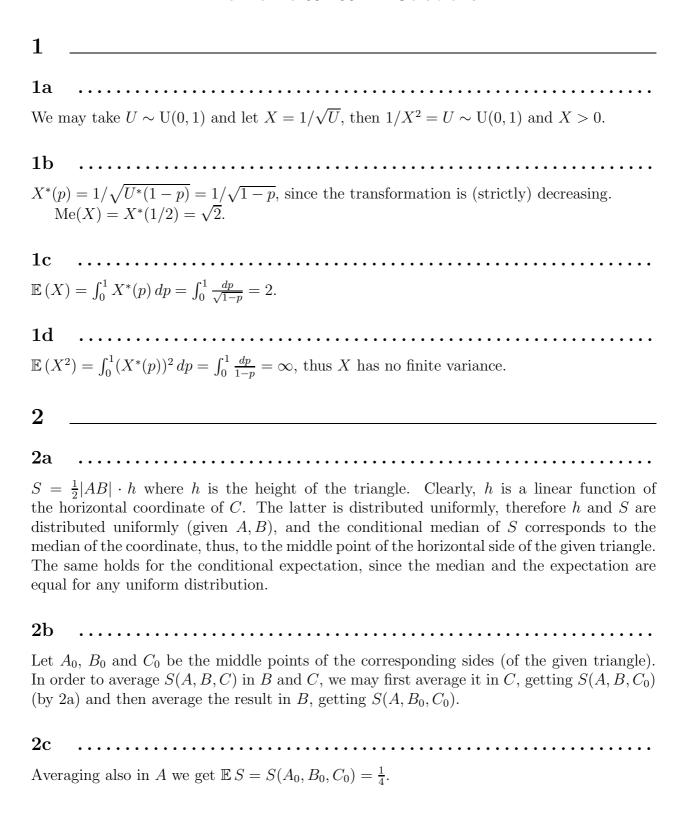
Exam of 27.09.2004 — Solutions



2d
The (unconditional) distribution of S is a mixture of uniform distributions. Thus, it has a density (a mixture of uniform densities), and has no atoms. The support evidently is $[0,1]$ The distribution is not uniform, since its expectation $(1/4)$ is not the middle point of the support.
2 e
$\mathbb{P}\left(D \in ABC \mid A, B, C\right) = S(A, B, C), \text{ therefore } \mathbb{P}\left(D \in ABC\right) = \mathbb{EP}\left(D \in ABC\right)$ $A, B, C = \mathbb{E}S(A, B, C) = 1/4.$
3
3a
Yes, $\frac{X_n}{n} \to 0$ a.s. Proof: $\mathbb{P}\left(X_n \geq \varepsilon n\right) \leq \frac{\mathbb{E} X_n ^2}{\varepsilon^2 n^2} = \frac{1}{\varepsilon^2 n^2}$; the series of these probabilities converges; the first Borel-Cantelli lemma gives $ X_n < \varepsilon n$ for all n large enough.
3b
Yes, $X_n - n \to -\infty$ a.s., since $\frac{X_n - n}{n} = \frac{X_n}{n} - 1 \to -1$.
3c
Yes, $X_n - \ln n \to -\infty$ a.s. Proof: $\mathbb{P}\left(X_n - \ln n \geq -C\right) = \mathbb{P}\left(X_n \geq \ln n - C\right) \leq \frac{\mathbb{E}e^{2X_n}}{e^{2\ln n - 2C}} = \text{const} \cdot \frac{1}{n^2}\mathbb{E}e^{2X_1}$. Taking into account that $\mathbb{E}e^{2X_1} < \infty$ we get a convergent series of probabilities (and continue similarly to 3a).
3d
No, $X_n - \ln \ln n$ does not tend to $-\infty$. Proof: $\limsup X_n / \sqrt{2 \ln n} = 1$ a.s., therefore $X_n > \sqrt{\ln n} > \ln \ln n$ for infinitely many n .