Exam of 31.08.1999 — Solutions

1

1a

The point (U, V) is distributed uniformly on the square $(10, 20) \times (15, 25)$ (in minutes; we omit 8 hours). The random variable $U - V$ is a linear function on the square, equal to $+5$ at one vertex, -5 on a diagonal, and -15 at the opposite vertex. Calculating needed areas (of triangles) we get

$$\mathbb{P}\big(U - V \ge a \big) = \frac{(5-a)^2}{2 \cdot 100} \quad \text{for } -5 \le a \le 5\,,$$
$$\mathbb{P}\big(U - V \le a \big) = \frac{(a+15)^2}{2 \cdot 100} \quad \text{for } -15 \le a \le -5\,.$$

X is a function of $U - V$, namely,

$$X = \begin{cases} U - V & \text{for } U - V \ge 0, \\ 0 & \text{for } U - V \le 0. \end{cases}$$

Therefore

$$F_X(0) = \mathbb{P}\big(X = 0 \big) = \mathbb{P}\big(U - V \le 0 \big) = \frac{7}{8}\,;$$
$$F_X(x) = \mathbb{P}\big(X \le x \big) = \mathbb{P}\big(U - V \le x \big) = 1 - \frac{(5-x)^2}{200} \quad \text{for } 0 \le x \le 5\,;$$

$F_X(x) = 0$ for $x < 0$, $F_X(x) = 1$ for $x \ge 5$.

X^* is inverse to F_X; for $p \in (7/8, 1)$ we have $1 - (5-x)^2/200 = p$; $x = 5 - \sqrt{200(1-p)}$. So,

$$X^*(p) = \begin{cases} 0 & \text{for } 0 < p \le 7/8; \\ 5 - \sqrt{200(1-p)} & \text{for } 7/8 \le p < 1; \end{cases}$$
$$x_{1/2} = X^*\Big(\frac{1}{2}\Big) = 0\,.$$

1b

Also Y is a function of $U - V$, namely,

$$Y = \begin{cases} V - U & \text{for } U - V \le 0, \\ 0 & \text{for } U - V \ge 0. \end{cases}$$

Therefore

$$F_Y(0) = \mathbb{P}\big(Y = 0 \big) = \mathbb{P}\big(U - V \ge 0 \big) = \frac{1}{8}\,;$$

$$F_Y(y) = \mathbb{P}\big(Y \le y \big) = \mathbb{P}\big(U - V \ge -y \big) = \begin{cases} \frac{(5+y)^2}{200} & \text{for } 0 \le y \le 5, \\ 1 - \frac{(15-y)^2}{200} & \text{for } 5 \le y \le 15; \end{cases}$$

$F_Y(y) = 0$ for $y < 0$, $F_Y(y) = 1$ for $y \ge 15$.

$$Y^*(p) = \begin{cases} 0 & \text{for } 0 < p \le 1/8; \\ \sqrt{200p} - 5 & \text{for } 1/8 \le p \le 1/2; \\ 15 - \sqrt{200(1-p)} & \text{for } 1/2 \le p < 1; \end{cases}$$

$$y_{1/2} = Y^*\Big(\frac{1}{2}\Big) = 5\,.$$

1c ....

$$F_X = \frac{7}{8} F_{X,\mathrm{d}} + \frac{1}{8} F_{X,\mathrm{ac}}\,;$$

$$F_{X,\mathrm{d}}(x) = \begin{cases} 0 & \text{for } x < 0, \\ 1 & \text{for } x \ge 0; \end{cases}$$

$$F_{X,\mathrm{ac}}(x) = 8\big(F_X(x) - \tfrac{7}{8} F_{X,\mathrm{d}}(x)\big) = \begin{cases} 0 & \text{for } x \le 0, \\ \frac{25-(5-x)^2}{25} & \text{for } 0 \le x \le 5, \\ 1 & \text{for } x \ge 5; \end{cases}$$

$$f_{X,\mathrm{ac}}(x) = F'_{X,\mathrm{ac}}(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{2(5-x)}{25} & \text{for } 0 < x < 5, \\ 0 & \text{for } x > 5. \end{cases}$$

1d

$$F_Y = \frac{1}{8} F_{Y,\mathrm{d}} + \frac{7}{8} F_{Y,\mathrm{ac}}\,;$$

$$F_{Y,\mathrm{d}}(y) = \begin{cases} 0 & \text{for } y < 0, \\ 1 & \text{for } y \ge 0; \end{cases}$$

$$F_{Y,\mathrm{ac}}(y) = \tfrac{8}{7}\big(F_Y(y) - \tfrac{1}{8}F_{Y,\mathrm{d}}(y)\big) = \begin{cases} 0 & \text{for } y \le 0, \\ \frac{(5+y)^2-25}{175} & \text{for } 0 \le y \le 5, \\ \frac{175-(15-y)^2}{175} & \text{for } 5 \le y \le 15, \\ 1 & \text{for } y \ge 15; \end{cases}$$

$$f_{Y,\mathrm{ac}}(y) = F'_{Y,\mathrm{ac}}(y) = \begin{cases} 0 & \text{for } y < 0, \\ \frac{2(5+y)}{175} & \text{for } 0 < y < 5, \\ \frac{2(15-y)}{175} & \text{for } 5 < y < 15, \\ 0 & \text{for } y > 15. \end{cases}$$

1e

First, $\mathbb{E}\,X = \int_0^1 X^*(p)\,dp = \int_{7/8}^1 \big(5 - \sqrt{200(1-p)}\big)\,dp = \big(5p + \sqrt{200}\cdot\frac{2}{3}(1-p)^{3/2}\big)\big|_{7/8}^1 = 5 + 0 - 5\cdot\frac{7}{8} - \sqrt{200}\cdot\frac{2}{3}\frac{1}{8\sqrt{8}} = 5\cdot\frac{1}{8} - \sqrt{25}\cdot\frac{2}{3}\cdot\frac{1}{8} = \frac{5}{8}\big(1-\frac{2}{3}\big) = \frac{5}{24}$.

Second, $\mathbb{E}\,X = \int x\,dF_X(x) = \frac{7}{8}\int x\,dF_{X,\mathrm{d}}(x) + \frac{1}{8}\int x\,dF_{X,\mathrm{ac}}(x) = 0 + \frac{1}{8}\int x f_{X,\mathrm{ac}}(x)\,dx = \frac{1}{8}\int_0^5 x\frac{2(5-x)}{25}\,dx = \frac{1}{8}\cdot\frac{2}{25}\int_0^5 (5x - x^2)\,dx = \frac{1}{100}\big(\frac{5x^2}{2} - \frac{x^3}{3}\big)\big|_0^5 = \frac{1}{100}\big(\frac{5^3}{2} - \frac{5^3}{3}\big) = \frac{5^3}{100}\cdot\frac{1}{6} = \frac{5}{4}\cdot\frac{1}{6} = \frac{5}{24}$.

1f

$Y - X = V - U$, since for $U - V \le 0$ we have $Y - X = (V-U) - 0 = V - U$, and for $U - V \ge 0$ we have $Y - X = 0 - (U - V) = V - U$. Therefore $\mathbb{E}\,Y - \mathbb{E}\,X = \mathbb{E}\,V - \mathbb{E}\,U = \frac{15+25}{2} - \frac{10+20}{2} = 5$, and so, $\mathbb{E}\,Y = 5 + \mathbb{E}\,X = 5 + \frac{5}{24}$.

2

2a

We know that X and Y are functions of $U - V$ (see 1a, 1b):

$$(X,Y) = \begin{cases} (U-V, 0) & \text{if } U - V \ge 0, \\ (0, V-U) & \text{if } U - V \le 0; \end{cases}$$

thus, the point (X,Y) always belongs to the union of the two axis. Therefore the 2-dim distribution has no absolutely continuous part. The random variable $U - V$ is nonatomic, and different values of $U - V$ lead to different (2-dim) values of (X,Y); therefore the 2-dim distribution has no atoms. It is singular.

2b

Yes, X and Y are dependent. Indeed,

$$0 = \mathbb{P}\big(X > 0, Y > 0 \big) \ne \underbrace{\mathbb{P}\big(X > 0 \big)}_{\ne 0} \underbrace{\mathbb{P}\big(Y > 0 \big)}_{\ne 0} .$$

2c

If $X = x > 0$ then necessarily $Y = 0$; thus,

$$F_{Y|X=x}(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y \ge 0 \end{cases}$$

for $x > 0$.

The condition $X = 0$ is equivalent to the condition $Y > 0$, thus $F_{Y|X=0}$ describes the conditional distribution of Y given that $Y > 0$, that is, Y does not belong to its set of atoms. We get just the nonatomic (absolutely continuous) part of the distribution of Y calculated in 1d. So,

$$F_{Y|X=0}(y) = F_{Y,\text{ac}}(y) = \begin{cases} 0 & \text{for } y \le 0, \\ \frac{(5+y)^2 - 25}{175} & \text{for } 0 \le y \le 5, \\ \frac{175 - (15-y)^2}{175} & \text{for } 5 \le y \le 15, \\ 1 & \text{for } y \ge 15. \end{cases}$$

3

3a

Yes, the distribution of X determines uniquely the distribution of Y. Indeed, the distribution of X determines X^* (except for values at jumps), therefore, Y^*, therefore, the distribution of Y.

No, the distribution of Y does not determine uniquely the distribution of X. Indeed, two increasing functions can be equal on $(\frac{1}{3}, \frac{2}{3})$ but quite different on $(0, \frac{1}{3})$ or/and $(\frac{2}{3}, 1)$.

3b

We know that for any a, the set $\{p \in (0,1) : X^*(p) \le a\}$ is either $\big(0, F_X(a)\big)$ or $(0, F_X(a)]$. The distinction is of no importance for the argument; I'll use the first possibility; the second one gives the same.

If a is such that $\frac{1}{3} < F_X(a) < \frac{2}{3}$ then $\{p \in (0,1) : Y^*(p) \le a\} = \{p \in (0,1) : \frac{1+p}{3} \in (0, F_X(a))\} = (0, 3F_X(a) - 1)$, thus, $F_Y(a) = 3F_X(a) - 1$. Otherwise, if $F_X(a) \le \frac{1}{3}$, we have $\{p \in (0,1) : Y^*(p) \le a\} = \{p \in (0,1) : \frac{1+p}{3} \in (0, F_X(a))\} = \emptyset$, thus, $F_Y(a) = 0$. Similarly, if

$F_X(a) \geq \frac{2}{3}$ then $F_Y(a) = 1$. So,

$$F_Y(y) = \begin{cases} 0, & \text{if } F_X(y) \leq \frac{1}{3}, \\ 3F_X(y) - 1, & \text{if } \frac{1}{3} \leq F_X(y) \leq \frac{1}{3}, \\ 1, & \text{if } F_X(y) \geq \frac{2}{3}. \end{cases}$$

3c . . . .

No, not every distribution is possible for Y, since not every increasing function on $(\frac{1}{3}, \frac{2}{3})$ can be extended to an increasing function on $(0, 1)$. Only a bounded function can be extended (indeed, say, $X^*(\frac{1}{6})$ is a lower bound for the whole Y^*). And every bounded function can be extended (say, as a constant on $(0, \frac{1}{3})$ and another constant on $(\frac{2}{3}, 1)$). So, a necessary and sufficient condition is, that Y^* is bounded; that is, $Y^*(0+) > -\infty$ and $Y^*(1-) < +\infty$; that is, Y has a bounded support.

3d . . . .

No. For example, let X^* be 0 on $(0, \frac{1}{3})$ and 1 on $(\frac{1}{3}, 1)$, then $\mathbb{E}\, X = \frac{2}{3}$ but $\mathbb{E}\, Y = 1$.

3e . . . .

Yes. Indeed, $\mathbb{E}\,|Y| = \int_0^1 |Y^*(p)|\, dp = \int_0^1 |X^*(\frac{1+p}{3})|\, dp = 3 \int_{1/3}^{2/3} |X^*(p)|\, dp \leq 3 \int_0^1 |X^*(p)|\, dp = 3\mathbb{E}\,|X| \leq 10\mathbb{E}\,|X|$.

4

4a . . . .

The joint distribution of X, Y is invariant under rotations. Vectors $\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$, $\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ are orthogonal unit vectors. Therefore the joint distribution of $\frac{3X-Y}{\sqrt{10}}$ and $\frac{X+3Y}{\sqrt{10}}$ is the same as the joint distribution of (X, Y). So,

$$\mathbb{E}\left(\cos(3X - Y)\sin(X + 3Y)\right) = \mathbb{E}\left(\cos(\sqrt{10}X)\sin(\sqrt{10}Y)\right) = \\ = \mathbb{E}\left(\cos(\sqrt{10}X)\right)\mathbb{E}\left(\sin(\sqrt{10}Y)\right) = 0\,,$$

since $\sin(\sqrt{10}Y)$ is symmetric w.r.t. 0.

5

5a . . . .

Denote by A_n the event $\max(X_{n^2+1}, X_{n^2+2}, \dots, X_{n^2+2n}) \leq 3$. We have $\mathbb{P}\left(A_n\right) = \mathbb{P}\left(X_{n^2+1} \leq 3, \dots, X_{n^2+2n} \leq 3\right) = \mathbb{P}\left(X_{n^2+1} \leq 3\right) \dots \mathbb{P}\left(X_{n^2+2n} \leq 3\right) = p^{2n}$, where $p = \mathbb{P}\left(X_1 \leq 3\right)$.

Thus,

$$\sum_n \mathbb{P}\left(A_n \right) < \infty \,.$$

By the first Borel-Cantelli lemma, only a finite number of events A_n occur (almost surely). It means existence of a (random) N such that $\forall n > N \quad \max(X_{n^2+1}, X_{n^2+2}, \dots, X_{n^2+2n}) > 3$.

5b ....

No, $n^2 + 2n$ cannot be replaced with $n^2 + 20$. Indeed, denote by B_n the event $\max(X_{n^2+1}, X_{n^2+2}, \dots, X_{n^2+20}) \le 3$. We have $\mathbb{P}\left(B_n \right) = p^{20}$ (where $p = \mathbb{P}\left(X_1 \le 3 \right)$, still). Thus,

$$\sum_n \mathbb{P}\left(B_n \right) = \infty \,.$$

Also, B_n are independent for n large enough (since $n^2 + 20 < (n+1)^2 + 1$). By the second Borel-Cantelli lemma, an infinite number of events B_n occur (almost surely). It means nonexistence of a (random) N such that $\forall n > N \quad \max(X_{n^2+1}, X_{n^2+2}, \dots, X_{n^2+20}) > 3$.