4 Mathematical expectation and integral

SUMMARY

4a Bounded case

Every bounded random variable X has its *expectation*

$$\mathbb{E}(X) = \int_0^1 X^*(p)\,dp\,,$$

just Riemann integral of the quantile function of X. Equivalently,

$$\mathbb{E}(X) = \int_0^\infty \Big(\mathbb{P}\big(X > x\big) - \mathbb{P}\big(X < -x\big)\Big)\,dx = -\underbrace{\int_{-\infty}^0 F_X(x)\,dx}_{\mathbb{E}(X_-)} + \underbrace{\int_0^\infty (1 - F_X(x))\,dx}_{\mathbb{E}(X_+)}\,;$$

integrands vanish for x large enough, since X is bounded.

Elementary case: X takes on only a finite number of values $x_1, \dots, x_n$, with probabilities $p_1, \dots, p_n$; then

$$\mathbb{E}(X) = x_1 p_1 + \dots + x_n p_n\,.$$

In particular, $\mathbb{E}(c\mathbf{1}_A) = c\mathbb{P}\big(A\big)$.

Smooth case: X has a continuous density f_X;

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} x\,f_X(x)\,dx\,;$$

the integrand vanishes for x large enough, since X is bounded.

If X and Y are identically distributed, then $\mathbb{E}(X) = \mathbb{E}(Y)$.

Comparison:

$$\begin{tikzcd}
 & X^* \le Y^* \arrow[dr, Rightarrow] \arrow[dd, Leftrightarrow] & \\
\mathbb{P}\big(X \le Y\big) = 1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \mathbb{E}(X) \le \mathbb{E}(Y) \\
 & F_X \ge F_Y \arrow[ur, Rightarrow] &
\end{tikzcd}$$

In particular: $\mathbb{P}\big(X \ge 0\big) = 1 \implies \mathbb{E}(X) \ge 0$, and $\mathbb{P}\big(a \le X \le b\big) = 1 \implies a \le \mathbb{E}(X) \le b$.

Markov inequality:

$$\mathbb{P}\big(X \ge 0\big) = 1 \quad\implies\quad \forall c \in (0,\infty)\;\; \mathbb{P}\big(X \ge c\big) \le \frac{1}{c}\mathbb{E}(X)\,.$$

In particular, if $\mathbb{P}\big(X \ge 0\big) = 1$ and $\mathbb{E}(X) = 0$ then $\mathbb{P}\big(X = 0\big) = 1$.

Linearity:

$$\mathbb{E}(cX) = c\mathbb{E}(X)\,; \qquad \mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)\,;$$

note that X, Y need not be independent. In particular,

$$\mathbb{E}\big(\mathbf{1}_{A_1} + \dots + \mathbf{1}_{A_n}\big) = \mathbb{P}\big(A_1\big) + \dots + \mathbb{P}\big(A_n\big)$$

for arbitrary events $A_1, \dots, A_n$.

Symmetry: if $(X - c)$ and $(c - X)$ are identically distributed, then $\mathbb{E}(X) = c$.

4b Unbounded case

$$\mathbb{E}\,X = \lim_{\varepsilon\to 0}\int_{\varepsilon}^{p_0} X^*(p)\,dp + \lim_{\varepsilon\to 0}\int_{p_0}^{1-\varepsilon} X^*(p)\,dp\,,$$

provided that both limits are finite; in that case X is *integrable.* The result does not depend on p_0. For the case (a number) + $(+\infty)$ one may say that $\mathbb{E}\,(X) = +\infty$; similarly, $(-\infty)+$ (a number) $= -\infty$; however, in the case of $(-\infty)+(+\infty)$, X has no expectation whatsoever (neither finite nor infinite). These 3 cases are all non-integrable.

The formulas for the bounded case remain true for unbounded *integrable* X, however, integrals are improper (rather than Riemann). Also,

$$\begin{aligned}\mathbb{E}\,|X| = \int_0^1 |X^*(p)|\,dp = \int_0^{\infty} \mathbb{P}\,\big(\,|X| > x\,\big)\,dx = \\ = +\underbrace{\int_{-\infty}^0 F_X(x)\,dx}_{\mathbb{E}\,(X_-)} + \underbrace{\int_0^{\infty} (1 - F_X(x))\,dx}_{\mathbb{E}\,(X_+)} \in [0,+\infty]\,,\end{aligned}$$

the three expressions being either all finite or all infinite; if they are finite, then X is integrable, otherwise X is non-integrable. For the smooth case, one more expression may be added to the list,

$$\mathbb{E}\,|X| = \int_{-\infty}^{+\infty} |x|\,f_X(x)\,dx \in [0,+\infty]\,.$$

For every piecewise monotone function $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}\,\varphi(X) = \int_0^1 \varphi\big(X^*(p)\big)\,dp$$

provided that φ is bounded on the support of X (thus $\varphi(X)$ is bounded, and $\varphi(X^*(\cdot))$ is Riemann integrable). Otherwise (for unbounded X), the integral is treated as improper; it converges if and only if $\varphi(X)$ is integrable. In any case, $\varphi(X)$ and $\varphi(X^*)$ are identically distributed.

For the elementary case,

$$\mathbb{E}\,\varphi(X) = \varphi(x_1)p_1 + \dots + \varphi(x_n)p_n\,.$$

For the smooth case,

$$\mathbb{E}\,\varphi(X) = \int_{-\infty}^{+\infty} \varphi(x) f_X(x)\,dx\,.$$

Especially useful are moments

$$\begin{aligned}\mathbb{E}\,X^{2n} &= \int_0^1 \big(X^*(p)\big)^{2n}\,dp \in [0,+\infty]\,,\\ \mathbb{E}\,X^{2n+1} &= \int_0^1 \big(X^*(p)\big)^{2n+1}\,dp \in \mathbb{R} \qquad \text{if } \int_0^1 |X^*(p)|^{2n+1}\,dp < \infty\,,\end{aligned}$$

and moment generating function

$$\mathrm{MGF}_X(t) = \mathbb{E}\, e^{tX} = \int_0^1 e^{tX^*(p)}\, dp \in (0, +\infty]\,;$$

for the smooth case,

$$\begin{aligned}
\mathbb{E}\, X^{2n} &= \int_{-\infty}^{+\infty} x^{2n} f_X(x)\, dx \in [0, +\infty]\,; \\
\mathbb{E}\, X^{2n+1} &= \int_{-\infty}^{+\infty} x^{2n+1} f_X(x)\, dx \in \mathbb{R} \qquad \text{if } \int_{-\infty}^{+\infty} |x|^{2n+1} f_X(x)\, dx < \infty\,; \\
\mathrm{MGF}_X(t) &= \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx \in (0, +\infty]\,.
\end{aligned}$$

If $\mathrm{MGF}_X(t) < \infty$ for $|t|$ small enough, then

$$\mathrm{MGF}_X(t) = 1 + t\, \mathbb{E}\,(X) + \frac{1}{2!} t^2\, \mathbb{E}\,(X^2) + \dots$$