5 Borel sets, functions and measures

SUMMARY

An elementary set is a union of a finite number of (disjoint, or arbitrary) intervals. (Two-dimensional intervals are basically rectangles.)

Classes $\mathcal{E}_1$ (1-dim), $\mathcal{E}_2$ (2-dim) of all elementary sets are algebras of sets.

Unions of countably many intervals belong to classes $(\mathcal{E}_1)_\sigma$, $(\mathcal{E}_2)_\sigma$. All open sets belong to the classes, and all sets of the form "$\exists n$ (a property of a finite number of digits)" belong. Their complements belong to classes $(\mathcal{E}_1)_\delta$, $(\mathcal{E}_2)_\delta$ (of intersections). All closed sets are here, and all sets of the form "$\forall n$ (a property of a finite number of digits)" are.

Iterating the operations we get higher classes, such as $(\mathcal{E}_1)_{\delta\sigma\delta}$. The event $(\alpha_1 + \cdots + \alpha_n)/n \to 1/2$ belongs to that class.

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Borel σ-fields $\mathcal{B}_1$ (1-dim), $\mathcal{B}_2$ (2-dim) are σ-fields generated by intervals or, equivalently, σ-fields generated by elementary sets, or monotone classes generated by elementary sets.

A function $X : \Omega \to \mathbb{R}$ is a random variable if and only if $\forall B \in \mathcal{B}_1 \;\; \underbrace{\{\omega : X(\omega) \in B\}}_{X^{-1}(B)} \in \mathcal{F}$.

A function $\varphi : \mathbb{R} \to \mathbb{R}$ is called a Borel function, if $\forall B \in \mathcal{B}_1 \;\; \varphi^{-1}(B) \in \mathcal{B}_1$. Similarly for $\mathbb{R}^2 \to \mathbb{R}$, $\mathbb{R} \to \mathbb{R}^2$, and $\mathbb{R}^2 \to \mathbb{R}^2$.

If $X : \Omega \to \mathbb{R}$ is a random variable and $\varphi : \mathbb{R} \to \mathbb{R}$ a Borel function, then $\varphi(X)$ is a random variable. The same for $\varphi(X, Y)$.

Continuous functions are Borel functions.

Limits of (convergent sequences of) Borel functions are Borel functions.

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Borel measures (finite, locally finite, probability measures) are measures on $\mathcal{B}_1$ (or $\mathcal{B}_2$).

A Borel probability measure μ on $\mathbb{R}$ is uniquely determined by its cumulative distribution function $F_\mu(x) = \mu\big((-\infty, x]\big)$.

Lebesgue measure: $\operatorname{mes}\big((a, b)\big) = b - a$. Its existence (without proof) and uniqueness.

Lebesgue integral: for a Borel function $f : \mathbb{R} \to \mathbb{R}$,

$$\underbrace{\int_{-\infty}^{+\infty} f(x)\,dx}_{\text{Lebesgue}} = \underbrace{\int_0^\infty \big(\operatorname{mes}\{x : f(x) > y\} - \operatorname{mes}\{x : f(x) < -y\}\big)\,dy}_{\text{improper Riemann}} ;$$

Lebesgue integrability: $\int_{-\infty}^{+\infty} |f(x)|\,dx = \int_0^\infty \big(\operatorname{mes}\{x : |f(x)| > y\}\,dy < \infty$.

Expectation as a kind of Lebesgue integral:

$$\mathbb{E}(X) = \int_0^\infty \big(P(X > x) - P(X < -x)\big)\,dx = \int_\Omega X\,dP\,.$$

Properties of Lebesgue integral (and expectation): linearity, positivity, strict positivity; Monotone convergence theorem; Dominated convergence theorem (without proofs).

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A small dictionary: measure theory / probability theory

measure space	probability space
Borel set	event
measure	probability
Borel function	random variable
Lebesgue integral	expectation
Almost everywhere (a.e.)	almost sure (a.s.)
binary digits	coin tossing