Exam of 04.02.2004 — Solutions

1

1a

$X = \sqrt{a^2 + b^2}$ where a is the horizontal coordinate of A, and b is the vertical coordinate of B. The probability of the event $X \le x$ is the area of the disk $\{(a, b) : a^2 + b^2 \le x^2\}$ intersected with the square $\{(a, b) : 0 \le a \le 1, 0 \le b \le 1\}$. For $x \le 1$ the area is $\frac{1}{4}\pi x^2$. The median $\mathrm{Me}(X) = \sqrt{2/\pi}$, since this number solves the equation $\frac{1}{4}\pi x^2 = \frac{1}{2}$ and belongs to $(0, 1)$.

1b

$F_X(x) = \frac{1}{4}\pi x^2$ for $x \in (0, 1)$, thus, $f_X(x) = (\frac{1}{4}\pi x^2)' = \frac{1}{2}\pi x$ for $x \in (0, 1)$.

1c

$\mathrm{Me}(X^2) = (\mathrm{Me}(X))^2 = 2/\pi$, since the function $x \mapsto x^2$ is (strictly) increasing on $(0, \infty)$. On the other hand, $\mathbb{E}\,(X^2) = \mathbb{E}\,(a^2 + b^2) = \mathbb{E}\,(a^2) + \mathbb{E}\,(b^2)$; $\mathbb{E}\,(a^2) = \int_0^1 a^2\,da = \frac{1}{3}$; $\mathbb{E}\,(b^2) = \frac{1}{3}$; $Ex(X^2) = \frac{2}{3}$.

Yes, $\mathbb{E}\,(X^2) > \mathrm{Me}(X^2)$.

1d

$|AB| \cdot |CD| < \frac{1}{2}(|AB|^2 + |CD|^2)$ a.s.; $\mathbb{E}\,|AB|^2 = \mathbb{E}\,(X^2) = \frac{2}{3}$; $\mathbb{E}\,|CD|^2 = \frac{2}{3}$; therefore $\mathbb{E}\,(|AB| \cdot |CD|) < \frac{2}{3}$.

2

2a

Yes, (*) implies (**). Proof: for every $\varepsilon > 0$,

$$\mathbb{P}\left(|X_1 \dots X_n| \ge \varepsilon \right) \le \frac{\mathbb{E}\,|X_1 \dots X_n|}{\varepsilon} = \frac{1}{\varepsilon}(\mathbb{E}\,|X_1|)^n\,,$$

therefore $\sum_n \mathbb{P}\left(|X_1 \dots X_n| \ge \varepsilon \right) < \infty$. By the first Borel-Cantelli lemma, $|X_1 \dots X_n| < \varepsilon$ for n large enough.

2b

No, (**) does not imply (*). A counterexample: $\mathbb{P}\left(X_1 = 0 \right) = \frac{1}{2}$, $\mathbb{P}\left(X_1 = 10 \right) = \frac{1}{2}$. Then $\mathbb{E}\,|X_1| = 5 > 1$, however, $X_1 \dots X_n \to 0$ a.s., since the event ‘all X_n are non-zero’ is of probability 0.

3

3a

The two triples (X_1, X_2, X_3) and (X_2, X_1, X_3) are identically distributed, therefore $\mathbb{P}\big(X_1 < X_2 < X_3 < x\big) = \mathbb{P}\big(X_2 < X_1 < X_3 < x\big)$. The same holds for all the 6 permutations. The sum of these 6 equal probabilities is equal to $\mathbb{P}\big(X_1 < x, X_2 < x, X_3 < x\big) = \mathbb{P}\big(X_1 < x\big) \cdot \mathbb{P}\big(X_2 < x\big) \cdot \mathbb{P}\big(X_3 < x\big) = x^3$, therefore each summand is equal to $x^3/6$.

3b

Y and A are independent, since $\mathbb{P}\big(A, Y \le y\big) = \mathbb{P}\big(X_1 < X_2 < \dots < X_{10} \le y\big) = y^{10}/10!$ (similarly to 3a), but also $\mathbb{P}\big(A\big)\mathbb{P}\big(Y \le y\big) = (1/10!)y^{10}$.

3c

$\mathbb{P}\big(N = 3 \bigm| X_3\big) = \mathbb{P}\big(X_1 < X_2 < X_3, X_3 \ge X_4 \bigm| X_3\big) = \mathbb{P}\big(X_1 < X_2 < X_3 \bigm| X_3\big)\ \mathbb{P}\big(X_4 \le X_3 \bigm| X_3\big) = \frac{1}{2}X_3^2 \cdot X_3 = \frac{1}{2}X_3^3$, therefore $\mathbb{P}\big(N = 3\big) = \mathbb{E}\,\mathbb{P}\big(N = 3 \bigm| X_3\big) = \mathbb{E}\,\frac{1}{2}X_3^3 = \frac{1}{2}\int_0^1 x^3\,dx = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$.

3d

$$f_{Y|N=3}(y) = f_{X_3|N=3}(y) = \frac{\mathbb{P}\big(N = 3 \bigm| X_3 = y\big)\ f_{X_3}(y)}{\mathbb{P}\big(N = 3\big)} = \frac{\frac{1}{2}y^3 \cdot 1}{\frac{1}{8}} = 4y^3$$

for $y \in (0, 1)$.

3e

As before, $\mathbb{P}\big(N = n \bigm| X_n\big) = \mathbb{P}\big(X_1 < \dots < X_{n-1} < X_n, X_n \ge X_{n+1} \bigm| X_n\big) = \mathbb{P}\big(X_1 < \dots < X_{n-1} < X_n \bigm| X_n\big)\ \mathbb{P}\big(X_{n+1} \le X_n \bigm| X_n\big) = \frac{1}{(n-1)!}X_n^{n-1} \cdot X_n = \frac{1}{(n-1)!}X_n^n$; $\mathbb{P}\big(N = n\big) = \mathbb{E}\,\mathbb{P}\big(N = n \bigm| X_n\big) = \mathbb{E}\,\frac{1}{(n-1)!}X_n^n = \frac{1}{(n-1)!}\int_0^1 x^n\,dx = \frac{1}{(n-1)!(n+1)}$;

$$f_{Y|N=n}(y) = f_{X_n|N=n}(y) = \frac{\mathbb{P}\big(N = n \bigm| X_n = y\big)\ f_{X_n}(y)}{\mathbb{P}\big(N = n\big)} = \frac{\frac{1}{(n-1)!}y^n \cdot 1}{\frac{1}{(n-1)!(n+1)}} = (n+1)y^n$$

for $y \in (0, 1)$.

3f

First, $f_Y(y) = \sum_{n=1}^\infty f_{Y|N=n}(y) \cdot \mathbb{P}\big(N = n\big) = \sum_{n=1}^\infty (n+1)y^n \cdot \frac{1}{(n-1)!(n+1)} = y\sum_{n=1}^\infty \frac{y^{n-1}}{(n-1)!} = ye^y$.

Second,

$$p_{N|Y=y}(n) = \frac{f_{Y|N=n}(y) \cdot \mathbb{P}\big(N = n\big)}{f_Y(y)} = \frac{(n+1)y^n \cdot \frac{1}{(n-1)!(n+1)}}{ye^y} = \frac{y^{n-1}}{(n-1)!}e^{-y}\,.$$

Third, $\mathbb{P}\left(N-1=k \;\middle|\; Y=y \right) = \mathbb{P}\left(N=k+1 \;\middle|\; Y=y \right) = \frac{y^k}{k!}e^{-y}$, which is the Poisson distribution with parameter y.

No, $\mathbb{P}\left(N=n \;\middle|\; Y=y \right) \neq \mathbb{P}\left(N=n \;\middle|\; X_n=y \right)$ in general, since $\frac{y^{n-1}}{(n-1)!}e^{-y} \neq \frac{1}{(n-1)!}y^n$.