2 Distribution functions and quantile functions

SUMMARY

Every random variable $X : \Omega \to \mathbb{R}$ has its *cumulative distribution function* $F_X : \mathbb{R} \to [0,1]$ and *quantile function* $X^* : (0,1) \to \mathbb{R}$. Functions X^* and F_X are inverse to each other as far as they are continuous and strictly increasing. Jumps of one function correspond to flats (constant intervals) of the other.

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A number x is called a *p-quantile* of X, if

$$\mathbb{P}\big(X < x \big) \le p \le \mathbb{P}\big(X \le x \big) ,$$

which is equivalent to

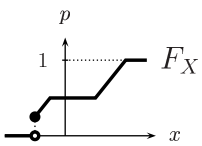
$$F_X(x-) \le p \le F_X(x)$$

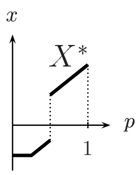
and also to

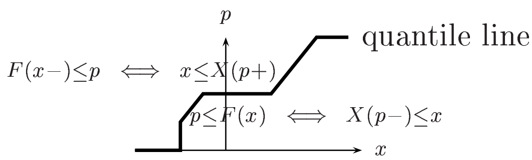
$$X^*(p-) \le x \le X^*(p+) .$$

The matter becomes simpler when F_X and X^* are continuous and strictly increasing:

$$F_X(x) = p ; \qquad X^*(p) = x .$$







2a Atoms and gaps

Each of the functions X^*, F_X may have discontinuities (jumps). A jump of X^* is described by p, x_1, x_2 such that

$$X^*(p-) = x_1 < x_2 = X^*(p+) ,$$

and corresponds to a constant interval of F_X, namely,

$$F_X(x_1) = p = F_X(x_2-) .$$

In that case, the interval (x_1, x_2) is called a *gap* of the distribution, since $\mathbb{P}\big(x_1 < X < x_2 \big) = 0$. A jump of F_X is described by x, p_1, p_2 such that

$$F_X(x-) = p_1 < p_2 = F_X(x) ,$$

and corresponds to a flat (constant interval) of X^*, namely,

$$X^*(p_1+) = x = X^*(p_2-) .$$

In that case, the number x is called an *atom* of the distribution, since $\mathbb{P}\big(X = x \big) = p_2 - p_1 > 0$.

Values of F_X at its jumping points are uniquely determined by right continuity: $F_X(x) = F_X(x+)$. In contrast, values of X^* at its jumping points remain arbitrary within corresponding intervals $[x_1, x_2]$.

Random variables $X_1 : \Omega_1 \to \mathbb{R}$, $X_2 : \Omega_2 \to \mathbb{R}$ are *identically distributed,* if $F_X = F_Y$, that is, $F_X(u) = F_Y(u)$ for all $u \in \mathbb{R}$. Or, equivalently, $X^*(p-) = Y^*(p-)$ and $X^*(p+) = Y^*(p+)$ for all $p \in (0,1)$.

2b Convergence in distribution, or weak convergence of distributions

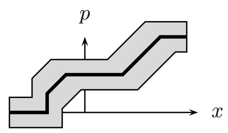
Given a sequence of random variables $X_n : \Omega_n \to \mathbb{R}$ and another random variable $X : \Omega \to \mathbb{R}$, we define *convergence in distribution*, $X_n \xrightarrow[n\to\infty]{\text{distr}} X$, by

$$X_n^*(p) \xrightarrow[n\to\infty]{} X^*(p) \qquad \text{for all } p \text{ satisfying } X^*(p-) = X^*(p+)$$

or equivalently by

$$F_{X_n}(x) \xrightarrow[n\to\infty]{} F_X(x) \qquad \text{for all } x \text{ satisfying } F_X(x-) = F_X(x).$$

Still equivalent: for every $\varepsilon > 0$, for all n large enough, the quantile line of X_n is within the ε-neighborhood of the quantile line of X.



Let X_n have the binomial distribution $B\big(n, \frac{\lambda}{n}\big)$, and X — the Poisson distribution $P(\lambda)$. Then $X_n \xrightarrow[n\to\infty]{\text{distr}} X$.

Let X_n be distributed uniformly on $\{1, 2, \dots, n\}$, and X — uniformly on $(0,1)$. Then $\frac{1}{n} X_n \xrightarrow[n\to\infty]{\text{distr}} X$.

Let X_n have the geometric distribution, $\mathbb{E} X_n = n$, and X — the exponential distribution, $\mathbb{E} X = 1$. Then $\frac{1}{n} X_n \xrightarrow[n\to\infty]{\text{distr}} X$.

Let X_n have the binomial distribution $B(n,p)$, and X — the normal distribution $N(0,1)$. Then $\dfrac{1}{\sqrt{np(1-p)}}(X_n - np) \xrightarrow[n\to\infty]{\text{distr}} X$.
