Exam of 12.07.1999 — Solutions

1

1a

$\mathbb{P}\left(R = 0 \right) = \mathbb{P}\left(Q \right) = \dfrac{\mathrm{mes}_2(Q)}{\mathrm{mes}_2(\Omega)} = \dfrac{1}{\pi + 5}$.

1b

The area of the r-neighborhood of Q is equal to $\pi r^2 + 4r + 1$, therefore $F_R(r) = \dfrac{\pi r^2 + 4r + 1}{\pi + 5}$ for $r \in [0, 1]$; $F_R(r) = 0$ for $r < 0$; and $F_R(r) = 1$ for $r \ge 1$.

1c

$F_R(r) = p \iff \pi r^2 + 4r + 1 - (\pi + 5)p = 0 \iff r = \frac{1}{2\pi}\left(-4 + \sqrt{16 - 4\pi(1 - (\pi + 5)p)}\right)$, thus $R^*(p) = \frac{1}{2\pi}\left(-4 + \sqrt{4\pi(\pi + 5)p - 4\pi + 16}\right)$ for $p \in \left(\frac{1}{\pi+5}, 1\right)$; also, $R^*(p) = 0$ for $p \in \left(0, \frac{1}{\pi+5}\right)$ according to the jump of F_R at 0.

1d

$r_{1/2} = R^*(1/2) = \frac{1}{2\pi}\left(-4 + \sqrt{4\pi(\pi + 5)/2 - 4\pi + 16}\right) = \frac{1}{2\pi}\left(-4 + \sqrt{2\pi^2 + 6\pi + 16}\right)$.

1e

F_R has a single jump at 0, and is smooth on $(0, 1)$, which means that the distribution has a discrete part and an absolutely continuous part, and no singular part. Namely, $p_{\mathrm{d}} = \mathbb{P}\left(R = 0 \right) = \dfrac{1}{\pi + 5}$; $p_{\mathrm{ac}} = 1 - p_{\mathrm{d}} = \dfrac{\pi + 4}{\pi + 5}$; $F_{R,\mathrm{d}}(r) = \begin{cases} 0 & \text{for } r < 0, \\ 1 & \text{for } r \ge 0; \end{cases}$; $F_{R,\mathrm{ac}}(r) = \dfrac{1}{p_{\mathrm{ac}}}\left(F(r) - p_{\mathrm{d}} F_{R,\mathrm{d}}(r)\right) = \dfrac{\pi r^2 + 4r}{\pi + 4}$ for $r \in [0, 1]$; $F_{R,\mathrm{ac}}(r) = 0$ for $r \le 0$, and $F_{R,\mathrm{ac}}(r) = 1$ for $r \ge 1$. The density: $f_{R,\mathrm{ac}}(r) = F'_{R,\mathrm{ac}}(r) = \dfrac{2\pi r + 4}{\pi + 4}$ for $r \in (0, 1)$.

1f

First, $\mathbb{E} R = \int_0^\infty \left(1 - F_R(r)\right) dr = \int_0^1 \left(1 - \frac{\pi r^2 + 4r + 1}{\pi + 5}\right) dr = 1 - \frac{1}{\pi+5}\left(\pi \cdot \frac{1}{3} + 4 \cdot \frac{1}{2} + 1\right) = \frac{1}{\pi+5}\left(\frac{2}{3}\pi + 2\right) = \dfrac{2(\pi + 3)}{3(\pi + 5)}$.

Second, $\mathbb{E}R = p_{\rm d}\int r\,dF_{R,\rm d}(r) + p_{\rm ac}\int f_{R,\rm ac}(r)\,dr$; however, $\int r\,dF_{R,\rm d}(r) = 0$, since the atom is at 0; so, $\mathbb{E}R = \frac{\pi+4}{\pi+5}\int_0^1 r\frac{2\pi r+4}{\pi+4}\,dr = \frac{1}{\pi+5}\big(2\pi\int_0^1 r^2\,dr + 4\int_0^1 r\,dr\big) = \frac{1}{\pi+5}\big(2\pi\cdot\frac13 + 4\cdot\frac12\big) = \dfrac{2(\pi+3)}{3(\pi+5)}$.

1g ....

$\mathbb{E}(1-R)^{-\alpha} = p_{\rm d}\int(1-r)^{-\alpha}\,dF_{R,\rm d}(r) + p_{\rm ac}\int(1-r)^{-\alpha}f_{R,\rm ac}(r)\,dr$; the discrete part converges for all α (since the atom is not at 1); so, only $(1-r)^{-\alpha}f_{R,\rm ac}(r)$ must be integrable when $r = 1-\varepsilon$, $\varepsilon\to 0+$. However, $f_{R,\rm ac}(1-\varepsilon)$ tends to a non-zero number, so, $\varepsilon^{-\alpha}$ must be integrable, which holds if and only if $\alpha < 1$.

2

2a ....

$P_{X,Y}$ contains a discrete part concentrated at vertices, a singular part concentrated on edges, and an absolutely continuous part within Q. Namely, $p_{\rm d} = \dfrac{\pi}{\pi+5}$, $p_{\rm s} = \dfrac{4}{\pi+5}$, $p_{\rm ac} = \dfrac{1}{\pi+5}$;

$$P_{X,Y,\rm d} = \left(\frac12\delta_{-1/2} + \frac12\delta_{1/2}\right)\otimes\left(\frac12\delta_{-1/2} + \frac12\delta_{1/2}\right);$$

$$P_{X,Y,\rm s} = \frac12\left(\frac12\delta_{-1/2} + \frac12\delta_{1/2}\right)\otimes U\big(-\tfrac12,\tfrac12\big) + \frac12 U\big(-\tfrac12,\tfrac12\big)\otimes\left(\frac12\delta_{-1/2} + \frac12\delta_{1/2}\right);$$

$$P_{X,Y,\rm ac} = U\big(-\tfrac12,\tfrac12\big)\otimes U\big(-\tfrac12,\tfrac12\big);$$

$$f_{X,Y,\rm ac}(x,y) = 1 \quad \text{for } (x,y)\in Q\,.$$

2b ....

X, Y are dependent, since (for example)

$$\mathbb{P}\big(\,X\in(-\tfrac12,\tfrac12)\;\&\;Y\in(-\tfrac12,\tfrac12)\,\big) = p_{\rm ac} = \frac{1}{\pi+5}\,,$$

$$\mathbb{P}\big(\,X\in(-\tfrac12,\tfrac12)\,\big) = \mathbb{P}\big(\,Y\in(-\tfrac12,\tfrac12)\,\big) = p_{\rm ac} + \frac12 p_{\rm s} = \frac{3}{\pi+5}\,;$$

$$\frac{1}{\pi+5} \ne \left(\frac{3}{\pi+5}\right)^2.$$

2c

We have

$$P_{X,Y} = U\big(-\tfrac12,\tfrac12\big) \otimes \bigg(\frac{1}{\pi+5}\delta_{-1/2} + \frac{1}{\pi+5}\delta_{1/2} + \frac{1}{\pi+5} U\big(-\tfrac12,\tfrac12\big) \bigg) + \\ + \bigg(\frac12\delta_{-1/2} + \frac12\delta_{1/2} \bigg) \otimes \bigg(\frac{\pi}{2(\pi+5)}\delta_{-1/2} + \frac{\pi}{2(\pi+5)}\delta_{1/2} + \frac{2}{\pi+5} U\big(-\tfrac12,\tfrac12\big) \bigg) = \\ = \frac{3}{\pi+5} U\big(-\tfrac12,\tfrac12\big) \otimes \bigg(\frac13\delta_{-1/2} + \frac13\delta_{1/2} + \frac13 U\big(-\tfrac12,\tfrac12\big) \bigg) + \\ + \frac{\pi+2}{\pi+5} \bigg(\frac12\delta_{-1/2} + \frac12\delta_{1/2} \bigg) \otimes \bigg(\frac{\pi}{2(\pi+2)}\delta_{-1/2} + \frac{\pi}{2(\pi+2)}\delta_{1/2} + \frac{2}{\pi+2} U\big(-\tfrac12,\tfrac12\big) \bigg),$$

which means that X, Y are conditionally independent given that $|X| < 1/2$, and also given that $|X| = 1/2$; the corresponding conditional distributions for Y are $\frac13\delta_{-1/2} + \frac13\delta_{1/2} + \frac13 U\big(-\frac12,\frac12\big)$ and $\frac{\pi}{2(\pi+2)}\delta_{-1/2} + \frac{\pi}{2(\pi+2)}\delta_{1/2} + \frac{2}{\pi+2} U\big(-\frac12,\frac12\big)$ respectively. So, for $|x| < 1/2$ we have

$$F_{Y|X=x}(y) = \begin{cases} 0 & \text{for } y < -1/2, \\ \dfrac12 + \dfrac13 y & \text{for } -1/2 \le y < 1/2, \\ 1 & \text{for } y \ge 1/2, \end{cases}$$

while for $|x| = 1/2$ we have

$$F_{Y|X=x}(y) = \begin{cases} 0 & \text{for } y < -1/2, \\ \dfrac12 + \dfrac{2}{\pi+2} y & \text{for } -1/2 \le y < 1/2, \\ 1 & \text{for } y \ge 1/2. \end{cases}$$

3

3a

F_α is a cumulative distribution function, since it satisfies the necessary and sufficient conditions, namely: $F_\alpha : \mathbb{R} \to [0,1]$ increases, $F_\alpha(-\infty) = 0$, $F_\alpha(+\infty) = 1$, and $F_\alpha(x+) = F_\alpha(x)$; all that follows from corresponding properties of F and the fact that the function $p \mapsto p^\alpha$ is continuous and increasing on $[0,1]$.

3b

If F is invertible, then F_α is also invertible, and[1] $F_\alpha(x) = p \iff \big(F(x)\big)^\alpha = p \iff F(x) = p^{1/\alpha} \iff x = F^{-1}(p^{1/\alpha}) \iff x = X_1^*(p^{1/\alpha})$, which means that

[1] Be careful not to confuse powers (like $p^{1/\alpha}$) and inverse functions (like F^{-1}). They *look* similarly, but are quite different in meaning and algebraic rules!

$X_\alpha^*(p) = X_1^*(p^{1/\alpha})$. In general (for non-invertible F) the formula still holds, but the proof is less straightforward: $X_\alpha^*(p-) \le x \le X_\alpha^*(p+) \iff F_\alpha(x-) \le p \le F_\alpha(x) \iff F(x-) \le p^{1/\alpha} \le F(x) \iff X_1^*(p^{1/\alpha}-) \le x \le X_1^*(p^{1/\alpha}+)$ for all x (and p), therefore $X_\alpha^*(p\pm) = X_1^*(p^{1/\alpha}\pm)$.

3c . . . .

Yes, $A_\alpha = A_1$, since $x \in A_1 \iff F(x-) < F(x) \iff \big(F(x-)\big)^\alpha < \big(F(x)\big)^\alpha \iff F_\alpha(x-) < F_\alpha(x) \iff x \in A_\alpha$. However, $\mathbb{P}\left(X_\alpha \in A_\alpha \right) \ne \mathbb{P}\left(X_1 \in A_1 \right)$ in general. Indeed, let F have a single jump, $F(x_0-) = p_1 < p_2 = F(x_0)$, then F_α also has a single jump, $F_\alpha(x_0-) = p_1^\alpha < p_2^\alpha = F_\alpha(x_0)$, so, $\mathbb{P}\left(X_1 \in A_1 \right) = p_2 - p_1$ while $\mathbb{P}\left(X_\alpha \in A_\alpha \right) = p_2^\alpha - p_1^\alpha$, which is not the same in general.

3d . . . .

Yes, F_{α_n} converges weakly to F_α, since $F_{\alpha_n}(x) = \big(F(x)\big)^{\alpha_n} \to \big(F(x)\big)^\alpha = F_\alpha(x)$ for all x, which is more than enough for weak convergence.

3e . . . .

It depends on the support of the distribution. If the support is bounded from above, that is, $\sup\{x : F(x) < 1\} = x_{\max} < \infty$, then $F_{\alpha_n} \to 0$ for all $x < x_{\max}$ and $F_{\alpha_n}(x) = 1$ for all $x \ge x_{\max}$, which means that F_{α_n} converges weakly to F concentrated at $x_{\max}$. However, let the support be unbounded from above, that is, $F(x) < 1$ for all $x \in \mathbb{R}$, then $F_{\alpha_n} \to 0$ for all x, and weak convergence is impossible. Indeed, assume that $F_{\alpha_n} \to F$ weakly, then $F(x) = 0$ for all x except for jumps of F, therefore for all x, in contradiction to the property $F(+\infty) = 1$ of every cumulative distribution function.

3f . . . .

Similarly to (e), using $\inf\{x : F(x) > 0\} = x_{\min}$. If $x_{\min} > -\infty$ then F_{α_n} converges weakly to the distribution concentrated at $x_{\min}$. Otherwise F_{α_n} cannot weakly converge.

3g . . . .

$\mathbb{P}\left(X_\alpha \in A_\alpha \right)$ is generally not monotone in α, since in the simple case considered in (c) it is $p_2^\alpha - p_1^\alpha$ which is generally not monotone.

Continuity of $\mathbb{P}\left(X_\alpha \in A_\alpha \right)$ follows from continuity of $p_2^\alpha - p_1^\alpha$, if A_1 is finite (the sum of a finite number of continuous functions is a continuous function). If A_1 is countable, $\mathbb{P}\left(X_\alpha \in A_\alpha \right)$ is still continuous, but the proof involves uniform convergence (I do not go into detail).

Limits of $p_2^\alpha - p_1^\alpha$ for $\alpha \to 0$ and $\alpha \to \infty$ are both 0 provided that $p_1, p_2 \in (0,1)$. However, if $p_1 < p_2 = 1$, the limit for $\alpha \to \infty$ is equal to 1. Therefore, if $\sup\{x : F(x) < 1\} = x_{\max} \in A_1$

then $\mathbb{P}\left(X_\alpha \in A_\alpha \right) \to 1$ for $\alpha \to \infty$. Similarly, if $\inf\{x : F(x) > 0\} = x_{\min} \in A_1$ then $\mathbb{P}\left(X_\alpha \in A_\alpha \right) \to 1$ for $\alpha \to 0$. If A_1 is finite and does not contain $x_{\min}$, nor $x_{\max}$, then both limits vanish. If A_1 is countable, limits need not exist (I do not go into detail).

4

4a

We have

$$U = 2X^2 + 2Y^2 + 2Z^2 - 2XY - 2YZ - 2XZ = 3(X^2 + Y^2 + Z^2) - (X + Y + Z)^2 \,;$$
$$\frac{1}{3} U = (X^2 + Y^2 + Z^2) - \left(\frac{X + Y + Z}{\sqrt{3}} \right)^2 ,$$

which is the squared distance[2] of the point (X, Y, Z) from the line $x = y = z$. Apply a rotation that transforms the line $x = y = z$ into, say, the axis $y = z = 0$. The squared distance of the point (X, Y, Z) from the axis is $Y^2 + Z^2$. Due to rotation invariance of the standard 3-dim normal distribution, $U/3$ is distributed like $Y^2 + Z^2$. However, $\frac{1}{2}Y^2 + \frac{1}{2}Z^2 \sim \text{Gamma}\big(\frac{1}{2}\big) * \text{Gamma}\big(\frac{1}{2}\big) = \text{Gamma}\big(1\big) = \text{Exp}(1)$. So, $\frac{1}{6}U \sim \text{Exp}(1)$, that is, $U \sim \text{Exp}(6)$, the exponential distribution with the mean 6.

5

5a

We have to prove that almost surely, only finitely many events

$$A_n = \{ X_{n^2} < \min(X_1, \dots, X_{n^2 - 1}) \}$$

occur. By the first Borel-Cantelli lemma, it suffices to prove that $\sum_n \mathbb{P}\left(A_n \right) < \infty$. We'll prove that $\mathbb{P}\left(A_n \right) \le 1/n^2$, which is enough, since $\sum_n \frac{1}{n^2} < \infty$.

The event A_n means that the sequence $X_1, \dots, X_{n^2}$ reaches its minimum only once, at the last element. It is one of n^2 events $X_1 < \min(X_2, \dots, X_{n^2})$, $X_2 < \min(X_1, X_3, \dots, X_{n^2})$, and so on; these n^2 events are equiprobable by symmetry, and disjoint. Therefore each of them is of probability[3] $\le 1/n^2$.

[2] Indeed, $x^2 + y^2 + z^2$ is the squared norm of the vector (x, y, z), and $(x + y + z)/\sqrt{3}$ is its scalar product by the unit vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, thus $\big((x + y + z)/\sqrt{3}\big)^2$ is the squared length of its projection onto the line.

[3] It can be $< 1/n^2$, since the given distribution can have atoms.

5b . . . .

In general, the answer is negative. For example, let $X_k \sim U(0,1)$, then $\inf(X_1, X_2, \dots) = 0$ almost surely,[4] while $\min(X_1, \dots, X_n) > 0$ for each n; it follows that $X_n < \min(X_1, \dots, X_{n-1})$ for infinitely many n.

However, the answer is positive if the given distribution has an atom, and the atom is the least point of the support. (I do not go into detail.)

[4]Indeed, $\sum_n \mathbb{P}\left(X_n < \varepsilon \right) = \infty$ for every $\varepsilon > 0$, and we may use the second Borel-Cantelli lemma.