Exam of 27.09.2004 — Solutions

1

1a

We may take $U \sim \mathrm{U}(0,1)$ and let $X = 1/\sqrt{U}$, then $1/X^2 = U \sim \mathrm{U}(0,1)$ and $X > 0$.

1b

$X^*(p) = 1/\sqrt{U^*(1-p)} = 1/\sqrt{1-p}$, since the transformation is (strictly) decreasing.
$\mathrm{Me}(X) = X^*(1/2) = \sqrt{2}$.

1c

$\mathbb{E}\,(X) = \int_0^1 X^*(p)\,dp = \int_0^1 \frac{dp}{\sqrt{1-p}} = 2$.

1d

$\mathbb{E}\,(X^2) = \int_0^1 (X^*(p))^2\,dp = \int_0^1 \frac{dp}{1-p} = \infty$, thus X has no finite variance.

2

2a

$S = \frac{1}{2}|AB| \cdot h$ where h is the height of the triangle. Clearly, h is a linear function of the horizontal coordinate of C. The latter is distributed uniformly, therefore h and S are distributed uniformly (given A, B), and the conditional median of S corresponds to the median of the coordinate, thus, to the middle point of the horizontal side of the given triangle. The same holds for the conditional expectation, since the median and the expectation are equal for any uniform distribution.

2b

Let A_0, B_0 and C_0 be the middle points of the corresponding sides (of the given triangle). In order to average $S(A,B,C)$ in B and C, we may first average it in C, getting $S(A,B,C_0)$ (by 2a) and then average the result in B, getting $S(A,B_0,C_0)$.

2c

Averaging also in A we get $\mathbb{E}\,S = S(A_0,B_0,C_0) = \frac{1}{4}$.

2d

The (unconditional) distribution of S is a mixture of uniform distributions. Thus, it has a density (a mixture of uniform densities), and has no atoms. The support evidently is $[0,1]$. The distribution is not uniform, since its expectation (1/4) is not the middle point of the support.

2e

$\mathbb{P}\,\big(\, D \in ABC \,\big|\, A, B, C \,\big) = S(A,B,C)$, therefore $\mathbb{P}\big(D \in ABC\big) = \mathbb{E}\,\mathbb{P}\,\big(\, D \in ABC \,\big|\, A, B, C \,\big) = \mathbb{E}\, S(A,B,C) = 1/4$.

3

3a

Yes, $\frac{X_n}{n} \to 0$ a.s. Proof: $\mathbb{P}\big(|X_n| \ge \varepsilon n\big) \le \frac{\mathbb{E}\,|X_n|^2}{\varepsilon^2 n^2} = \frac{1}{\varepsilon^2 n^2}$; the series of these probabilities converges; the first Borel-Cantelli lemma gives $|X_n| < \varepsilon n$ for all n large enough.

3b

Yes, $X_n - n \to -\infty$ a.s., since $\frac{X_n - n}{n} = \frac{X_n}{n} - 1 \to -1$.

3c

Yes, $X_n - \ln n \to -\infty$ a.s. Proof: $\mathbb{P}\big(X_n - \ln n \ge -C\big) = \mathbb{P}\big(X_n \ge \ln n - C\big) \le \frac{\mathbb{E}\, e^{2X_n}}{e^{2\ln n - 2C}} = \text{const} \cdot \frac{1}{n^2} \mathbb{E}\, e^{2X_1}$. Taking into account that $\mathbb{E}\, e^{2X_1} < \infty$ we get a convergent series of probabilities (and continue similarly to 3a).

3d

No, $X_n - \ln\ln n$ does not tend to $-\infty$. Proof: $\limsup X_n / \sqrt{2 \ln n} = 1$ a.s., therefore $X_n > \sqrt{\ln n} > \ln\ln n$ for infinitely many n.