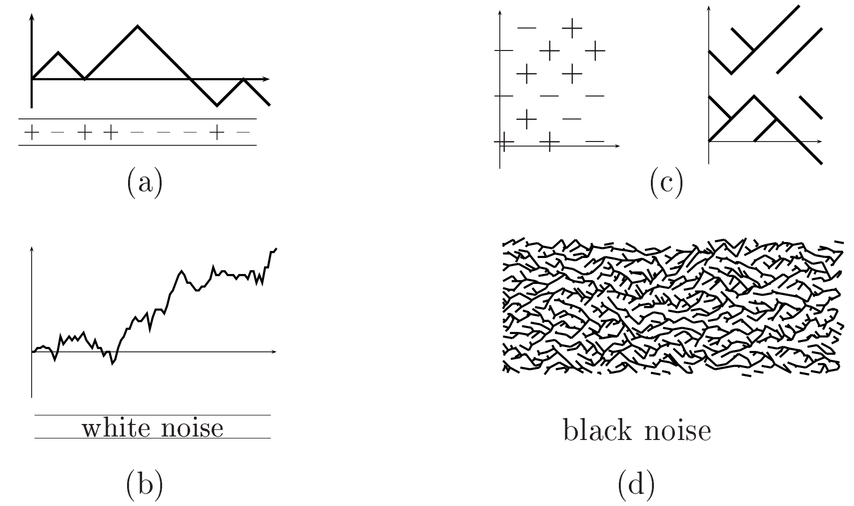
9 The Brownian web as a black noise

9a The Brownian web as a stochastic flow

We know that a one-dimensional array of random signs can produce various noises in the scaling limit. However, I still do not know, whether it can produce a black noise,[1] or not. This is why we turn to a two-dimensional array of random signs.



One-dimensional array of random signs produces a random walk (a) that converges to Brownian motion (b). Two-dimensional array of random signs produces a system of coalescing random walks (c) that converges to Brownian web (d).

The Brownian web was investigated by Arratia, Toth, Werner, Soucaliuc, and recently by Fontes et al.[2]

For an example of a black noise, we do not need convergence of the discrete model to the Brownian web, but only the web itself. Also, we treat the web as a collection of random maps rather than a random geometric configuration on the plane.

In order to keep finite everything that can be kept finite, we consider Brownian motions in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ rather than the line $\mathbb{R}$. That is, points x and $x+1$ are treated as the same point. (Equivalently, you may use the circle on the complex plane via $e^{2\pi i x}$.)

First, we define a pair of *coalescing* random paths starting from given points $X_1(0), X_2(0) \in \mathbb{T}$. Namely,

$$\begin{aligned}
\tau_{12} &= \min\{t \in [0,\infty) : X_1(0) + B_1(t) = X_2(0) + B_2(t)\}\,; \\
X_1(t) &= X_1(0) + B_1(t)\,; \\
X_2(t) &= \begin{cases} X_2(0) + B_2(t) & \text{for } t \le \tau_{12}, \\ X_1(t) & \text{for } t \ge \tau_{12}; \end{cases}
\end{aligned}$$

[1]To be defined in 9d1.

[2]L.R.G. Fontes, M. Isopi, C.M. Newman, K. Ravishankar, "The Brownian web", arXiv:math.PR/0203184. (Other references may be found therein.)

here $B_1(\cdot), B_2(\cdot)$ are independent Brownian motions; the equality is treated mod$\mathbb{Z}$, that is, $(X_1(0) + B_1(\tau_{12})) - (X_2(0) + B_2(\tau_{12})) \in \mathbb{Z}$. The construction is asymmetric: when paths meet, the second one joins the first. But it does not matter; priority does not influence the *distribution* of the two-dimensional process (X_1, X_2).

For a third path, the procedure is a bit more complicated:

$$\tau_{13} = \min\{t \in [0,\infty) : X_1(t) = X_3(0) + B_3(t)\} ;$$
$$\tau_{23} = \min\{t \in [0,\infty) : X_2(t) = X_3(0) + B_3(t)\} ;$$
$$X_3(t) = \begin{cases} X_3(0) + B_3(t) & \text{if } t \le \min(\tau_{13}, \tau_{23}), \\ X_1(t) & \text{if } t \ge \min(\tau_{13}, \tau_{23}) = \tau_{13}; \\ X_2(t) & \text{if } t \ge \min(\tau_{13}, \tau_{23}) = \tau_{23}. \end{cases}$$

And so on. The number $\eta_n(t)$ of *different* points among $X_1(t), \dots, X_n(t)$ is a random process, integer-valued, decreasing.

9a1 Exercise. $\mathbb{P}\big(\eta_n\big(\frac{c_1}{n^2}\big) = n \big) \le e^{-c_2 n}$ for some absolute constants $c_1, c_2 \in (0,\infty)$.

Prove it.

Hint. Let $0 < X_1(0) < \dots < X_{2m}(0) < 1$.

(a) Before the first coalescence, $X_k(t) = X_k(0) + B_k(t)$ for all k, and processes $X_2 - X_1, \dots, X_{2m} - X_{2m-1}$ are independent.

(b) $\mathbb{P}\big(X_2(0) + B_2(\cdot) > X_1(0) + B_1(\cdot) \text{ on } [0,t] \big) = 2\Phi\big(\frac{X_2(0)-X_1(0)}{\sqrt{2t}}\big) - 1$.

(c) The product of probabilities is maximal when $X_2(0) - X_1(0) = X_4(0) - X_3(0) = \dots = X_{2m}(0) - X_{2m-1}(0) = \frac{1}{m}$.

9a2 Exercise. $\mathbb{P}\big(\eta_n\big(\frac{c_1}{m}\big) \ge m \big) \le e^{-c_2 m}$ for some absolute constants $c_1, c_2 \in (0,\infty)$.

Prove it.

Hint. $\eta_n\big(\frac{c_1}{k+1}\big) - \eta_n\big(\frac{c_1}{k}\big) \ge 1$ for all k, except for an event of probability $\le \sum_k e^{-c_2 k}$.

We may choose an infinite sequence $X_1(0), X_2(0), \dots \in \mathbb{T}$ and consider $\eta_\infty(t)$, the number of different points among all $X_k(t)$.

9a3 Exercise. $\mathbb{P}\big(\eta_\infty\big(\frac{c_1}{m}\big) \ge m \big) \le e^{-c_2 m}$ for all m; here c_1, c_2 are the same as in 9a2.

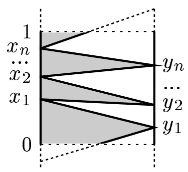
Prove it.

Hint: $\eta_n(t) \uparrow \eta_\infty(t)$ for $n \to \infty$.

9a4 Exercise. $\mathbb{P}\big(\eta_\infty(t) < \infty \text{ for all } t > 0 \big) = 1$. Moreover, $\mathbb{E}\,\eta_\infty(t) < \infty$ for all $t > 0$.

Prove it.

We choose a *dense* (in $\mathbb{T}$) sequence $(X_k(0))_{k=1}^\infty$, and after a given time $t > 0$ we get a finite number of points, $X_k(t) \in \{y_1, \dots, y_n\}$ for all k. For every $l \in \{1, \dots, n\}$ the set of $X_k(0)$ for all k such that $X_k(t) = y_l$ is dense in an interval, and we get a random step function $\mathbb{T} \to \mathbb{T}$.



$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n} : \mathbb{T} \to \mathbb{T},$$
$$x_1 < \dots < x_n < x_1, \; y_1 < \dots < y_n < y_1 \text{ (cyclically)},$$
$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n}(x) = y_{k+1} \text{ for } x \in (x_k, x_{k+1}] \,.$$

In fact, the value at x_k does not matter; we let it be y_k for convenience, but equally well it could be y_{k+1}, or remain undefined. Points $x_1, \dots, x_n$ will be called left critical points of the map, while $y_1, \dots, y_n$ are right critical points.

We introduce the set G_∞ consisting of all step functions $\mathbb{T} \to \mathbb{T}$ and in addition, the identity function. If $f, g \in G_\infty$ then $g \circ f \in G_\infty$, thus G_∞ is a semigroup. It consists of pieces of dimensions $2, 4, 6, \dots$ and the identity. Similarly to G_3 (recall 7b), G_∞ is not a topological semigroup, since the composition is discontinuous.

The distribution of the random map is a probability measure μ_t on G_∞. It can be shown that μ_t does not depend on the choice of a dense countable set $\{X_k(0)\} \subset \mathbb{T}$.

Similarly to 8a, the Brownian web may be described by random maps

$$\xi_{s,t} : \Omega \to G_\infty \, , \qquad \xi_{s,t} = f^{y_1(s,t),\dots,y_{n(s,t)}(s,t)}_{x_1(s,t),\dots,x_{n(s,t)}(s,t)} \, ,$$

and for any $r < s < t$,

$$\begin{gathered} \xi_{r,s} \text{ and } \xi_{s,t} \text{ are independent,} \\ \xi_{s,t} \circ \xi_{r,s} = \xi_{r,t} \text{ almost surely.} \end{gathered}$$

Moreover, $\xi_{t_1,t_2}, \xi_{t_2,t_3}, \dots, \xi_{t_{n-1},t_n}$ are independent whenever $t_1 < \dots < t_n$. The distribution of $\xi_{s,t}$ is the probability measure μ_{t-s} on G_∞.

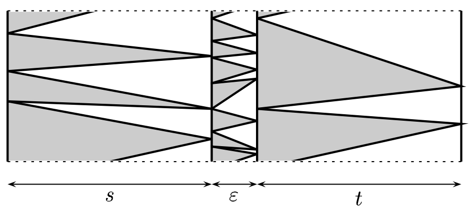
9b Nothing in the first chaos

Similarly to 8a we consider the sub-σ-field $\mathcal{F}_{s,t}$ generated by $\xi_{u,v}$ for all $(u,v) \subset (s,t)$. Again, $\mathcal{F}_{r,t} = \mathcal{F}_{r,s} \otimes \mathcal{F}_{s,t}$, and so, we have a factorization; let us call it *the web factorization.* Similarly to 8c1, the web factorization is continuous. Thus, its first chaos is described by 8b6. And again, we may restrict ourselves to (8c2):

$$\begin{gathered} X = \varphi(\xi_{0,1}), \qquad \mathbb{E} X = 0, \qquad \mathbb{P}\big(-1 \le X \le 1 \big) = 1 \, ; \\ \lim_{n\to\infty} \sum_{k=1}^{2^n} \mathbb{E}\big(X \,\big|\, \mathcal{F}_{(k-1)2^{-n},k2^{-n}} \big) \text{ is the projection of } X \text{ to } H_1(0,1) \, . \end{gathered}$$

As before,

$$\begin{aligned} \mathbb{E}\big(\varphi(\xi_{0,1}) \,\big|\, \mathcal{F}_{s,s+\varepsilon} \big) &= \mathbb{E}\big(\varphi(\xi_{s+\varepsilon,1} \circ \xi_{s,s+\varepsilon} \circ \xi_{0,s}) \,\big|\, \mathcal{F}_{s,s+\varepsilon} \big) = \\ &= \iint \varphi(h \circ g \circ f) \, d\mu_s(f) d\mu_t(h) = \alpha(g) \, , \end{aligned}$$



where $g = \xi_{s,s+\varepsilon}$. Note that now the situation on the whole circle looks rather similar to the situation on the bottom (near 0) of the picture on page 80. We may guess that the first chaos vanishes!

However, there (on page 80), c_2 was mostly 0. Now it is not so simple. It is wise to start with the discrete model. Here, g corresponds to a single column of random signs. Most of them do not influence $h \circ g \circ f$; that is the key to the problem.

Here are some quite general statements (unrelated to the Brownian web). Roughly they say that a small subsample is nearly independent of a large sample.

9b1 Exercise. Let $\tau_1, \dots, \tau_n : \Omega \to \{-1, +1\}$ be random signs (independent, ± 1 with probabilities 50%, 50%), and $X : \Omega \to \{1, \dots, n\}$ be a random variable independent of $\tau_1, \dots, \tau_n$ and distributed uniformly on $\{1, \dots, n\}$. Then

$$\operatorname{Corr}\bigl(\varphi(X, \tau_X), \psi(\tau_1, \dots, \tau_n)\bigr) \le \frac{1}{\sqrt{n}}$$

for every functions $\varphi : \{1, \dots, n\} \times \{-1, +1\} \to \mathbb{R}$, $\psi : \{-1, +1\}^n \to \mathbb{R}$ such that the correlation coefficient is well-defined (recall 7e2).

Prove it.

Hint. $\mathbb{E}\bigl(\varphi(X, \tau_X)\bigr) = \frac{1}{n} \sum_{k,a} \varphi(k, a)$, let it vanish; $\mathbb{E}\bigl(\varphi(X, \tau_X) \,\big|\, \tau_1, \dots, \tau_n \bigr) = \frac{1}{n} \sum_k \varphi(k, \tau_k)$; $\|\varphi(X, \tau_X)\| = \frac{1}{2n} \sum_{k,a} \varphi^2(k, a)$; $\|\mathbb{E}\bigl(\varphi(X, \tau_X) \,\big|\, \tau_1, \dots, \tau_n \bigr)\| = \frac{1}{\sqrt{n}} \|\varphi(X, \tau_X)\|$.

9b2 Exercise. Let $\tau_1, \dots, \tau_n : \Omega \to \{-1, +1\}$ be random signs (as before), and $X_1, \dots, X_n : \Omega \to \{0, 1\}$ random variables such that the two random vectors $(\tau_1, \dots, \tau_n)$ and $(X_1, \dots, X_n)$ are independent. (Dependence between $X_1, \dots, X_n$ is allowed.) Then

$$\operatorname{Corr}\bigl(\varphi(X_1, \dots, X_n; X_1\tau_1, \dots, X_n\tau_n),\ \psi(\tau_1, \dots, \tau_n)\bigr) \le \sqrt{\max_k \mathbb{P}\bigl(X_k = 1 \bigr)}$$

for every functions φ, ψ (such that the correlation coefficient is well-defined).

Prove it.

Hint. Consider the linear operator $\psi(\tau_1, \dots, \tau_n) \mapsto \mathbb{E}\bigl(\psi(\tau_1, \dots, \tau_n) \,\big|\, X_1, \dots, X_n; X_1\tau_1, \dots, X_n\tau_n \bigr)$ on monomials $\psi(\tau_1, \dots, \tau_n) = \tau_{k_1} \dots \tau_{k_i}$. Check that they remain orthogonal, and their norms decrease at least by $\sqrt{\max_k \mathbb{P}\bigl(X_k = 1 \bigr)}$.

We return to the Brownian web, or rather to its discrete counterpart. Recall that g corresponds to a single column of n random signs, or n random lines (recall fig. (c) on page 91). Each line influences $h \circ g \circ f$ only if its left endpoint is a right critical point of f, and its right endpoint is a left critical point of g. These two events are independent. Each one is of probability $O(1/n) = O(\sqrt{\varepsilon})$. Indeed, the expected number of critical points is $O(1)$, which follows from 9a4; and the probability is the same for each point, due to symmetry (invariance w.r.t. rotations of the circle $\mathbb{T}$ by $1/n$).

All that, combined with 9b2, gives

$$\operatorname{Corr}\bigl(\varphi(h \circ g \circ f), \psi(g)\bigr) = O(\sqrt{\varepsilon})\,.$$

Therefore $\|\mathbb{E}\bigl(\varphi(h \circ g \circ f) \,\big|\, g \bigr)\| = O(\sqrt{\varepsilon})$, that is, $\|\mathbb{E}\bigl(\varphi(\xi_{0,1}) \,\big|\, \mathcal{F}_{s,s+\varepsilon} \bigr)\| = O(\sqrt{\varepsilon})$. But... it does not exclude the first chaos! It is just the same as for the usual Brownian motion.

Well, we must be more clever. Recall that we have not only $\mathbb{E}\,|X|^2 \le 1$ but also $\mathbb{P}\bigl(-1 \le X \le 1 \bigr) = 1$. Let $A \subset \Omega$ be the event "at least one of the n random signs influences $h \circ g \circ f$"

(that is, at least one right critical point of f is $\sqrt{\varepsilon}$-close to at least one left critical point of h). We have

$$
\begin{gathered}
X = X \cdot \mathbf{1}_A + X \cdot \mathbf{1}_{\Omega \setminus A} \, ; \\
\| X \cdot \mathbf{1}_A \| \le \sqrt{ \mathbb{P} \big(A \big) } = \sqrt{ n \cdot O(1/n) \cdot O(1/n) } = O(1/\sqrt{n}) = O(\varepsilon^{1/4}) \, ; \\
X = X - \mathbb{E} X = \big(X \cdot \mathbf{1}_A - \mathbb{E} (X \cdot \mathbf{1}_A) \big) + \big(X \cdot \mathbf{1}_{\Omega \setminus A} - \mathbb{E} (X \cdot \mathbf{1}_{\Omega \setminus A}) \big) \, ; \\
\| \mathbb{E} \big(X \cdot \mathbf{1}_A \, \big| \, g \big) - \mathbb{E} (X \cdot \mathbf{1}_A) \| = O(\sqrt{\varepsilon}) \cdot \| X \cdot \mathbf{1}_A \| = O(\varepsilon^{3/4}) \, .
\end{gathered}
\tag{9b3}
$$

Indeed, $\operatorname{Corr}\big(X \cdot \mathbf{1}_A, \psi(g) \big) = O(\sqrt{\varepsilon})$. Further,

$$
\begin{aligned}
\mathbb{E} \big(X \cdot \mathbf{1}_{\Omega \setminus A} \, \big| \, g \big) &= \mathbb{E} (X \cdot \mathbf{1}_{\Omega \setminus A}) \, ; \\
\mathbb{E} \big(X \, \big| \, g \big) &= \mathbb{E} \big(X \cdot \mathbf{1}_A \, \big| \, g \big) - \mathbb{E} (X \cdot \mathbf{1}_A) \, ; \\
\| \mathbb{E} \big(X \, \big| \, g \big) \| &= O(\varepsilon^{3/4}) \, ,
\end{aligned}
\tag{9b4}
$$

that is, $\| \mathbb{E} \big(\varphi(\xi_{0,1}) \, \big| \, \mathcal{F}_{s,s+\varepsilon} \big) \| = O(\varepsilon^{3/4}) = o(\sqrt{\varepsilon})$, and so, the first chaos must vanish... provided that our arguments work for continuous time. Do they, really?

First of all we generalize 9b2 from random signs to continuous random variables.

9b5 Exercise. Let $(X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be two independent random vectors, $X_k : \Omega \to \{0, 1\}$, $Y_k : \Omega \to \mathbb{R}$, and $Y_1, \dots, Y_n$ are independent random variables. (Dependence between $X_1, \dots, X_n$ is allowed.) Then

$$
\operatorname{Corr}\big(\varphi(X_1, \dots, X_n; X_1 Y_1, \dots, X_n Y_n), \, \psi(Y_1, \dots, Y_n) \big) \le \sqrt{ \max_k \mathbb{P} \big(X_k = 1 \big) }
$$

for every functions φ, ψ (such that the correlation coefficient is well-defined).

Prove it. What about Y_k taking on values in spaces more general than $\mathbb{R}$?

Hint. Similarly to 9b2, but monomials are of the form $\psi_1(Y_{k_1}) \dots \psi_i(Y_{k_i})$, $\mathbb{E} \, \psi_j(Y_{k_j}) = 0$; recall the hint to 7e3.

In discrete time, values $g(x)$ for different x are independent, but in continuous time they are not. We feel that $g(x_1), g(x_2)$ are nearly independent if $|x_1 - x_2| \gg \sqrt{\varepsilon}$; what of it? Fortunately, the gap between $O(\varepsilon^{3/4})$ and $o(\sqrt{\varepsilon})$ is at our disposal. We may divide the circle $\mathbb{T}$ into n equal pieces by n equidistant points $X_1(s), \dots, X_n(s)$, $n \sim \varepsilon^{-\gamma}$; the constant γ will be chosen later. Corresponding paths $X_k(u)$, $u \in [s, s+\varepsilon]$ do not intersect with probability $\ge 1 - n \cdot 2 \big(1 - \Phi\big(\frac{1/n}{\sqrt{2\varepsilon}} \big) \big)$, exponentially close to 1 if $\gamma < 0.5$, since $\frac{1/n}{\sqrt{2\varepsilon}} \sim \text{const} \cdot \varepsilon^{-(0.5-\gamma)}$ (and of course, $1 - \Phi$ decreases exponentially). Hopefully we may assume that they do not intersect, and moreover, $|X_k(u) - X_k(s)| \le \frac{1}{3n}$ for $u \in [s, s+\varepsilon]$, $k = 1, \dots, n$. These paths divide g into n *conditionally independent* pieces. Such a piece influences $h \circ g \circ f$ only if it contains both a right critical point of f and a left critical point of h. These two events are independent, and each one is of probability $O(1/n) = O(\varepsilon^\gamma)$. In terms of 9b5 it means $\operatorname{Corr}(\dots) \le \sqrt{O(\varepsilon^\gamma) \cdot O(\varepsilon^\gamma)} = O(\varepsilon^\gamma)$. The event A (namely, "at least one piece influences") is of probability $\mathbb{P} \big(A \big) \le n \cdot O(1/n) \cdot O(1/n) = O(1/n) = O(\varepsilon^\gamma)$. Similarly to (9b3), (9b4) we get $\| X \cdot \mathbf{1}_A \| = O(\varepsilon^{\gamma/2})$; $\| \mathbb{E} \big(X \cdot \mathbf{1}_A \, \big| \, g \big) - \mathbb{E} (X \cdot \mathbf{1}_A) \| = O(\varepsilon^\gamma) \cdot O(\varepsilon^{\gamma/2})$;

$\|\mathbb{E}\left(X \,\middle|\, g \right)\| = O(\varepsilon^{3\gamma/2})$. It remains to choose γ such that $\gamma < 0.5$ and $3\gamma/2 > 1/2$, tha is, $\gamma \in \left(\frac13, \frac12\right)$, and we get $\|\mathbb{E}\left(X \,\middle|\, g \right)\| = o(\sqrt{\varepsilon})\dots$ though, two problems remain. One problem is, conditioning on n paths $X_k(\cdot)$. The other problem is, the (small but positive) probability of intersections between these paths.

It is possible to get (unconditionally) independent pieces of the Brownian web itself. However, it is technically simpler to use the construction of the Brownian web out of independent Brownian motions. We introduce, for each $k = 1, \dots, n$, the σ-field $\mathcal{E}_k$ generated by Brownian motions $B_l(\cdot)|_{[0,\varepsilon]}$ for all l such that $X_l(s) \in [x_k, x_{k+1})$. Then $\mathcal{E}_1, \dots, \mathcal{E}_n$ are independent, and g is measurable w.r.t. $\mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n$. The dense sequence $\big(\mathcal{X}_k(s)\big)_{k=1}^\infty$ may be chosen at will; we choose $X_k(s) = x_k$ for $k = 1, \dots, n$ (the tail remains arbitrary).

Introduce for $k = 1, \dots, n$

$$\tilde B_k(u) = \begin{cases} B_k(u) & \text{if } \max_{[0,\varepsilon]} |B_k(\cdot)| \le \frac{1}{3n}, \\ 0 & \text{otherwise;} \end{cases}$$

of course, $\tilde B_k$ is not a Brownian motion, but anyway, we may replace $B_1, \dots, B_n$ with $\tilde B_1, \dots, \tilde B_n$ in the construction of g, thus constructing another random map $\tilde g$. The distribution of $\tilde g$ differs from the distribution of g. Especially, $|\tilde g(x_k) - x_k| \le \frac{1}{3n}$ with probability 1; in contrast, $|g(x_k) - x_k|$ exceeds $\frac{1}{3n}$ with a positive (but small) probability.

We consider $\tilde X = \varphi(h \circ \tilde g \circ f)$, treating $\tilde g$ as a random variable measurable w.r.t. $\mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n$, and apply 9b5 (together with estimations outlined before), getting

$$\|\mathbb{E}\left(\tilde X \,\middle|\, \mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n \right) - \mathbb{E}(\tilde X)\| = o(\sqrt{\varepsilon})\,.$$

On the other hand,

$$\mathbb{P}\left(\tilde X \ne X \right) \le \mathbb{P}\left(\tilde g \ne g \right) = O(\varepsilon^p) \quad \text{for every } p\,,$$

since it is $O\big(\exp(-\varepsilon^{-2(0.5-\gamma)})\big)$. Taking into account that $|X| \le 1$, $|\tilde X| \le 1$ always, we get

$$\begin{gathered} \|\tilde X - X\| = O(\varepsilon^p) \quad \text{for every } p\,; \\ \|\mathbb{E}\left(X \,\middle|\, \mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n \right)\| \le \\ \le \|\mathbb{E}\left(\tilde X \,\middle|\, \mathcal{E}_1 \otimes \dots \otimes \mathcal{E}_n \right) - \mathbb{E}(\tilde X)\| + \|\tilde X - X - \mathbb{E}(\tilde X - X)\| = o(\sqrt{\varepsilon})\,; \\ \|\mathbb{E}\left(X \,\middle|\, g \right)\| = o(\sqrt{\varepsilon})\,. \end{gathered}$$

So,

$$H_1(0,1) = \{0\}\,.$$

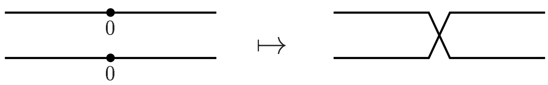
The web factorization admits no L_2^0-decomposable processes (except for 0).

9c Stability and sensitivity

Let $(\mathcal{F}_{s,t})_{s\le t}$ be a factorization of a probability space $(\Omega, \mathcal{F}, P)$. We have $\mathcal{F} = \mathcal{F}_{-\infty,0} \otimes \mathcal{F}_{0,\infty}$. As was explained in 5b, Ω may be thought of as the product, $\Omega = \Omega_{-\infty,0} \times \Omega_{0,\infty}$ (more exactly, $(\Omega, \mathcal{F}, P)$ is the product…). Each $\omega \in \Omega$ becomes a pair $(\omega^{\text{past}}, \omega^{\text{future}})$, $\omega^{\text{past}} \in \Omega_{-\infty,0}$,

$\omega^{\text{future}} \in \Omega_{0,\infty}$. On the space $\Omega^2 = \Omega \times \Omega$ of pairs (ω_1, ω_2) $(\omega_1, \omega_2 \in \Omega)$ we have a *crossover,* a measure preserving map

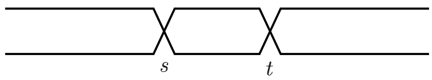
$$\beta : \Omega^2 \to \Omega^2\,,$$
$$\beta\big((\omega_1^{\text{past}}, \omega_1^{\text{future}}), (\omega_2^{\text{past}}, \omega_2^{\text{future}})\big) = \big((\omega_1^{\text{past}}, \omega_2^{\text{future}}), (\omega_2^{\text{past}}, \omega_1^{\text{future}})\big)\,.$$



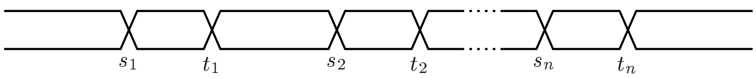
9c1 Example. For the Brownian factorization,

$$B_1(t) \circ \beta = \begin{cases} B_1(t) & \text{for } t < 0, \\ B_2(t) & \text{for } t > 0; \end{cases} \qquad B_2(t) \circ \beta = \begin{cases} B_2(t) & \text{for } t < 0, \\ B_1(t) & \text{for } t > 0. \end{cases}$$

Similarly, a crossover $\beta_E : \Omega^2 \to \Omega^2$ may be defined for an interval $E = (s,t) \subset \mathbb{R}$,



or the union of a finite number of intervals $E = (s_1, t_1) \cup \dots \cup (s_n, t_n)$.

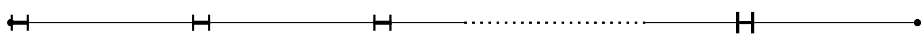


9c2 Example. For the Brownian factorization,

$$B_1(10) \circ \beta_{(1,2)\cup(5,6)} = B_1(1) + B_2(2) - B_2(1) + B_1(5) - B_1(2) + B_2(6) - B_2(5) + B_1(10) - B_1(6)\,.$$

Especially, consider

$$E_{\varepsilon,n} = \big(0, \frac{\varepsilon}{n}\big) \cup \big(\frac{1}{n}, \frac{1+\varepsilon}{n}\big) \cup \dots \cup \big(\frac{n-1}{n}, \frac{n-1+\varepsilon}{n}\big)\,;$$
$$\operatorname{mes} E_{\varepsilon,n} = \varepsilon\,;$$
$$\beta_{\varepsilon,n} = \beta_{E_{\varepsilon,n}}\,.$$



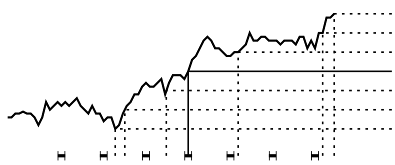
9c3 Exercise. For the Brownian factorization,
(a) $\sup_n \|B_1(t) \circ \beta_{\varepsilon,n} - B_1(t)\| \to 0$ for $\varepsilon \to 0$;
(b) $\sup_n \|\big(\int \varphi(t)\, dB_1(t)\big) \circ \beta_{\varepsilon,n} - \int \varphi(t)\, dB_1(t)\| \to 0$ for $\varepsilon \to 0$;
(c) $\sup_n \|X \circ \beta_{\varepsilon,n} - X\| \xrightarrow[\varepsilon\to 0]{} 0$ for every $X = \iint_{s<t} \varphi(s,t)\, dB_1(s) dB_2(t)$.
Prove it. What about higher chaos?

Let us try it for the Brownian web. For now, we consider a single path $X_1(\cdot)$, starting at a given point $X_1(0)$; we compare it with $Y_1(t) = X_1(t) \circ \beta_{\varepsilon,n}$ (you see, we deal with two independent copies of the Brownian web). On each interval of $E_{\varepsilon,n}$ the two processes (X_1 and Y_1) move independently; I mean, their increments are independent. On each interval of $[0,1] \setminus E_{\varepsilon,n}$ the two processes coalesce; namely, they move independently until/unless they meet; after the meeting (if any), they move together till the end of the time interval. The process $Z(t) = \frac{1}{\sqrt{2}}(X_1(t) - Y_1(t))$ is somewhat similar to sticky Brownian motion; outside the origin it moves like a Brownian motion, but at the origin it is trapped till the end of the interval of $[0,1] \setminus E_{\varepsilon,n}$. We may examine such a process by the approach of 7d:

$$B(t) - |Z(t)| = \min\bigl(B(t), \inf\{x : \sigma(x) \in E_{\varepsilon,n}\}\bigr) \,; \\ \sigma(x) = \max\{t : B(t) = x\} \,;$$

here x runs over $[\min_{[0,t]} B(\cdot), B(t)]$; recall 7d4. The problem is that events $\sigma(x) \in E_{\varepsilon,n}$ for different x are not independent. We feel that they are nearly independent if n is large enough, and x_1, x_2 are not too close. We choose a (small) $\delta > 0$ and consider

$$T_k = \sigma(B(t) - k\delta) \,,$$



$k = 0, 1, 2, \dots$ Increments $T_k - T_{k+1}$ are independent identically distributed random variables.[3] Their distribution can be found similarly to 7d:

$$\mathbb{P}\bigl(T_k - T_{k+1} > u \bigr) = \mathbb{P}\bigl(\max_{[0,u]} B(\cdot) < \delta \bigr) = 2\Phi\Bigl(\frac{\delta}{\sqrt{u}}\Bigr) - 1 \,;$$

$$f_{T_k - T_{k+1}}(u) = -\frac{d}{du}\biggl(2\Phi\Bigl(\frac{\delta}{\sqrt{u}}\Bigr) - 1\biggr) = 2\Phi'\Bigl(\frac{\delta}{\sqrt{u}}\Bigr) \cdot \delta \cdot \frac{1}{2} u^{-3/2} = \frac{\delta}{\sqrt{2\pi}\, u^{3/2}} \exp\Bigl(-\frac{\delta^2}{u} \Bigr) \,.$$

We do not need the formula for the density, but we need to know that a density exists. It follows (using independence) that $T_1, \dots, T_k$ have a joint density, for every k. Further, it follows that there exists a conditional density of $T_1, \dots, T_k$ given that $T_k > 0$, $T_{k+1} < 0$.

9c4 Exercise. Let X be a random variable having a density, and $A \subset \mathbb{R}$ a Borel set, $\operatorname{mes} A \ne 0$, such that $\forall x \ \bigl(x \in A \iff x + 1 \in A\bigr)$. Then $\mathbb{P}\bigl(X \in \varepsilon A \bigr) \to \operatorname{mes}\bigl(A \cap (0,1)\bigr)$ for $\varepsilon \to 0$, and the conditional distribution of X given $X \in \varepsilon A$ (that is, $\frac{1}{\varepsilon} X \in A$) converges (weakly) for $\varepsilon \to 0$ to the (unconditional) distribution of X.

Prove it. What about arbitrary distributions? Or, arbitrary nonatomic distributions?

Hint. Approximate (in $L_1(\mathbb{R})$) the density by step functions.

In other words, X and $\mathbf{1}_{\varepsilon A}(X)$ are asymptotically independent (for $\varepsilon \to 0$).

9c5 Exercise. Let random variables $X_1, \dots, X_n$ have a joint density, and A is as in 9c4. Then the random vector $(X_1, \dots, X_n)$ and random variables $\mathbf{1}_{\varepsilon A}(X_1), \dots, \mathbf{1}_{\varepsilon A}(X_n)$ are asymptotically independent (for $\varepsilon \to 0$). Also, $\lim_{\varepsilon \to 0} \mathbb{P}\bigl(X_k \in \varepsilon A \bigr) = \operatorname{mes}\bigl(A \cap (0,1)\bigr)$ for each k.

Formulate it exactly, and prove.

Conditionally, given that $T_k > 0$, $T_{k+1} < 0$ for a given k, we see that events $T_1 \in E_{\varepsilon,n}$, $\dots$, $T_k \in E_{\varepsilon,n}$ are asymptotically independent, of probability (nearly) ε each, provided that n is large enough (for given δ, ε).

[3] Here we consider $B(\cdot)$ on $(-\infty, t]$ rather than $[0,t]$, and $\sigma(\cdot)$ on $(-\infty, B(t)]$.

It follows that $\min\{k : T_k \in E_{\varepsilon,n}\}$ is distributed approximately geometrically, $\mathrm{G}(\varepsilon)$. Typically, $\min\{k : T_k \in E_{\varepsilon,n}\} = O(1/\varepsilon)$. However, we need maximal (rather than minimal) k, which is similar:

$$\max\{k : T_k \in E_{\varepsilon,n}\} = \max\{k : T_k > 0\} - O\Big(\frac{1}{\varepsilon}\Big) \, ;$$
$$B(t) - |Z(t)| = \min_{[0,t]} B(\cdot) + O\Big(\frac{\delta}{\varepsilon}\Big)$$

with high probability, provided that n is large enough. Note that n tends to infiinity, ε stays constant, and δ is arbitrary. In the limit $(n \to \infty)$ we get $B(t) - |Z(t)| = \min_{[0,t]} B(\cdot)$, which means that the trap becomes ineffective.[4]

For every $\varepsilon > 0$, the effect of the trap tends to 0 for $n \to \infty$.

We see that processes $X_1(\cdot)$ and $Y_1(\cdot) = X_1(\cdot) \circ \beta_{\varepsilon,n}$ are asymptotically independent for $n \to \infty$. It holds for every $\varepsilon > 0$.

Such a behavior is called *sensitivity.* Random variables $X_1(t)$ are sensitive. We may know, say, 99% of the random data (namely, the Brownian web outside $E_{0.01,n}$), and strangely enough, it does not help us, if we need to know the value of a sensitive random variable. The missing 1% of data is crucial, if it is scattered in time uniformly enough.

In contrast, for the Brownian motion, lack of a small fraction of data causes only a small error (recall 9c3), no matter how is it scattered in time. Such a behavior is called *stability.*

Clearly, a random variable cannot be both stable and sensitive (unless it is constant).

A single path $X_1(\cdot)$ of the Brownian web is distributed like the usual Brownian motion $B(\cdot)$;[5] nevertheless $X_1(\cdot)$ is sensitive, but $B(\cdot)$ is stable. A paradox? Note however that $B(t) - B(s)$ is measurable w.r.t. $\mathcal{F}^{\mathrm{Brown}}_{s,t}$, while $X_1(t) - X_1(s)$ is not measurable w.r.t. $\mathcal{F}_{s,t}$ (but only w.r.t. $\mathcal{F}_{0,t}$; think, why).

Well, a single path $X_1(\cdot)$ is only a small part of the Brownian web. Consider two paths $X_1(\cdot), X_2(\cdot)$ starting from two different points $X_1(0), X_2(0)$. We know that each one is sensitive, but maybe some function of the two (say, their difference) is not sensitive, and even stable?

We consider four processes, $X_1(\cdot), X_2(\cdot)$ and $Y_1(\cdot) = X_1(\cdot) \circ \beta_{\varepsilon,n}$, $Y_2(\cdot) = X_2(\cdot) \circ \beta_{\varepsilon,n}$. During some time, $X_1(t) \ne X_2(t)$ and $Y_1(t) \ne Y_2(t)$. A trap manifests itself when $X_1(t) = Y_1(t)$, or $X_1(t) = Y_2(t)$, or $X_2(t) = Y_1(t)$, or $X_2(t) = Y_2(t)$. However, the trap is ineffective (for $n \to \infty$), as we know from the former analysis. A new problem could appear in the case of multiple collision, say, $X_1(t) = Y_1(t) = Y_2(t)$. However, it never happens. Three independent Brownian motions never meet all together. By using this fact, one can show that the argument about the ineffective trap for two particles remains in force for more particles. Sometimes X_1 and X_2 coalesce, and then we deal with three particles; and so on.

Everything is sensitive in the factorization of the Brownian web. Combined with 9c3, it shows that the factorization does not admit a decomposable process distributed like the Brownian motion (which is already known to us, see 9b).

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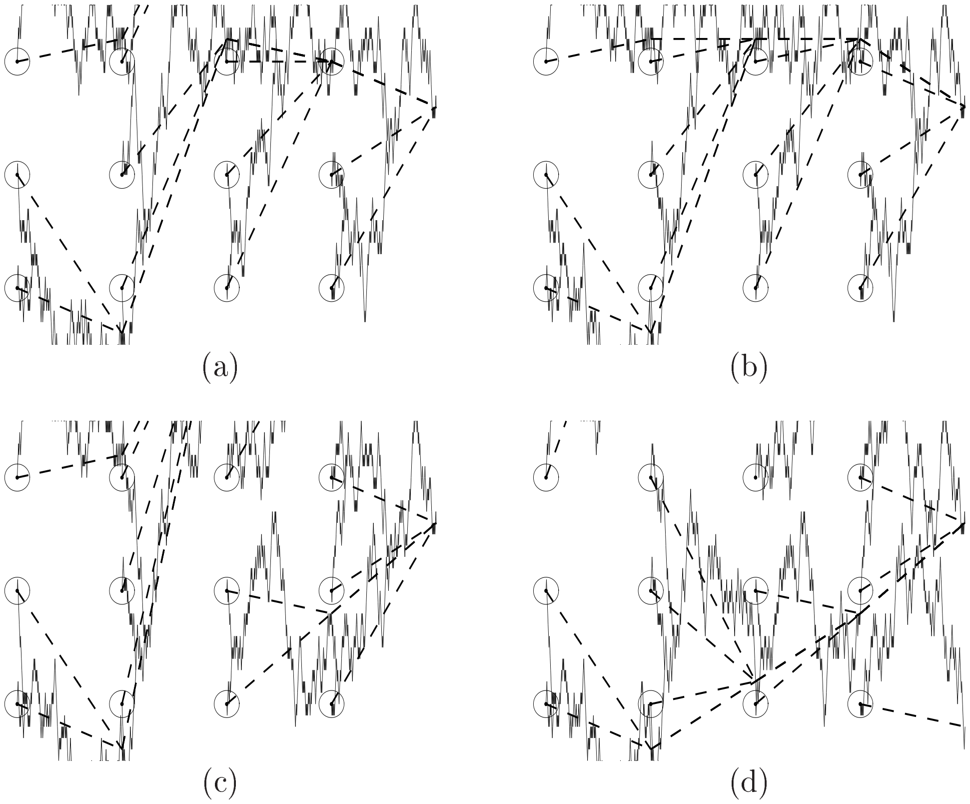
[4]And no wonder! An effective trap is given by the model of 7b. There, the fraction of maps f_* (releasing from the trap) is $\sqrt{\varepsilon}$, which tends to 0 in the scaling limit. Here, the fraction of points of $E_{\varepsilon,n}$ (releasing from the trap) is ε, for the continuous model (no scaling limit needed), and it does not tend to 0 in the limit $n \to \infty$.

[5]Mapped to the circle, but is does not matter.



An illustration. Coalescing walks on the grid 1000×30, starting from $4 \times 3 = 12$ points (circled). Unperturbed array of random signs (a). Perturbed array: each random sign is flipped with probability 0.025 (b). Further perturbation of the same type (c). Still further (d).

9d Black noises, and all that

9d1 Definition. A noise $\big((\Omega, \mathcal{F}, P), (\mathcal{F}_{s,t})_{s<t}, (\alpha_t)_{t\in\mathbb{R}}\big)$ is called *black,* if every L_2^0-decomposable process $(X_{s,t})_{s\le t}$ vanishes (that is, $\|X_{s,t}\| = 0$ for all s, t).

It is easy to upgrade the web factorization (introduced in 9b) to a noise, by constructing the group of time shifts (α_t), similarly to 8d1. That is the *noise of coalescence.*

9d2 Theorem. The noise of coalescence is black.

For the proof see 9b.

9d3 Proposition. For every noise, the following four items define one and the same sub-σ-field $\mathcal{F}_{\text{stable}} \subset \mathcal{F}$.

(a) $\mathcal{F}_{\text{stable}}$ is generated by all (real-valued) L_2^0-decomposable processes, in other words, by the first chaos (recall 8b1).

(b) $\mathcal{F}_{\text{stable}}$ is generated by all decomposable processes (not just L_2^0; see 8b1).

(c) $\mathcal{F}_{\text{stable}}$ is generated by all complex-valued *multiplicative* decomposable processes; these are $(X_{s,t})_{s\le t}$ such that $X_{s,t} : \Omega \to \mathbf{C}$ is measurable w.r.t. $\mathcal{F}_{s,t}$, and $X_{r,s}X_{s,t} = X_{r,t}$ a.s., whenever $r < s < t$.

I give no proof.[6]

Why call it "stable"? Wait a little…

9d4 Exercise. The noise of coalescence admits no decomposable processes, neither additive nor multiplicative (in the sense of 9d3(c)), neither square integrable nor non-integrable, neither real nor complex.

Prove it (using 9d2, 9d3).

9d5 Proposition. The following item may be added to equivalent definitions of $\mathcal{F}_{\text{stable}}$:

$\mathcal{F}_{\text{stable}}$ is generated by all decomposable processes $(X_{s,t})_{s<t}$ that are either Brownian or Poisson; that is, for each process, either $X_{s,t} \sim \mathrm{N}(0, t-s)$ for all $s < t$, or $X_{s,t} \sim \mathrm{Poisson}\big(\lambda(t-s)\big)$ for all $s < t$ (the parameter $\lambda \in (0,\infty)$ may depend on the process).

I give no proof.[7]

For example: the main result of 9c shows that $\mathcal{F}_{\text{stable}}$ of 9d5 is trivial for the noise of coalescence; by 9d5, $\mathcal{F}_{\text{stable}}$ of 9d3 is also trivial; the main result of 9b follows.

Every noise has its *stable part;* it is the noise $\big((\Omega, \mathcal{F}_{\text{stable}}, P), (\mathcal{F}_{s,t} \cap \mathcal{F}_{\text{stable}})_{s<t}, (\alpha_t)_{t\in\mathbb{R}}\big)$. If the stable part is equal to the whole noise, it means a classical noise. If the stable part is trivial, it means a black noise.

The stable part of a noise is its greatest classical subnoise.

The stable part of the noise of stickiness is Brownian. It is contained in every part of the noise. The noise is not classical, but contains no black part.

The idea of stability and sensitivity was explained in 9c in terms of losing a small fraction of data according to a deterministic pattern (the set $E_{\varepsilon,n}$). Another approach: each portion of data is lost with a small probability, independently of others. It works for factorizations, not only noises. A random variable X is stable if and only if $X = \mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right)$, that is, X is measurable w.r.t. $\mathcal{F}_{\text{stable}}$ (which motivates the notation). Also, a random variable X is sensitive if and only if $\mathbb{E}\left(X \,\middle|\, \mathcal{F}_{\text{stable}} \right) = 0$.[8]

Existence of black noises was proven first (on completely different ideas) by Tsirelson and Vershik ("Examples…"). The very idea of a nonclassical continuous product (of whatever) was suggested to me by Anatoly Vershik.

Another kind of a black noise of coalescence was found by Shinzo Watanabe.[9]

These examples may be called toy models. A really important example of a nonclassical noise was found recently by Yves Le Jan and Oliver Raimond.[10]

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[6] See Theorem 1.7 in:
B.S. Tsirelson and A.M. Vershik, "Examples of nonlinear continuous tensor products of measure spaces and non-Fock factorizations", Reviews in Mathematical Physics **10**:1 (1998), 81–145.

[7] See Lemma 2.9 in:
B. Tsirelson, "Unitary Brownian motions are linearizable", arXiv:math.PR/9806112.
(It is written for the Brownian component, but the idea works also for the Poisson component.) By the way, Lemma 2.1 there states that the factorization of any noise is continuous.

[8] See Definition 2.13 in:
B. Tsirelson, "Noise sensitivity on continuous products: an answer to an old question of J. Feldman", arXiv:math.PR/9907011.

[9] S. Watanabe, "A simple example of black noise", Bull. Sci. Math. **125**:6/7, 605–622.

[10] Y. Le Yan, O. Raimond, "Flows, coalescence and noise", arXiv:math.PR/0203221.