Exam of 22.11.1999 — Solutions

1

1a

The random variable X is a function of another random variable, the time of the bird's arrival. The function is equal to 15 during 10 sec ($10 = 25 - 15$); then the function decreases linearly to 0 during 15 sec; then it remains 0 during 20 ($= (5+25+5) - 15$) sec; then it increases linearly to 15 during 15 sec; that is the whole period (60 sec). Thus, $X = 15$ with probability $10/60 = 1/6$, and $X = 0$ with probability $20/60 = 1/3$. The rest of the probability ($1/2 = 1 - 1/6 - 1/3 = (15+15)/60$) is spread uniformly on $(0, 15)$. So,

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{1}{3} + \frac{x}{30} & \text{for } 0 \le x < 15, \\ 1 & \text{for } x \ge 15; \end{cases} \qquad X^*(p) = \begin{cases} 0 & \text{for } 0 < p < 1/3, \\ 30\bigl(p - \frac{1}{3}\bigr) & \text{for } 1/3 < p < 5/6, \\ 15 & \text{for } 5/6 < p < 1. \end{cases}$$

1b

$x_{1/2} = X^*(1/2) = 30\bigl(\frac{1}{2} - \frac{1}{3}\bigr) = 5$;
$x_{1/4} = X^*(1/4) = 0$;
$x_{3/4} = X^*(3/4) = 30\bigl(\frac{3}{4} - \frac{1}{3}\bigr) = 12.5$;
the quantiles are unique, since X^* is continuous at these points.

1c

The support: $[0, 15]$. Discrete, singular and absolutely continuous components:
$p_{\mathrm{d}} = 1/2$, $p_{\mathrm{s}} = 0$, $\quad p_{\mathrm{ac}} = 1/2$;

$$F_{X,\mathrm{d}}(x) = \begin{cases} 0 & \text{for } x < 0, \\ 2/3 & \text{for } 0 \le x < 15, \\ 1 & \text{for } x \ge 15; \end{cases} \qquad F_{X,\mathrm{ac}}(x) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{x}{15} & \text{for } 0 \le x \le 15, \\ 1 & \text{for } x \ge 15; \end{cases}$$

$f_{X,\mathrm{ac}}(x) = 1/15$ for $0 < x < 15$ (and 0 otherwise).

1d

Any two of the following 3 ways (though, integration in time can be used, too).

First, $\mathbb{E}(X) = \int_0^1 X^*(p)\,dp = 0 \cdot \frac{1}{3} + 30 \int_{1/3}^{5/6} (p - \frac{1}{3})\,dp + 15 \cdot \frac{1}{6} = 0 + 30 \cdot \frac{1}{8} + 15 \cdot \frac{1}{6} = 3.75 + 2.5 = 6.25$.

Second, $\mathbb{E}(X) = \int_0^\infty \bigl(1 - F_X(x)\bigr)\,dx = \int_0^{15} \bigl(1 - \frac{1}{3} - \frac{x}{30}\bigr)\,dx = \frac{2}{3} \cdot 15 - \frac{1}{30} \cdot \frac{x^2}{2}\Big|_0^{15} = 10 - 3.75 = 6.25$.

Third, $\mathbb{E}(X) = \int x\,dF_X(x) = p_{\mathrm{d}} \int x\,dF_{X,\mathrm{d}}(x) + p_{\mathrm{ac}} \int x\,dF_{X,\mathrm{ac}}(x) = \frac{1}{2} \cdot \bigl(0 \cdot \frac{2}{3} + 15 \cdot \frac{1}{3}\bigr) + \frac{1}{2} \cdot \frac{15}{2} = 2.5 + 3.75 = 6.25$.

1e

No, $\frac{1}{X-5}$ is non-integrable, since $\int \left|\frac{1}{x-5}\right| dF_X(x) = \infty$. Indeed, the discrete part contributes a finite number, but the absolutely continuous part contributes $\frac{1}{2}\int_0^{15} \frac{1}{|x-5|} \cdot \frac{1}{15}\,dx = \frac{1}{30}\int_0^{15} \frac{dx}{|x-5|} = \infty$.

3

3a

Like 1a, consider the pair (X, Z) as a function of time. It is equal to $(15, 0)$ during 10 sec; then it moves linearly to $(10, 0)$ during 5 sec; then it moves linearly to $(0, 10)$ during 10 sec; then it moves linearly to $(0, 15)$ during 5 sec; then it stays at $(0, 15)$ during 10 sec. The second part is symmetric: to $(0, 10)$ during 5 sec, to $(10, 0)$ during 10 sec, and to $(15, 0)$ during 5 sec. That is the whole period (60 sec). Thus, the two-dimensional distribution has two atoms, at $(15, 0)$ and $(0, 15)$, each of probability $10/60 = 1/6$. The rest of probability is distributed continuously on straight segments. So, $p_{\mathrm{d}} = 1/3$, $p_{\mathrm{s}} = 2/3$, $p_{\mathrm{ac}} = 0$. The support is a polygonal line $(0, 15)$ – $(0, 10)$ – $(10, 0)$ – $(15, 0)$.

3b

$\mathbb{E}X = 6.25$ (see 1d); $\mathbb{E}Z = 6.25$ by symmetry. There are several ways of calculating $\mathbb{E}(XZ)$; I do it by representing both X and Z as functions of T, the random time of the bird's arrival; $\mathbb{E}(XZ) = \frac{1}{60}\int_0^{60} X(t)Z(t)\,dt = \frac{1}{60}\bigl(\int_0^{10} + \int_{10}^{15} + \int_{15}^{25} + \int_{25}^{30} + \int_{30}^{40} + \int_{40}^{45} + \int_{45}^{55} + \int_{55}^{60}\bigr) \overset{\text{by symmatry}}{=} \frac{1}{30}\bigl(\int_0^{10} + \int_{10}^{15} + \int_{15}^{25} + \int_{25}^{30}\bigr) \overset{\text{drop terms with } XZ=0}{=} \frac{1}{30}\int_{15}^{25} \dots = \frac{1}{30}\int_0^{10} (10 - t)t\,dt = \frac{1}{30}\bigl(10 \cdot \frac{t^2}{2} - \frac{t^3}{3}\bigr)\Big|_0^{10} = \frac{1}{30}(500 - \frac{1000}{3}) = \frac{50}{9}$. So, $\mathrm{Cov}(X, Z) = \frac{50}{9} - (\frac{25}{4})^2 < 0$. Random variables X, Z are correlated, therefore dependent.

3c

According to (3a), Z is a function of X, except for $X = 0$. For every $x \in (0, 15)$ the conditional distribution of Z is concentrated at a single point, $10 - x$ for $x \in (0, 10)$, but 0 for $x \in (10, 15)$. However, for $x = 0$ the conditional distribution of Z contains an atom at 15 of probability $10/20 = 1/2$; the rest of conditional probability $1/2 = 1 - 1/2 = (5 + 5)/20$

is distributed uniformly on $(10,15)$. So,

$$F_{Z|X=x}(z) = \begin{cases} 0 & \text{for } z < 0, \\ 1 & \text{for } z \ge 0, \end{cases} \qquad \text{when } x \in (10,15)\,;$$

$$F_{Z|X=x}(z) = \begin{cases} 0 & \text{for } z < 10-x, \\ 1 & \text{for } z \ge 10-x, \end{cases} \qquad \text{when } x \in (0,10)\,;$$

$$F_{Z|X=0}(z) = \begin{cases} 0 & \text{for } z \le 10, \\ \dfrac{z-10}{10} & \text{for } 10 \le z < 15, \\ 1 & \text{for } z \ge 15. \end{cases}$$

4

4a

We know that $\frac12(X_1^2+\dots+X_{2n}^2)$ is distributed Gamma(n), that is, has the density $\frac{1}{(n-1)!}x^{n-1}e^{-x}$. Therefore

$$\mathbb{P}\left(\sqrt{X_1^2+\dots+X_{2n}^2} \le r\right) = \mathbb{P}\left(\tfrac12(X_1^2+\dots+X_{2n}^2) \le r^2/2\right) =$$
$$= \frac{1}{(n-1)!}\int_0^{r^2/2} u^{n-1}e^{-u}\,du \le \frac{1}{(n-1)!}\int_0^{r^2/2} u^{n-1}\,du =$$
$$= \frac{1}{(n-1)!}\frac{u^n}{n}\bigg|_0^{r^2/2} = \frac{u^n}{n!}\bigg|_0^{r^2/2} = \frac{1}{n!}(r^2/2)^n\,.$$

4b

$$\mathbb{P}\left(\sqrt{X_1^2+\dots+X_{2n}^2} \le r\right) = \frac{1}{(n-1)!}\int_0^{r^2/2} u^{n-1}e^{-u}\,du \ge$$
$$\ge e^{-r^2/2}\frac{1}{(n-1)!}\int_0^{r^2/2} u^{n-1}\,du = e^{-r^2/2}\frac{1}{n!}(r^2/2)^n\,;$$

so, $e^{-r^2/2} \le n!\left(\frac{2}{r^2}\right)^n\mathbb{P}\left(\sqrt{\dots} \le r\right) \le 1$, therefore it converges to 1 for $r \to 0$.

4c

$\mathbb{P}\left(\sqrt{\dots} \le r\right)$ is the integral of the $2n$-dimensional density over the $2n$-dimensional ball. For small r the density is close to its value at the origin, $(1/\sqrt{2\pi})^{2n} = (2\pi)^{-n}$. Thus,

$$\lim_{r\to0}\frac{1}{V_{2n,r}}\mathbb{P}\left(\sqrt{\dots} \le r\right) = (2\pi)^{-n}\,,$$

where $V_{2n,r}$ is the volume of the $2n$-dimensional ball of radius r; clearly, $V_{2n,r} = r^{2n} V_{2n,1}$. Combining it with (4b) we get $\dfrac{1}{r^{2n} V_{2n,1}} \dfrac{1}{n!} \left(\dfrac{r^2}{2} \right)^n \to (2\pi)^{-n}$, which means that

$$V_{2n,1} = \frac{(2\pi)^n}{2^n n!} = \frac{\pi^n}{n!} \, ; \qquad V_{2n,r} = \frac{\pi^n}{n!} r^{2n} \, .$$

In particular, $V_{2,r} = \pi r^2$, as it should be.

5

For the standard two-dimensional normal distribution, $\mathbb{P} \left(|Y| \le \varepsilon |X| \right)$ in polar coordinates $X = R \cos \Phi$, $Y = R \sin \Phi$ becomes simply $|\sin \Phi| \le \varepsilon |\cos \Phi|$, that is, $|\tan \Phi| \le \varepsilon$, which holds with probability $\frac{2}{\pi} \arctan \varepsilon$, since $\Phi \sim \mathrm{U}(0, 2\pi)$.

5a

No. $\sum_n \mathbb{P} \left(n|X_{2n+1}| \le |X_{2n}| \right) = \infty$, since $\mathbb{P} \left(|X_{2n+1}| \le \frac{1}{n} |X_{2n}| \right) = \frac{2}{\pi} \arctan \frac{1}{n} \sim \text{const} \cdot \frac{1}{n}$; by the second Borel-Cantelli lemma, $n|X_{2n+1}| \le |X_{2n}|$ for infinitely many n, almost surely.

5b

Yes. $\sum_n \mathbb{P} \left(n^2 |X_{2n+1}| \le |X_{2n}| \right) < \infty$, since $\mathbb{P} \left(|X_{2n+1}| \le \frac{1}{n^2} |X_{2n}| \right) \sim \text{const} \cdot \frac{1}{n^2}$. By the first Borel-Cantelli lemma, $n^2 |X_{2n+1}| \le |X_{2n}|$ only for finitely many n, almost surely.

5c

No. In contrast to 5a, we cannot apply the second Borel-Cantelli lemma, since events $n|X_{n+1}| > |X_n|$ are dependent. However, the negative answer to 5a implies the negative answer to 5c. The inequality violated infinitely often for $n = 2, 4, 6, \ldots$ cannot hold for all $n > N$.

5d

Yes, similarly to 5b. Events are dependent, but it is OK for the *first* Borel-Cantelli lemma.