5 Borel sets, functions and measures

Two cases are studied in parallel in this section, the two-dimensional case (new) and the one-dimensional case (treated before in a simplistic manner). Full generality will be achieved by using some theorems of *measure theory* (formulated here without proofs).

5a Intervals and elementary sets

Some definitions.

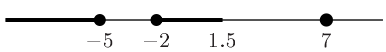
A *one-dimensional interval* is, by definition, a subset of $\mathbb{R}$ of one of the following forms:[1]

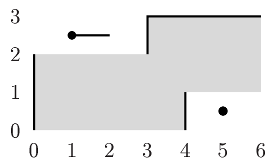
$$(a,b); \quad [a,b]; \quad [a,b); \quad (a,b]; \quad (-\infty,b); \quad (-\infty,b]; \quad [a,+\infty); \quad (a,+\infty);$$
$$(-\infty,+\infty)=\mathbb{R}; \quad \emptyset \text{ (the empty set)}.$$

A singleton $\{a\}$ is an interval, since $\{a\}=[a,a]$.

A *two-dimensional interval* is a (cartesian) product of two one-dimensional intervals. For example: $[-1,+2]\times(0,\frac{1}{2})$, or $(-\infty,0]\times\{\pi\}$.

A one-dimensional *elementary set* is a union of a finite number of one-dimensional intervals. Similarly, a two-dimensional elementary set is a union of a finite number of two-dimensional intervals. Examples: $(-\infty,-5]\cup[-2,1.5)\cup\{7\}$ (one-dim); $[0,4]\times(0,2)\cup[3,6)\times(1,3]\cup[1,2)\times\{2.5\}\cup\{5\}\times\{0.5\}$ (two-dim).





Every elementary set can be represented as a union of a finite number of *disjoint* intervals (which is trivial in dimension 1, and a bit more complicated in dimension 2).

Denote the class of all 1-dimensional elementary sets by $\mathcal{E}$ or $\mathcal{E}_1$, and the class of all 2-dimensional elementary sets by $\mathcal{E}_2$. Both are algebras (fields) of sets (recall 1a1), that is,

$$\begin{array}{lll}
& \emptyset,\ \mathbb{R}\in\mathcal{E}_1\,; & \emptyset,\ \mathbb{R}^2\in\mathcal{E}_2\,;\\
\text{(5a1)} & E\in\mathcal{E}_1 \implies \mathbb{R}\setminus E\in\mathcal{E}_1\,; & E\in\mathcal{E}_2 \implies \mathbb{R}^2\setminus E\in\mathcal{E}_2\,;\\
& E,F\in\mathcal{E}_1 \implies E\cap F, E\cup F\in\mathcal{E}_1\,; & E,F\in\mathcal{E}_2 \implies E\cap F, E\cup F\in\mathcal{E}_2\,.
\end{array}$$

Each 1-dimensional interval has its length, called also its (Lebesgue) *measure:* $\operatorname{mes}\big([a,b]\big)=b-a$; $\operatorname{mes}\big((a,b)\big)=b-a$; $\operatorname{mes}\big((-\infty,b]\big)=\infty$; $\operatorname{mes}\big(\{a\}\big)=0$; etc. Every 1-dimensional elementary set $E\in\mathcal{E}_1$ has its measure $\operatorname{mes}(E)$, equal to the sum of measures of its (disjoint) intervals. Every 2-dimensional elementary set $E\in\mathcal{E}_2$ has its measure $\operatorname{mes}_2(E)$, equal to its area. An elementary set has many partitions into (disjoint) intervals; they all give the

[1]These are exactly the one-dimensional *convex* sets, as well as the one-dimensional *connected* sets, but we do not need it.

same measure (I omit the proof).[2] We have finitely additive measures $\operatorname{mes} : \mathcal{E}_1 \to [0, +\infty]$, $\operatorname{mes}_2 : \mathcal{E}_2 \to [0, +\infty]$.

5b Non-elementary sets: the first step

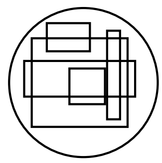
A disk does not belong to $\mathcal{E}_2$, but still, it should have an area!

Define

(5b1)
$$(\mathcal{E}_1)_\sigma = \{ E_1 \cup E_2 \cup \dots : E_1, E_2, \dots \in \mathcal{E}_1 \} \, ,$$
$$(\mathcal{E}_2)_\sigma = \{ E_1 \cup E_2 \cup \dots : E_1, E_2, \dots \in \mathcal{E}_2 \} \, .$$

An open disk belongs to $(\mathcal{E}_2)_\sigma$, which is a special case of the following result. Recall that a set is called *open,* if it contains a neighborhood of each point of the set.

5b2 Lemma. Every 1-dimensional open set belongs to $(\mathcal{E}_1)_\sigma$. Every 2-dimensional open set belongs to $(\mathcal{E}_2)_\sigma$.



Proof. (I prove the 2-dim case; 1-dim case is similar but simpler.) Let $U \subset \mathbb{R}^2$ be an open set. The set of all $(a, b) \times (c, d) \subset U$ with rational a, b, c, d is countable; their union belongs to $(\mathcal{E}_2)_\sigma$. However, the union is equal to U. Indeed, being open, U contains a neighborhood of each point of U; the neighborhood may be chosen as a rectangle $(a, b) \times (c, d)$ with rational a, b, c, d. $\square$

Every $A \in (\mathcal{E}_2)_\sigma$ can be represented as $E_1 \uplus E_2 \uplus \dots$, $E_k \in \mathcal{E}_2$, or alternatively, as the limit of an increasing sequence of elementary sets.

If $E_n, F_n \in \mathcal{E}_2$, $E_n \uparrow A$, $F_n \uparrow B$, and $\lim_{n\to\infty} \operatorname{mes} E_n \ne \lim_{n\to\infty} \operatorname{mes} F_n$, does it mean that $A \ne B$? The question is quite nontrivial, even in dimension 1.[3] Fortunately, the answer is positive, which is proven by measure theory.[4] This is why we may define $\operatorname{mes}_2 A$ for $A \in (\mathcal{E}_2)_\sigma$ as $\lim_{n\to\infty} \operatorname{mes} E_n$, where $(E_1, E_2, \dots)$ is *any* sequence of elementary sets increasing to A.

5b3 Note. Every countable set $A \subset \mathbb{R}$ belongs to $(\mathcal{E}_1)_\sigma$, and $\operatorname{mes}(A) = 0$. Every countable set $A \subset \mathbb{R}^2$ belongs to $(\mathcal{E}_2)_\sigma$, and $\operatorname{mes}_2(A) = 0$.

Countable sets can be dense (recall rational numbers); anyway, they are negligible w.r.t. Lebesgue measure.

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[2]Ancient Greeks used only rational numbers. Let $\mathbb{Q}$ be space of all rational numbers, $E_1, \dots, E_m \subset \mathbb{Q}$ be disjoint intervals, and also $F_1, \dots, F_n \subset \mathbb{Q}$ be disjoint intervals. If $\operatorname{mes}(E_1) + \dots + \operatorname{mes}(E_m) \ne \operatorname{mes}(F_1) + \dots + \operatorname{mes}(F_n)$, then necessarily $E_1 \uplus \dots \uplus E_m \ne F_1 \uplus \dots \uplus F_n$. Perform a thought experiment: replace $\mathbb{Q}$ with some *non-dense* set. The statement becomes wrong!

Elementary measure theory can be built over $\mathbb{Q}$, since rational numbers are dense.

[3]This time, rational numbers do not suffice. It may happen that A and B differ only on irrational numbers, that is, $A \cap \mathbb{Q} = B \cap \mathbb{Q}$.

Ancient Greeks could not build our probability theory over $\mathbb{Q}$. Something is wrong with rational numbers!

[4]The proof uses compactness. An interval $[a, b] \subset \mathbb{R}$ is *compact;* its rational counterpart $[a, b] \cap \mathbb{Q}$ is not compact. That is the failure of rational numbers.

Non-elementary sets appear naturally when dealing with infinite sequences. A number $\omega \in (0,1)$ has its binary digits

$$\omega = \big(0.\alpha_1\alpha_2\dots\big)_2 = \sum_{k=1}^{\infty} \frac{\alpha_k}{2^k}\,, \qquad \alpha_k = \alpha_k(\omega) \in \{0,1\}\,,$$

as well as decimal digits,

$$\omega = \big(0.\beta_1\beta_2\dots\big)_{10} = \sum_{k=1}^{\infty} \frac{\beta_k}{10^k}\,, \qquad \beta_k = \beta_k(\omega) \in \{0,1,2,3,4,5,6,7,8,9\}$$

(and the same for other bases). Probabilistically, binary digits are just an (infinitely long) coin tossing; $\mathbb{P}\,\big(\,\alpha_1 = 0\,\big) = \operatorname{mes}\{\omega \in (0,1) : \alpha_1(\omega) = 0\} = \operatorname{mes}\big((0,\frac12)\big) = 1/2$, $\mathbb{P}\,\big(\,\alpha_2 = 0 \;\big|\; \alpha_1 = 0\,\big) = \dfrac{\operatorname{mes}\big((0,\frac14)\big)}{\operatorname{mes}\big((0,\frac12)\big)} = 1/2$, etc. Probabilistic intuition tells us that $(\alpha_1 + \dots + \alpha_n)/n$ is usually close to $1/2$ for large n. Thus, the event $\alpha_1 + \dots + \alpha_{1000} < 1000/4$ is a large deviation. What about the probability

$$\mathbb{P}\,\big(\,\exists n \ge 1000 \quad \alpha_1 + \dots + \alpha_n < n/4\,\big) = ? \tag{5b4}$$

Before trying to calculate it, we should ask: is it well-defined? (Otherwise, trying to calculate it we'll encounter paradoxes!) For each n, the event[5] $E_n = \{\alpha_1 + \dots + \alpha_n < n/4\}$ is elementary (think, why). Therefore, the event $A = E_{1000} \cup E_{1001} \cup \dots$ belongs to $(\mathcal{E}_1)_\sigma$. (Note that A is not elementary; in fact, it is dense!) We'll return to the point later.

Generally, events of the form

$$\exists n \quad \big(\text{ a property of } \alpha_1,\dots,\alpha_n\,\big) \tag{5b5}$$

belong to $(\mathcal{E}_1)_\sigma$.

5c More complicated sets

The class $(\mathcal{E}_2)_\sigma$ (as well as $(\mathcal{E}_1)_\sigma$) satisfies[6] (recall (5a1))

$$\begin{aligned} &\emptyset,\ \mathbb{R}^2 \in (\mathcal{E}_2)_\sigma\,;\\ &A,B \in (\mathcal{E}_2)_\sigma \quad\Longrightarrow\quad A\cap B \in (\mathcal{E}_2)_\sigma\,;\\ &A_1,A_2,\dots \in (\mathcal{E}_2)_\sigma \quad\Longrightarrow\quad (A_1 \cup A_2 \cup \dots) \in (\mathcal{E}_2)_\sigma\,. \end{aligned} \tag{5c1}$$

However, $A \in (\mathcal{E}_2)_\sigma$ does not imply $\mathbb{R}^2 \setminus A \in (\mathcal{E}_2)_\sigma$. In particular, closed sets[7] in general do not belong to $(\mathcal{E}_2)_\sigma$, though they are complements of open sets, therefore (by 5b2) complements

[5] That is, the set $E_n = \{\omega \in (0,1) : \alpha_1(\omega) + \dots + \alpha_n(\omega) < n/4\}$.

[6] For the intersection, note that $E_n \uparrow A$ and $F_n \uparrow B$ imply $E_n \cap F_n \uparrow A \cap B$. For the union, note that a sequence of sequences can be lined up into a single sequence.

[7] Recall that a set A is called *closed,* if $x_n \in A$, $x_n \to x$ imply $x \in A$. A set is closed if and only if its complement is open.

of $(\mathcal{E}_2)_\sigma$-sets. Say, a closed rectangle belongs to $(\mathcal{E}_2)_\sigma$ if and only if its sides are parallel to coordinate axes.[8] Also, a closed disk does not belong to $(\mathcal{E}_2)_\sigma$.

Similarly to (5b1) we may define

(5c2)
$$\begin{aligned}(\mathcal{E}_1)_\delta &= \{E_1 \cap E_2 \cap \dots : E_1, E_2, \dots \in \mathcal{E}_1\}\,,\\ (\mathcal{E}_2)_\delta &= \{E_1 \cap E_2 \cap \dots : E_1, E_2, \dots \in \mathcal{E}_2\}\,,\end{aligned}$$

then

$$A \in (\mathcal{E}_2)_\sigma \quad \iff \quad \mathbb{R}^2 \setminus A \in (\mathcal{E}_2)_\delta\,;$$

closed sets belong to $(\mathcal{E}_2)_\delta$, but open sets, in general, do not belong. Similarly to (5b5), events of the form

(5c3)
$$\forall n \quad \big(\text{ a property of } \alpha_1, \dots, \alpha_n\,\big)$$

belong to $(\mathcal{E}_1)_\delta$ but, in general, do not belong to $(\mathcal{E}_1)_\sigma$.

If $A \in (\mathcal{E}_2)_\sigma$ and $B \in (\mathcal{E}_2)_\delta$ then $A \cap B$ and $A \cup B$, in general, are neither in $(\mathcal{E}_2)_\sigma$ nor $(\mathcal{E}_2)_\delta$.

Some quite important sets are essentially more complicated than all considered before. For example (recall (5b4)), the set

(5c4)
$$\{\omega \in (0,1) : \exists N\, \forall n \ge N\ \alpha_1 + \dots + \alpha_n \ge n/4\}$$

belongs to the class

(5c5)
$$(\mathcal{E}_1)_{\delta\sigma} = \{A_1 \cup A_2 \cup \dots : A_1, A_2, \dots \in (\mathcal{E}_1)_\delta\}\,,$$

while the set

(5c6)
$$\begin{aligned}&\left\{\omega \in (0,1) : \frac{\alpha_1 + \dots + \alpha_n}{n} \xrightarrow[n\to\infty]{} \frac12 \right\} = \\ &\qquad = \left\{ \forall k\ \exists N\ \forall n \ge N \quad \frac12 - \frac1k \le \frac{\alpha_1 + \dots + \alpha_n}{n} \le \frac12 + \frac1k \right\}\end{aligned}$$

belongs to the class $(\mathcal{E}_1)_{\delta\sigma\delta}$. Note the simple relation between a sequence of quantifiers ($\forall\exists\forall$) and the type of the set ($\delta\sigma\delta$). We have a *tower of classes*

(5c7)
$$\begin{tikzcd}
 & (\mathcal{E}_2)_\sigma \arrow[r, dashed] \arrow[dr] & (\mathcal{E}_2)_{\delta\sigma} \arrow[r, dashed] \arrow[dr] & (\mathcal{E}_2)_{\sigma\delta\sigma} \arrow[r, dashed] \arrow[dr] & \cdots \\
\mathcal{E}_2 \arrow[ur] \arrow[dr] & & & & \\
 & (\mathcal{E}_2)_\delta \arrow[r, dashed] \arrow[uur] & (\mathcal{E}_2)_{\sigma\delta} \arrow[r, dashed] \arrow[uur] & (\mathcal{E}_2)_{\delta\sigma\delta} \arrow[r, dashed] \arrow[uur] & \cdots
\end{tikzcd}$$

the arrows being directed toward larger classes. It is known (and far from being evident) that all these classes differ. The union $(\mathcal{E}_2)_\infty$ of the classes satisfies

(5c8)
$$\begin{aligned}A \in (\mathcal{E}_2)_\infty \quad &\Longrightarrow \quad \mathbb{R}^2 \setminus A \in (\mathcal{E}_2)_\infty\,,\\ A, B \in (\mathcal{E}_2)_\infty \quad &\Longrightarrow \quad A \cap B, A \cup B \in (\mathcal{E}_2)_\infty\,.\end{aligned}$$

[8]If they are not parallel, then an elementary set inside the rectangle contains only finite number of its boundary points. However, the whole boundary is not countable!

Still not a σ-field! If $A_1 \in \mathcal{E}_2$, $A_2 \in (\mathcal{E}_2)_\sigma$, $A_3 \in (\mathcal{E}_2)_{\delta\sigma}$, $A_4 \in (\mathcal{E}_2)_{\sigma\delta\sigma}$, ... then, in general, $A_1 \cup A_2 \cup \dots$ does not belong to $(\mathcal{E}_2)_\infty$. Rather, it belongs to $(\mathcal{E}_2)_{\infty\sigma}$; we start to understand that (5c7) is only a miserable part of a giant tower, too vast and complicated for being exhausted by a sequence.

Fortunately, we can avoid the giant tower. To this end, first of all, we must abandon the hope of a general form of a set. You see,

$$\begin{aligned} &\text{sets of } (\mathcal{E}_2)_\sigma \text{ are of the form } \cup_k E_k \,, \\ &\text{sets of } (\mathcal{E}_2)_{\sigma\delta} \text{ are of the form } \cap_k \cup_l E_{kl} \,, \\ &\text{sets of } (\mathcal{E}_2)_{\sigma\delta\sigma} \text{ are of the form } \cup_k \cap_l \cup_m E_{klm} \,, \end{aligned} \tag{5c9}$$

and so on;[9] however, it does not work on higher levels of the giant tower.

In the next section the whole tower will be treated at once, without dividing it into levels.

5d Borel sets

5d1 Definition. (a) *Borel σ-fields* $\mathcal{B}_1$ (one-dimensional) and $\mathcal{B}_2$ (two-dimensional) are σ-fields generated by $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively.

(b) A *Borel set* is a set belonging to the Borel σ-field.

Recall some similar but simpler definitions. Say, from linear algebra; the linear subspace L *generated by* given vectors $x_1, \dots, x_n$ can be defined in two equivalent ways:

(a) the intersection of all linear subspaces that contain $x_1, \dots, x_n$;

(b) $\{c_1 x_1 + \dots + c_n x_n : c_1, \dots, c_n \in \mathbb{R}\}$.

Item (b) is the general form of a vector of the generated subspace. Unfortunately, such a constructive description is not available for Borel sets. Item (a) involves the (vast and seemingly irrelevant) collection of *all* linear subspaces that contain $x_1, \dots, x_n$, and gives no clear idea, which vectors belong to the generated space. Anyway, the intersection of a family of linear spaces is a linear space, be the family small or vast.

The σ-field *generated* by a class is the intersection of all σ-fields that contain the class. Intersection of a family of σ-fields is a σ-field (check it), be the family small or vast.

Intuitively, a set is a Borel set, if (and only if) it can be constructed from intervals by iterated operations of complement, finite or countable intersection, and finite or countable union.

5d2 Definition. A class $\mathcal{A}$ of sets[10] is called *monotone,* if

$$\begin{aligned} A_1, A_2, \dots \in \mathcal{A},\ A_n \uparrow A \quad &\Longrightarrow \quad A \in \mathcal{A}\,; \\ A_1, A_2, \dots \in \mathcal{A},\ A_n \downarrow A \quad &\Longrightarrow \quad A \in \mathcal{A}\,. \end{aligned}$$

Intervals (both 1-dim and 2–dim) are a monotone class, but not an algebra. Elementary sets are an algebra, but not a monotone class.

5d3 Exercise. Prove that an algebra is monotone if and only if it is a σ-field.

[9] For example, (5c6) is $\cap_k \cup_N \cap_n E_{kNn}$, where $E_{kNn} = \bigl\{ \frac12 - \frac1k \le \frac{\alpha_1 + \dots + \alpha_n}{n} \le \frac12 + \frac1k \bigr\}$.

[10] More exactly, of subsets of a given set.

5d4 Theorem. (*"Monotone class theorem"*) If $\mathcal{A}$ is an algebra, then the monotone class generated by $\mathcal{A}$ is also an algebra.

(I give no proof.)

5d5 Exercise. If $\mathcal{A}$ is an algebra, then the monotone class generated by $\mathcal{A}$ is equal to the σ-field generated by $\mathcal{A}$. Prove it (using Monotone class theorem).

So, Borel σ-fields $\mathcal{B}_1$, $\mathcal{B}_2$ may be defined as monotone classes generated by $\mathcal{E}_1, \mathcal{E}_2$ respectively.

Intuitively, a set is a Borel set, if (and only if) it can be constructed from elementary sets[11] by iterated limits of monotone sequences.

Especially, all open sets, all closed sets, and all countable sets are Borel sets. It is quite difficult, to construct an example of a non-Borel set!

5d6 Proposition. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $X : \Omega \to \mathbb{R}$ a random variable. Then for every Borel set $B \subset \mathbb{R}$, the set

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$

is an event, that is, belongs to the σ-field $\mathcal{F}$.

Proof. Consider the class $\mathcal{B}_X$ of all Borel sets $B \subset \mathbb{R}$ such that $X^{-1}(B) \in \mathcal{F}$; we have to prove that $\mathcal{B}_X$ is the whole $\mathcal{B}_1$. The "inverse image" operation $X^{-1}(\cdot)$ preserves main operations on sets:[12]

$$\begin{aligned}
&X^{-1}(\emptyset) = \emptyset\,, \qquad X^{-1}(\mathbb{R}) = \Omega\,,\\
&X^{-1}(\mathbb{R} \setminus B) = \Omega \setminus X^{-1}(B)\,,\\
&X^{-1}(A \cap B) = X^{-1}(A) \cap X^{-1}(B)\,, \qquad X^{-1}(A \cup B) = X^{-1}(A) \cup X^{-1}(B)\,,\\
&X^{-1}(B_1 \cup B_2 \cup \dots) = X^{-1}(B_1) \cup X^{-1}(B_2) \cup \dots
\end{aligned}$$

Combining it with the fact that $\mathcal{F}$ is a σ-field, we conclude that $\mathcal{B}_X$ is a σ-field.[13]

The σ-field $\mathcal{B}_X$ contains all intervals (recall 1a2). Therefore, $\mathcal{B}_X$ contains the σ-field generated by intervals. So, $\mathcal{B}_X$ contains all Borel sets. □

So, Def. 1a2 is equivalent to the following.

5d7 Definition. A *random variable* is a function $X : \Omega \to \mathbb{R}$ that is $\mathcal{F}$-measurable, which means

$$\forall B \in \mathcal{B}_1 \quad \{\omega : X(\omega) \in B\} \in \mathcal{F}\,.$$

Now we may ask non-elementary questions about random variiables; not just $\mathbb{P}\big(X \in [a,b]\big)$, but also, say, $\mathbb{P}\big(\, X \text{ is rational } \big)$, etc.[14]

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[11] Not just intervals!

[12] Note that the "image" operation is worse; say, $X(A \cap B) \neq X(A) \cap X(B)$, in general.

[13] Check it!

[14] Provided that we have a probability space, of course.

5e Borel functions

5e1 Definition. (a) A function $\varphi : \mathbb{R} \to \mathbb{R}$ is a *Borel function,* if $\forall B \in \mathcal{B}_1 \quad \varphi^{-1}(B) \in \mathcal{B}_1$.
(b) A function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is Borel, if $\forall B \in \mathcal{B}_1 \quad \varphi^{-1}(B) \in \mathcal{B}_2$.
(c) A map $\varphi : \mathbb{R} \to \mathbb{R}^2$ is Borel, if $\forall B \in \mathcal{B}_2 \quad \varphi^{-1}(B) \in \mathcal{B}_1$.
(d) A map $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is Borel, if $\forall B \in \mathcal{B}_2 \quad \varphi^{-1}(B) \in \mathcal{B}_2$.

(As you probably guess, 5e1(a–d) and 5d7 are special cases of a more general notion of a Borel map from one Borel space to another.)

5e2 Lemma. Let $X : \Omega \to \mathbb{R}$ be a random variable, and $\varphi : \mathbb{R} \to \mathbb{R}$ a Borel function. Then $\varphi(X)$ is a random variable.[15]

Proof. We use 5d7; let $B \in \mathcal{B}_1$, then

$$\{\omega : \varphi(X(\omega)) \in B\} = \{\omega : X(\omega) \in \underbrace{\varphi^{-1}(B)}_{\in \mathcal{B}_1}\} \in \mathcal{F}\,.$$

□

5e3 Lemma. Every continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ is a Borel function.

Proof. Consider the class $\mathcal{B}_\varphi$ of all Borel sets $B \subset \mathbb{R}$ such that $\varphi^{-1}(B) \in \mathcal{B}_1$; we have to prove that $\mathcal{B}_\varphi$ is the whole $\mathcal{B}_1$. Similarly to the proof of 5d6, $\mathcal{B}_\varphi$ is a σ-field. However, every open interval (a, b) belongs to $\mathcal{B}_\varphi$, since $\varphi^{-1}\big((a, b)\big)$ is an open set.[16] □

5e4 Corollary. A continuous function of a random variable is a random variable.

(Note that a continuous function need not be piecewise monotone.)

5e5 Lemma. Let $\varphi_1, \varphi_2, \dots : \mathbb{R} \to \mathbb{R}$ be Borel functions, $\varphi : \mathbb{R} \to \mathbb{R}$, and $\varphi_n(x) \uparrow \varphi(x)$ for every $x \in \mathbb{R}$. Then φ is a Borel function.[17]

Proof. For every a,

$$\{x : \varphi(x) \le a\} = \{x : \forall n \;\; \varphi_n(x) \le a\} = \bigcap_n \{x : \varphi_n(x) \le a\} \in \mathcal{B}_1\,;$$

similarly to the proof of 5e3, $(-\infty, a] \in \mathcal{B}_\varphi$, therefore $\mathcal{B}_\varphi$ is the whole $\mathcal{B}_1$. □

5e6 Lemma. Let $\varphi_1, \varphi_2, \dots : \mathbb{R} \to \mathbb{R}$ be Borel functions, $\varphi : \mathbb{R} \to \mathbb{R}$, and $\varphi_n(x) \to \varphi(x)$ for every $x \in \mathbb{R}$. Then φ is a Borel function.[18]

Proof. $\varphi(x) = \lim_{n\to\infty} \sup\{\varphi_n(x), \varphi_{n+1}(x), \dots\}$; apply 5e5 twice. □

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[15] As before, $\varphi(X)$ means the same as $\varphi \circ X$, or $\omega \mapsto \varphi(X(\omega))$, or $\Omega \xrightarrow{X} \mathbb{R} \xrightarrow{\varphi} \mathbb{R}$.
[16] Check it.
[17] Note that convergence need not be uniform.
[18] Note that convergence need not be uniform.

5e7 Exercise. Calculate the function

$$\varphi(x) = \lim_{n\to\infty} \lim_{k\to\infty} \cos^{2k}(\pi n! x) \, .$$

Is φ a Borel function?

5e8 Exercise. Let $X, Y : \Omega \to \mathbb{R}$ be random variables. Then for every Borel set $B \subset \mathbb{R}^2$, the set

$$\{\omega \in \Omega : \big(X(\omega), Y(\omega) \big) \in B\}$$

is an event. Prove it. (Hint: similarly to 5d6...)

5e9 Exercise. Let $X, Y : \Omega \to \mathbb{R}$ be random variables, and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a Borel function. Then $\varphi(X, Y)$ is a random variable. Prove it. (Hint: similarly to 5e2...)

5e10 Exercise. Every continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a Borel function. Prove it. (Hint: similarly to 5e3...)

5e11 Corollary. Let $X, Y : \Omega \to \mathbb{R}$ be random variables. Then $X + Y$ and XY are random variables.

5e12 Exercise. Consider X/Y assuming that $\mathbb{P}\big(Y = 0 \big) = 0$.

5f Borel measures, Lebesgue measure

A probability measure was defined in 1a1 on an arbitrary σ-field $\mathcal{F}$. Especially, a probability measure on $\mathcal{B}_1$ (or $\mathcal{B}_2$) is called a Borel probability measure. Waiving the normalization $P(\Omega) = 1$ we get a more general notion of a *finite Borel measure:*

$$\begin{aligned} &\mu : \mathcal{B}_1 \to [0, +\infty) \, , \\ &\mu(A \uplus B) = \mu(A) + \mu(B) \, , \\ &\mu(A_1 \uplus A_2 \uplus \dots) = \mu(A_1) + \mu(A_2) + \dots \end{aligned} \tag{5f1}$$

Similarly to (2a2), σ-additivity implies continuity:

$$\begin{aligned} B_n \uparrow B \quad &\Longrightarrow \quad \mu(B_n) \uparrow \mu(B) \, , \\ B_n \downarrow B \quad &\Longrightarrow \quad \mu(B_n) \downarrow \mu(B) \, . \end{aligned} \tag{5f2}$$

Waiving finiteness, we get a *locally finite Borel measure:*

$$\begin{aligned} &\mu : \mathcal{B}_1 \to [0, +\infty] \, , \\ &\mu\big((a, b)\big) < \infty \qquad \text{whenever } -\infty < a < b < +\infty \, , \\ &\mu(A \uplus B) = \mu(A) + \mu(B) \, , \\ &\mu(A_1 \uplus A_2 \uplus \dots) = \mu(A_1) + \mu(A_2) + \dots \end{aligned} \tag{5f3}$$

There are also signed measures, σ-finite measures, vector-valued measures etc., but we need only

- Borel *probability* measures,
- a single locally finite Borel measure, the famous Lebesgue measure.

Every Borel probability measure μ has its cumulative distribution function

$$F_\mu(x) = \mu\big((-\infty, x]\big) \,. \tag{5f4}$$

Similarly to (2a3), F_μ determines uniquely $\mu(E)$ for all elementary sets E.

5f5 Lemma. If two Borel probability measures coincide on all intervals, then they coincide everywhere (on all Borel sets).

Proof. The class $\{B \in \mathcal{B}_1 : \mu(B) = \nu(B)\}$ is monotone by 5f2, therefore, by Monotone class theorem 5d4, it is the whole $\mathcal{B}_1$. ☐

5f6 Corollary. $F_\mu = F_\nu \quad \Longrightarrow \quad \mu = \nu$.

Locally finite measures violate 5f2; true, upward continuity still holds, but downward continuity fails; say, $[n, \infty) \downarrow \emptyset$, but $\mu\big([n,\infty)\big)$ may be $+\infty$ for all n. Nevertheless:

5f7 Exercise. If two locally finite Borel measures coincide on all bounded intervals, then they coincide everywhere (on all Borel sets). Prove it. (Hint: measures $\mu_n(B) = \mu(B \cap [-n, n])$, $\nu_n(B) = \nu(B \cap [-n, n])$ are finite, and coincide an all intervals.)

5f8 Definition. Lebesgue measure (denoted mes) is a locally finite Borel measure on $\mathcal{B}_1$ satisfying

$$\operatorname{mes}\big((a,b)\big) = b - a$$

whenever $-\infty < a < b < \infty$.

Uniqueness of Lebesgue measure follows from 5f7,[19] but its existence is quite nontrivial.[20]

5f9 Theorem. Lebesgue measure exists.

The proof is given by measure theory.

Now (at last!) we are in position to give an example of a nondiscrete probability space $(\Omega, \mathcal{F}, P)$:

$$\begin{aligned} &\Omega = (0,1) \,, \\ &\mathcal{F} = \mathcal{B}_1|_\Omega = \{B \in \mathcal{B}_1 : B \subset (0,1)\} \,, \\ &P = \operatorname{mes}|_\Omega, \quad \text{that is, } P(B) = \operatorname{mes}(B) \text{ for } B \in \mathcal{F} \,. \end{aligned} \tag{5f10}$$

This is the probability space meant in Sections 1–4.

[19]Intuitively: a Borel set results from elementary sets by iterated monotone limits, and its measure is the corresponding iterated limit of (elementary) measures.

In the discrete case, probability of a set is the sum of probabilities of its points. Accordingly, all sets have probabilities.

In the continuous case, probability (or measure) of a set does not arise from its points. Rather, it arises from its relation to intervals. If a set is not related to intervals, it has no probability (or measure) at all.

A point is not a meter; an interval is a meter.

[20]Recall 5b.