{E}\, f_{Y|X}(y) =$ $\int \underbrace{f_{Y|X=x}(y)}_{=\frac{f_{X,Y}(x,y)}{f_X(x)}} f_X(x)\, dx.$	$f_{Y|X=x}(y) f_X(x) =$ $f_{X|Y=y}(x) f_Y(y).$
Singular 1: X abs. cont., Y discrete	$p_Y(y) = \mathbb{E}\, p_{Y|X}(y) =$ $\int p_{Y|X=x}(y) f_X(x)\, dx;$ $f_X(x) = \mathbb{E}\, f_{X|Y}(x) =$ $\sum_y f_{X|Y=y}(x) p_Y(y).$	$p_{Y|X=x}(y) f_X(x) =$ $f_{X|Y=y}(x) p_Y(y).$
Singular 2: X abs.cont., $Y = \mathbf{1}_A$	$\mathbb{P}\left(A \right) = \mathbb{E}\mathbb{P}\left(A \,\middle|\, X \right) =$ $\int \mathbb{P}\left(A \,\middle|\, X = x \right) f_X(x)\, dx;$ $f_X(x) = f_{X|A}(x)\mathbb{P}\left(A \right) +$ $f_{X|\overline{A}}(x)(1 - \mathbb{P}\left(A \right)).$	$\mathbb{P}\left(A \,\middle|\, X = x \right) f_X(x) =$ $f_{X|A}(x)\mathbb{P}\left(A \right).$
Singular 3: X abs.cont., $Y = \varphi(X)$, φ smooth	$f_Y(y) = \sum_{x \in \varphi^{-1}(y)} \frac{1}{\varphi'(x)} f_X(x).$	$\frac{1}{\varphi'(x)} f_X(x) =$ $\mathbb{P}\left(X = x \,\middle|\, Y = \varphi(x) \right)$ $f_Y(y).$