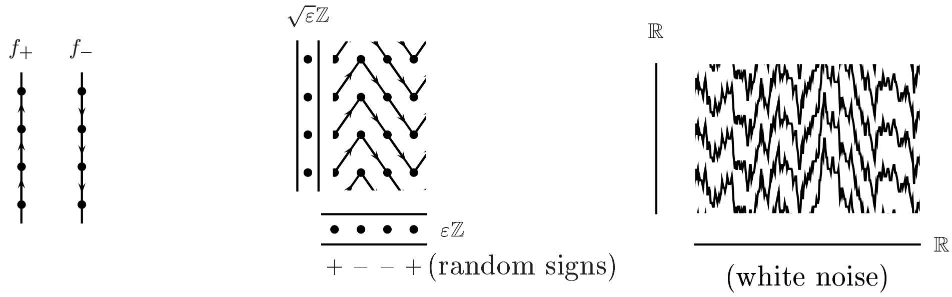
6 Some non-smooth stochastic flows: reflection

6a Usual Brownian flow

Stochastic flows will give us interesting examples of nonclassical noises. However, we start with a very simple (and classical) case.

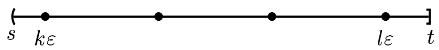
In discrete time, $t \in \varepsilon\mathbb{Z}$, we consider random signs $\tau(k\varepsilon)$ as before (independent equiprobable ± 1), and random maps

(6a1)
$$\xi_{k\varepsilon} = \begin{cases} f_+ & \text{if } \tau(k\varepsilon) = +1, \\ f_- & \text{if } \tau(k\varepsilon) = -1; \end{cases}$$
$$f_+, f_- : \mathbb{R} \to \mathbb{R}, \quad f_+(x) = x + \sqrt{\varepsilon}, \quad f_-(x) = x - \sqrt{\varepsilon}\,.$$

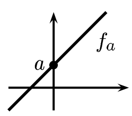


Imagine that for any $s < t$ we can measure the composition $\xi_{s,t} : \mathbb{R} \to \mathbb{R}$ defined as[1]

(6a2) $$\xi_{s,t} = \xi_{l\varepsilon} \circ \xi_{(l-1)\varepsilon} \circ \dots \circ \xi_{(k+1)\varepsilon} \circ \xi_{k\varepsilon} \quad \text{for } (k-1)\varepsilon \le s < k\varepsilon\,,\ l\varepsilon \le t < (l+1)\varepsilon\,.$$



Note that $f_- \circ f_+ = f_+ \circ f_- = \mathrm{id}$ (the identity mapping). Therefore, every composition (say, $f_- \circ f_+ \circ f_+ \circ f_- \circ f_+$) boils down to f_+^n, or f_-^n, or id. All maps $\xi_{s,t}$ belong to a one-parameter family,

(6a3)
$$\xi_{s,t} = f_a\,, \quad a = \sqrt{\varepsilon}\big(\tau(k\varepsilon) + \dots + \tau(l\varepsilon)\big) = \sqrt{\varepsilon} \sum_{i : i\varepsilon \in (s,t]} \tau(i\varepsilon)\,,$$
$$f_a(x) = x + a\,;$$


$$f_a \circ f_b = f_b \circ f_a = f_{a+b}\,;$$

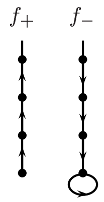
measuring $\xi_{s,t}$ means measuring $a = \sqrt{\varepsilon}\big(\tau(k\varepsilon) + \dots + \tau(l\varepsilon)\big)$, which is a special case of 'observables' $\sqrt{\varepsilon} \sum \varphi(k\varepsilon)\tau(k\varepsilon)$ introduced in Sect. 1. Clearly, the scaling limit is basically the Brownian motion,
$$\xi_{s,t} = f_{B(t)-B(s)}\,.$$

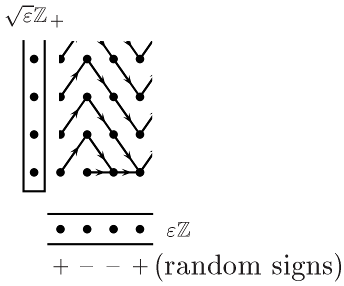
[1] Composition is understood as $(g \circ f)(x) = g\big(f(x)\big)$ (note the order).

6b Reflecting Brownian flow

We replace (6a1) with

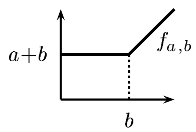
(6b1)
$$\xi_{k\varepsilon} = \begin{cases} f_+ & \text{if } \tau(k\varepsilon) = +1, \\ f_- & \text{if } \tau(k\varepsilon) = -1; \end{cases}$$
$$f_+, f_- : [0,\infty) \to [0,\infty)\,, \quad f_+(x) = x + \sqrt{\varepsilon}, \quad f_-(x) = \max(0, x - \sqrt{\varepsilon})\,.$$





Note that $f_- \circ f_+ = \mathrm{id}$ (but $f_+ \circ f_- \ne \mathrm{id}$). Therefore every composition boils down to some $f_+^m \circ f_-^n$. The maps f_-, f_+, as well as their compositions $\xi_{s,t}$ (defined like (6a2)) belong to a two-parameter family $f_{a,b} : [0,\infty) \to [0,\infty)$,

(6b2)
$$f_{a,b}(x) = \begin{cases} x + a & \text{if } x \ge b, \\ a + b & \text{if } 0 \le x \le b \end{cases}$$



for $b \ge 0$, $a + b \ge 0$, as we'll see now.

6b3 Exercise.

(a) $$f_+ = f_{\sqrt{\varepsilon},0}; \quad f_- = f_{-\sqrt{\varepsilon},\sqrt{\varepsilon}}\,;$$
(b) $$f_+^n = f_{n\sqrt{\varepsilon},0}; \quad f_-^n = f_{-n\sqrt{\varepsilon},n\sqrt{\varepsilon}}\,;$$
(c) $$f_+^m \circ f_-^n = f_{(m-n)\sqrt{\varepsilon},n\sqrt{\varepsilon}}\,;$$
(d) $$f_{a,b} = f_{a+b,0} \circ f_{-b,b}\,;$$
(e) $$f_{a_2,0} \circ f_{a_1,0} = f_{a_1+a_2,0}; \quad f_{-b_2,b_2} \circ f_{-b_1,b_1} = f_{-b_1-b_2,b_1+b_2}\,;$$
(f) $$f_{-b,b} \circ f_{b,0} = \mathrm{id}\,;$$
(g) $$f_{-b,b} \circ f_{a,0} = \begin{cases} f_{a-b,0} & \text{if } a \ge b, \\ f_{a-b,b-a} & \text{if } a \le b; \end{cases}$$
(h) $$f_{a_2,b_2} \circ f_{a_1,b_1} = f_{a,b} \quad \text{where } a = a_1 + a_2, b = \max(b_1, b_2 - a_1)\,.$$

Prove it.

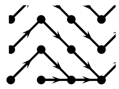
You see, our non-commutative two-dimensional semigroup is generated by two (commutative) one-parameter semigroups (see 6b3(e)) with a (quite simple and natural) relation (see 6b3(f)).

Measuring $\xi_{s,t}$ means measuring the corresponding parameters a, b.

6b4 Exercise. $\xi_{s,t} = f_{a,b}$ where $a = \sqrt{\varepsilon} \sum_{i:i\varepsilon\in(s,t]} \tau(i\varepsilon)$ is the same as in (6a3), and

$$b = -\sqrt{\varepsilon} \min_{m=k-1,k,k+1,\dots,l} \big(\tau(k\varepsilon) + \tau((k+1)\varepsilon) + \dots + \tau(m\varepsilon) \big)$$

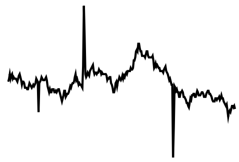
for $(k-1)\varepsilon \le s < k\varepsilon$, $l\varepsilon \le t < (l+1)\varepsilon$ (if $m = k-1$, the empty sum is 0).
Prove it.

Hint. Either use 6b3(h), or just look: 

We see that $\xi_{s,t} = f_{a(s,t),b(s,t)}$ where $a(s,t)$ is given by 6b4, and $b(s,t) = -\min_{u\in(s,t]} a(s,u)$. We guess that in the scaling limit

$$\text{(6b5)} \qquad \begin{aligned} a(s,t) &= B(t) - B(s)\,, \\ b(s,t) &= -\min_{u\in[s,t]} \big(B(u) - B(s) \big)\,. \end{aligned}$$

No problems with $a(s,t)$. However, $b(s,t)$ is a new kind of 'observable'. The random walk (the discrete counterpart of the Brownian motion) converges in distribution to the Brownian motion, as far as a finite set of points $t_1 < \dots < t_n$ is considered. Linear (or even nonlinear) integrals are also admissible, but the minimum is a challenge. The random walk moves by $\sqrt{\varepsilon}$ during the time ε, which means a high speed $1/\sqrt{\varepsilon}$. If the random walk has narrow peaks (at random points, of course),



then probably the minimum does not fit into the Brownian scaling limit.[2]

Fortunately, such peaks do not appear. A Brownian path is not at all differentiable, moreover,

$$\liminf_{t\to 0+} \frac{B(t)}{\sqrt{t}} = -\infty\,, \quad \limsup_{t\to 0+} \frac{B(t)}{\sqrt{t}} = +\infty\,;$$

however, it is continuous, moreover,

$$\sup_{0\le s<t\le 1} \frac{B(t)-B(s)}{(t-s)^{1/3}} < \infty \quad \text{a.s.}$$

A similar estimation holds for the random walk *uniformly in* ε. This is why the following (well-known) result holds.

6b6 Proposition. Let $f : C[0,1] \to \mathbb{R}$ be a bounded continuous function on the space $C[0,1]$,[3] $B(\cdot)$ the Brownian motion, and $B_\varepsilon(\cdot)$ its piecewise linear discrete counterpart.[4] Then

$$\mathbb{E}\, f\big(B_\varepsilon(\cdot)\big) \to \mathbb{E}\, f\big(B(\cdot)\big) \quad \text{for } \varepsilon \to 0\,.$$

[2] Similarly to the Poisson process as considered in 4a.

[3] $C[0,1]$ is the Banach space of all continuous functions $g : [0,1] \to [0,1]$ with the norm $\|g\| = \max_{t\in[0,1]} |g(t)|$.

[4] $B_\varepsilon\big((k+1)\varepsilon\big) - B_\varepsilon(k\varepsilon) = \sqrt{\varepsilon}\,\tau(k\varepsilon)$, and B_ε is linear on $[k\varepsilon, (k+1)\varepsilon]$.

6b7 Exercise. Formulate and prove (using 6b6) a correct interpretation of the incorrect relation

$$\operatorname{Lim} f\big(B_\varepsilon(\cdot)\big) = f\big(B(\cdot)\big)$$

for a continuous $f : C[0,1] \to \mathbb{R}$.

Hint: recall 1b6, 1b7.

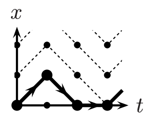
Does 6b6 contain our former 'linear' result 1b6, or even its 'nonlinear' generalization (mentioned in 3c)? To some extent. The linear stochastic integral $\int_0^1 \varphi(x)\, dB(x)$ is continuous (in B) on $C[0,1]$ if and only if φ is a function of bounded variation.[5]

So, in the scaling limit we get (6b5).

In discrete time, the random process

$$\begin{gathered} X(t) = \xi_{0,t}(0) = f_{a(0,t),b(0,t)}(0) = a(0,t) + b(0,t)\,, \\ X(n\varepsilon) = \sqrt{\varepsilon} \max_{k=1,\dots,n,n+1} \big(\tau(k\varepsilon) + \dots + \tau(n\varepsilon)\big) \end{gathered} \tag{6b8}$$

is the reflecting random walk.



In the scaling limit it becomes

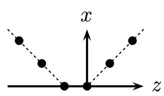
$$X(t) = a(0,t) + b(0,t) = B(t) - \min_{s\in[0,t]} B(s)\,.$$

On the other hand, the reflecting random walk $X(\cdot)$ is distributed like a function of the usual random walk

$$\begin{gathered} Z(t) = a(0,t)\,, \\ Z(n\varepsilon) = \sqrt{\varepsilon}\big(\tau(\varepsilon) + \dots + \tau(n\varepsilon)\big)\,. \end{gathered} \tag{6b9}$$

Namely,

$$X(\cdot) \sim \left| Z(\cdot) + \frac{\sqrt{\varepsilon}}{2} \right| - \frac{\sqrt{\varepsilon}}{2}\,. \tag{6b10}$$



6b11 Exercise. Prove that, indeed, these two processes are identically distributed.

Hint: for each process, find the conditional distribution of the next value (at $(k+1)\varepsilon$), given the past (at $\varepsilon, 2\varepsilon, \dots, k\varepsilon$).

In the scaling limit, Z becomes the Brownian motion B, and we get $X(\cdot) \sim |B(\cdot)|$. So, we have two candidates to 'reflecting Brownian motion':

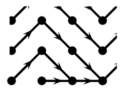
$$\begin{gathered} X(t) = B(t) - \min_{s\in[0,t]} B(s)\,; \\ X(t) = |B(t)|\,; \end{gathered} \tag{6b12}$$

these are different functions of $B(\cdot)$, of course; however, they are identically distributed; thus, we have two equivalent definitions of the *distribution* of the reflecting Brownian motion.

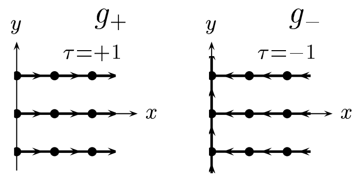
[5]Maybe, after a correction on a negligible set.

6c Counting reflections

Having the discrete reflecting flow,



we want to introduce a new 'observable' that counts reflections. We can do it by considering such a two-dimensional stochastic flow:



(6c1)
$$
\begin{gathered}
\xi_{k\varepsilon} = \begin{cases} g_+ & \text{if } \tau(k\varepsilon) = +1, \\ g_- & \text{if } \tau(k\varepsilon) = -1; \end{cases} \\
g_+, g_- : \sqrt{\varepsilon}\mathbb{Z}_+ \times \sqrt{\varepsilon}\mathbb{Z} \to \sqrt{\varepsilon}\mathbb{Z}_+ \times \sqrt{\varepsilon}\mathbb{Z} \, , \\
g_+(m\sqrt{\varepsilon}, n\sqrt{\varepsilon}) = \big((m+1)\sqrt{\varepsilon}, n\sqrt{\varepsilon} \big) \, ; \\
g_-(m\sqrt{\varepsilon}, n\sqrt{\varepsilon}) = \big((m-1)\sqrt{\varepsilon}, n\sqrt{\varepsilon} \big) \quad \text{if } m > 0 \, ; \\
g_-(0, n\sqrt{\varepsilon}) = \big(0, (n+1)\sqrt{\varepsilon} \big) \, .
\end{gathered}
$$

You see, the x-projection is the (discrete) reflecting flow, while y counts reflections of x. Note also that $x - y$ is just the 'usual flow' of 6a.

Though, we need not restrict ourselves to lattice points;

(6c2)
$$
\begin{gathered}
g_+, g_- : [0, \infty) \times \mathbb{R} \to [0, \infty) \times \mathbb{R} \, , \\
g_+(x, y) = \big(x + \sqrt{\varepsilon}, y \big) \, ; \\
g_-(x, y) = \big(x - \sqrt{\varepsilon}, y \big) \quad \text{if } x \ge \sqrt{\varepsilon} \, ; \\
g_-(x, y) = \big(0, y + \sqrt{\varepsilon} - x \big) \quad \text{if } x \le \sqrt{\varepsilon} \, .
\end{gathered}
$$

Similarly to 6b, we have $g_- \circ g_+ = \mathrm{id}$ (but $g_+ \circ g_- \ne \mathrm{id}$). The maps g_-, g_+, as well as their compositions $\xi_{s,t}$ (defined like (6a2)) belong to a two-parameter family $g_{a,b} : [0, \infty) \times \mathbb{R} \to [0, \infty) \times \mathbb{R}$,

$$
g_{a,b}(x, y) = \begin{cases} (x + a, y) & \text{if } x \ge b, \\ (a + b, y + b - x) & \text{if } 0 \le x \le b \end{cases}
$$

for $b \ge 0$, $a + b \ge 0$, as we'll see now.

6c3 Exercise.

(a)
$$
g_+ = g_{\sqrt{\varepsilon},0}; \quad g_- = g_{-\sqrt{\varepsilon},\sqrt{\varepsilon}} \, ;
$$

(b)
$$
g_+^n = g_{n\sqrt{\varepsilon},0}; \quad g_-^n = g_{-n\sqrt{\varepsilon},n\sqrt{\varepsilon}} \, ;
$$

(c)
$$
g_+^m \circ g_-^n = g_{(m-n)\sqrt{\varepsilon},n\sqrt{\varepsilon}} \, ;
$$

(d) $$g_{a,b} = g_{a+b,0} \circ g_{-b,b}\,;$$

(e) $$g_{a_2,0} \circ g_{a_1,0} = g_{a_1+a_2,0}; \quad g_{-b_2,b_2} \circ g_{-b_1,b_1} = g_{-b_1-b_2,b_1+b_2}\,;$$

(f) $$g_{-b,b} \circ g_{b,0} = \mathrm{id}\,;$$

(g) $$g_{-b,b} \circ g_{a,0} = \begin{cases} g_{a-b,0} & \text{if } a \ge b, \\ g_{a-b,b-a} & \text{if } a \le b; \end{cases}$$

(h) $$g_{a_2,b_2} \circ g_{a_1,b_1} = g_{a,b} \quad \text{where } a = a_1 + a_2, b = \max(b_1, b_2 - a_1)\,.$$

Prove it.

We see that the two semigroups, $(f_{a,b})$ and $(g_{a,b})$ are isomorphic, the isomorphism being simply $f_{a,b} \leftrightarrow g_{a,b}$. In other words, the same abstract semigroup acts on $[0,\infty)$ (by $f_{a,b}$) and on $[0,\infty) \times \mathbb{R}$ (by $g_{a,b}$). We see also that 6b4 is still applicable:

$$\xi_{s,t} = g_{a(s,t),b(s,t)}\,,$$
$$a(s,t) = \sqrt{\varepsilon}\big(\tau(k\varepsilon) + \dots + \tau(l\varepsilon)\big)\,,$$
$$b(s,t) = -\sqrt{\varepsilon} \min_{m=k-1,k,k+1,\dots,l} \big(\tau(k\varepsilon) + \tau((k+1)\varepsilon) + \dots + \tau(m\varepsilon)\big)\,,$$
$$(k-1)\varepsilon \le s < k\varepsilon,\ l\varepsilon \le t < (l+1)\varepsilon\,.$$

Note that the map $g_{a,b}$ is uniquely determined by the point $g_{a,b}(0,0)$. Therefore, in order to find the distribution of the random map $\xi_{s,t}$, it suffices to find the distribution of the random point

(6c4) $$(X,Y) = \xi_{s,t}(0,0) = g_{a,b}(0,0) = (a+b, b)\,.$$

Denote by n the number of points in $(s,t] \cap \varepsilon\mathbb{Z}$. We treat X, Y as functions of n random signs, therefore, random variables. Note that $X \ge 0$, $Y \ge 0$.

6c5 Exercise. $\dfrac{X-Y}{2\sqrt{\varepsilon}} + \dfrac{n}{2} \sim \mathrm{Binom}\big(n, \frac12\big)$, that is,

$$\mathbb{P}\big(\, X - Y = (-n+2k)\sqrt{\varepsilon}\,\big) = 2^{-n}\binom{n}{k} = \frac{n!}{2^n k!(n-k)!} \quad \text{for } k = 0,1,\dots,n\,.$$

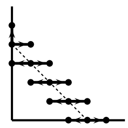
Prove it.

Hint: $X - Y = \sqrt{\varepsilon}\big(\tau(k\varepsilon) + \dots + \tau(l\varepsilon)\big)$.

6c6 Exercise. The probability $\mathbb{P}\big(\, X = l\sqrt{\varepsilon}, Y = (k-l)\sqrt{\varepsilon}\,\big)$ does not depend on $l \in \{0,1,\dots,k\}$.

Prove it.

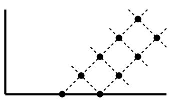
Hint: induction in n.



6c7 Exercise. $\mathbb{P}\big(\, X = k\sqrt{\varepsilon}, Y = 0\,\big) = \mathbb{P}\big(\, X - Y = k\sqrt{\varepsilon}\,\big) - \mathbb{P}\big(\, X - Y = (k+2)\sqrt{\varepsilon}\,\big)$.

Prove it.

Hint: use 6c6.



It follows that

$$\mathbb{P}\left(X = k\sqrt{\varepsilon}, Y = l\sqrt{\varepsilon}\right) = \frac{n!}{2^n} \frac{k+l+1}{\left(\frac{n+k+l}{2}+1\right)!\left(\frac{n-k-l}{2}\right)!}$$

for $k \ge 0$, $l \ge 0$, $k+l \le n$ such that $n-k-l$ is even.

The scaling limit can be found now via the Stirling formula. However, the result can be guessed easily: $X-Y$ becomes normal $\mathrm{N}(0, t-s)$;[6] and 6c7 turns into[7]

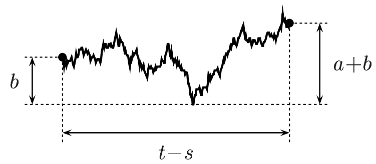
$$f_{X,Y}(x,0) = -2f'_{X-Y}(x) = -2\frac{d}{dx}\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{x^2}{2(t-s)}\right);$$

so,

$$f_{X,Y}(x,y) = \frac{2(x+y)}{\sqrt{2\pi}(t-s)^{3/2}}\exp\left(-\frac{(x+y)^2}{2(t-s)}\right). \tag{6c8}$$

That is the joint density of random variables $X = a(s,t)+b(s,t)$ and $Y = b(s,t)$, recall (6c4). It gives us the joint density of $a(s,t) = B(t) - B(s)$ and $b(s,t) = -\min_{u\in[s,t]}\big(B(u)-B(s)\big)$ (recall (6b5)):[8]

$$f_{a(s,t),b(s,t)}(a,b) = \frac{2(a+2b)}{\sqrt{2\pi}(t-s)^{3/2}}\exp\left(-\frac{(a+2b)^2}{2(t-s)}\right). \tag{6c9}$$



Note also that X and Y are identically distributed,[9] and X is distributed like $|B(t-s)|$ (recall (6b12)); thus,

$$\begin{aligned} f_X(x) &= \frac{2}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{x^2}{2(t-s)}\right) \quad \text{for } x\in(0,\infty)\,, \\ f_Y(y) &= \frac{2}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{y^2}{2(t-s)}\right) \quad \text{for } y\in(0,\infty)\,. \end{aligned} \tag{6c10}$$

6c11 Exercise. Derive (6c10) from (6c8) just by integration.

6c12 Exercise. In discrete time,

$$\mathbb{P}\left(X = k\sqrt{\varepsilon}\right) = \mathbb{P}\left(Y = k\sqrt{\varepsilon}\right) = \begin{cases} 2^{-n}\binom{n}{(n+k)/2} & \text{for } n+k \text{ even,} \\ 2^{-n}\binom{n}{(n+k+1)/2} & \text{for } n+k \text{ odd.} \end{cases}$$

Prove it.

Hint: use 6c6 and (6b10).

[6] $\mathrm{Var}(X-Y) = \varepsilon n = t-s+O(\varepsilon)$.

[7] Here f_{X-Y} is the (one-dimensional) density of (the distribution of) $X-Y$, and $f_{X,Y}$ is the two-dimensional density of (X,Y).

[8] The Jacobian $\frac{\partial(x,y)}{\partial(a,b)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$.

[9] Moreover, the pairs (X,Y) and (Y,X) are identically distributed (that is, have the same two-dimensional distribution), which follows from 6c6.

6d Local time

Returning to the idea of counting reflections, we see that in continuous time, the random process $X(t) = a(0,t)+b(0,t)$ is the reflecting Brownian motion, while the process $Y(t) = b(0,t)$ counts its reflections. The process $Y(\cdot)$ is called the *local time* of the reflecting Brownian motion $X(\cdot)$.

6d1 Exercise. Given a value $x = X(1)$ of the reflecting Brownian motion (at $t = 1$), the (conditional) density of the local time $y = Y(1)$ is

$$(x+y)\exp\Big(-\frac{y^2}{2} - xy\Big) .$$

Prove it. Try to explain intuitively, why small y are improbable for small x, but highly probable for large x.

Hint: use 6c8.

The local time, is it a function of the reflecting Brownian motion? The (evident) positive answer in discrete time says nothing about continuous time.[10] Both $X(\cdot)$ and $Y(\cdot)$ are functions of $B(\cdot) = a(0,\cdot) = X(\cdot)-Y(\cdot)$; however, is $Y(\cdot)$ a function of $X(\cdot)$? We know that $Y(1)$ is not a function of $X(1)$, but is it a function of the whole path $X(\cdot)$?

For every $\delta > 0$ the process[11]

$$Y_\delta(t) = \frac{1}{\delta}\operatorname{mes}\{s \in (0,t) : X(t) \le \delta\} = \frac{1}{\delta}\int_0^t \mathbf{1}_{[0,\delta]}\big(X(s)\big)\,ds$$

is a function of $X(\cdot)$. Maybe, it converges (when $\delta \to 0$) to the local time $Y(t)$? If it does, then $Y(\cdot)$ is a function of $X(\cdot)$.

The discrete counterpart of Y_δ is

$$Y_{\delta,\varepsilon}(t) = \frac{\varepsilon}{\delta}\cdot \#\{k : k\varepsilon \in [0,t], X_\varepsilon(k\varepsilon) \le \delta\} = \frac{\varepsilon}{\delta}\sum_{k:k\varepsilon\le t} \mathbf{1}_{[0,\delta]}\big(X_\varepsilon(k\varepsilon)\big) ,$$

where $X_\varepsilon(\cdot)$ is defined by (6b8), or by (6b9)–(6b10), which is the same for now, since only the distribution of $X_\varepsilon(\cdot)$ is relevant to our question: the local time *near* 0, is it close to the local time *at* 0? Is $Y_{\delta,\varepsilon}(\cdot)$ close to $Y_\varepsilon(\cdot)$? I mean that δ is small, but ε is much smaller ($\sqrt{\varepsilon} \ll \delta$); and[12]

$$Y_\varepsilon(t) = \sqrt{\varepsilon}\cdot \#\{k : k\varepsilon \in (0,t], X_\varepsilon((k-1)\varepsilon) = X_\varepsilon(k\varepsilon) = 0\} ,$$

the counter of reflections. Another natural discrete-time counterpart of the local time is

$$L_\varepsilon(t) = \frac{1}{2}\sqrt{\varepsilon}\cdot \#\{k : k\varepsilon \in [0,t], X_\varepsilon(k\varepsilon) = 0\} ,$$

the counter of visits to the origin. Sometimes (for some paths of $X_\varepsilon(\cdot)$) these Y_ε and L_ε are not close at all. Indeed, it may happen that $X_\varepsilon(\cdot)$ visits 0 many times, but every time leaves 0 immediately, without reflection. However, such a behavior is improbable, as we'll see now.

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[10] Recall 4a, and other cases.

[11] "mes" stands for the Lebesgue measure.

[12] Sorry, the new notation "Y_ε" conflicts with the old "Y_δ". Anyway, both Y_ε and Y_δ will be abandoned (replaced with L_ε and L_{φ_δ} respectively).

6d2 Exercise. The random process

$$M(k\varepsilon) = Y_\varepsilon(k\varepsilon) - L_\varepsilon((k-1)\varepsilon) \quad \text{for } k > 0, \quad \text{and } M(0) = 0\,,$$

is a martingale. That is,[13]

$$\mathbb{E}\left(M((k+1)\varepsilon) \,\middle|\, X_\varepsilon(0), X_\varepsilon(\varepsilon), \dots, X_\varepsilon(k\varepsilon) \right) = M(k\varepsilon)\,.$$

Prove it.

Hint: just consider the two possibilities, $\tau((k+1)\varepsilon) = \pm 1$.

6d3 Exercise.

$$\|M(k\varepsilon)\|^2 = \sum_{i=0}^{k-1} \|M((i+1)\varepsilon) - M(i\varepsilon)\|^2\,.$$

Prove it. (Each norm is taken in L_2 on the corresponding probability space.)

Hint: Martingale differences $M((k+1)\varepsilon) - M(k\varepsilon)$ are orthogonal; moreover, $M((k+1)\varepsilon) - M(k\varepsilon)$ is orthogonal to all functions of $X_\varepsilon(0), X_\varepsilon(\varepsilon), \dots, X_\varepsilon(k\varepsilon)$.

6d4 Exercise.

$$\|M((k+1)\varepsilon) - M(k\varepsilon)\|^2 = \frac{\sqrt{\varepsilon}}{2} \mathbb{E}\left(L_\varepsilon(k\varepsilon) - L_\varepsilon((k-1)\varepsilon) \right).$$

Prove it. (Here $L_\varepsilon(-\varepsilon) = 0$.)

Hint: both are equal to $\frac{\varepsilon}{4}\mathbb{P}\left(X_\varepsilon(k\varepsilon) = 0 \right)$.

We have

$$\|M((k+1)\varepsilon)\|^2 = \frac{\sqrt{\varepsilon}}{2} \mathbb{E}\, L_\varepsilon(k\varepsilon)\,;$$

$$\|Y_\varepsilon(t) - L_\varepsilon(t-\varepsilon)\| = \|M(t)\| = \sqrt{\frac{\sqrt{\varepsilon}}{2} \mathbb{E}\, L_\varepsilon(t-\varepsilon)}\,;$$

is $\mathbb{E}\, L_\varepsilon(t-\varepsilon)$ bounded when $\varepsilon \to 0$? We guess that $\mathbb{E}\, L_\varepsilon(t-\varepsilon) \to \mathbb{E}\, Y(t)$ for $\varepsilon \to 0$, but that is not proven yet. Rather, we know that the scaling limit of $Y_\varepsilon(t)$ is $Y(t)$, and $\mathbb{E}\, Y(t) < \infty$ (see (6c10)); still, it does not ensure that $\mathbb{E}\, Y_\varepsilon(t) \to \mathbb{E}\, Y(t)$.

6d5 Exercise. $\sup_{\varepsilon \in (0,1]} \|Y_\varepsilon(t)\| < \infty$.

Prove it.

Hint: use 6c12.

We have

$$\|L_\varepsilon(t-\varepsilon)\| \le \|Y_\varepsilon(t)\| + \sqrt{\frac{\sqrt{\varepsilon}}{2} \mathbb{E}\, L_\varepsilon(t-\varepsilon)} \le \left(\sup_\varepsilon \|Y_\varepsilon(t)\| \right) + \sqrt{\frac{\sqrt{\varepsilon}}{2} \|L_\varepsilon(t-\varepsilon)\|}\,,$$

[13] And, of course, $M(k\varepsilon)$ is a function of $X_\varepsilon(0), X_\varepsilon(\varepsilon), \dots, X_\varepsilon(k\varepsilon)$. It is a martingale w.r.t. the natural filtration of $X_\varepsilon(\cdot)$.

therefore $\sup_\varepsilon \|L_\varepsilon(t-\varepsilon)\| < \infty$ (think, why), and $\|M(t)\| = O(\varepsilon^{1/4})$ uniformly in t on bounded intervals. So,

(6d6) $$\|Y_\varepsilon(t) - L_\varepsilon(t)\| \to 0 \quad \text{for } \varepsilon \to 0 \, .$$

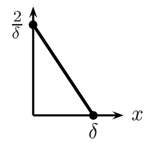
In the scaling limit, Y_ε and L_ε become the same, — the local time $Y(t)$, denoted traditionally by $L(t)$. We abandon Y_ε and use L_ε instead. For now we do not know, whether L_ε is close to $Y_{\delta,\varepsilon}$, or not.

Unfortunately, $Y_\delta(t)$ is a discontinuous function of a path $X(\cdot) \in C[0,t]$, which complicates the transition $Y_{\delta,\varepsilon} \to Y_\delta$. It is better to abandon $Y_{\delta,\varepsilon}$ and use instead

$$L_{\varphi_\delta,\varepsilon}(t) = \varepsilon \sum_{k:k\varepsilon\le t} \varphi_\delta\big(X_\varepsilon(k\varepsilon)\big)$$

where $\varphi_\delta : [0,\infty) \to [0,\infty)$ is a continuous function such that $\int_0^\infty \varphi_\delta(x)\,dx = 1$, and φ_δ is concentrated on $(0,\delta)$. Say, we may take

$$\varphi_\delta(x) = \frac{2}{\delta^2}(\delta - x) \, .$$



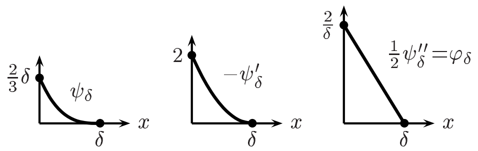
The scaling limit of $L_{\varphi_\delta,\varepsilon}$ is

$$L_{\varphi_\delta}(t) = \int_0^t \varphi_\delta\big(X(s)\big)\,ds \, .$$

Note that sometimes (for some paths of $X_\varepsilon(\cdot)$) these $L_{\varphi_\delta,\varepsilon}$ and L_ε are not close at all. Indeed, it may happen that $X_\varepsilon(\cdot)$ spends a long time near 0 without hitting 0. Still, we may hope that such behavior is improbable. How could we prove it?

Here is a trick that helps. We consider the process $\psi_\delta\big(X(\cdot)\big)$, where $\psi_\delta : [0,\infty) \to [0,\infty)$ is a smooth function concentrated on $[0,\delta]$ and such that

$$\frac12 \frac{d^2}{dx^2}\psi_\delta(x) = \varphi_\delta(x) \, .$$



Say, for $\varphi_\delta(x) = \frac{2}{\delta^2}(\delta - x)$ we have $\psi_\delta(x) = \frac{2}{3\delta^2}(\delta - x)^3$. Let us use just these functions.

6d7 Exercise.

$$\begin{aligned}\mathbb{E}\Big(\, \psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big) - \psi_\delta\big(X_\varepsilon(k\varepsilon)\big) \,\Big|\, X_\varepsilon(0), X_\varepsilon(\varepsilon), \dots, X_\varepsilon(k\varepsilon) \,\Big) &= \\ = \mathbb{E}\Big(\, \psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big) \,\Big|\, X_\varepsilon(0), X_\varepsilon(\varepsilon), \dots, X_\varepsilon(k\varepsilon) \,\Big) - \psi_\delta\big(X_\varepsilon(k\varepsilon)\big) &= \varepsilon\varphi_{\delta,\varepsilon}\big(X_\varepsilon(k\varepsilon)\big) \, ,\end{aligned}$$

where $\varphi_{\delta,\varepsilon} : \varepsilon\mathbb{Z}_+ \to \mathbb{R}$ satisfies

$$\begin{aligned}\varphi_{\delta,\varepsilon}(0) &= \frac{1}{2\sqrt{\varepsilon}}\psi'_\delta(0) + o\Big(\frac{1}{\sqrt{\varepsilon}}\Big) = -\frac{1}{\sqrt{\varepsilon}} + o\Big(\frac{1}{\sqrt{\varepsilon}}\Big) \, , \\ \varphi_{\delta,\varepsilon}(k\varepsilon) &= \frac12\psi''_\delta(k\varepsilon) + o(1) = \varphi_\delta(k\varepsilon) + o(1) \quad \text{if } k > 0\end{aligned}$$

for $\varepsilon \to 0$. (This "$o(1)$" is uniform in $x = k\varepsilon$, but not in δ.)

Prove it.

Hint: check the two possibilities $\tau\big((k+1)\varepsilon\big) = \pm 1$, and use the Taylor formula.

Thus, the process

$$M_{\delta,\varepsilon}(k\varepsilon) = \psi_\delta\big(X_\varepsilon(k\varepsilon)\big) - \varepsilon \sum_{i=0}^{k-1} \varphi_{\delta,\varepsilon}\big(X_\varepsilon(i\varepsilon)\big)$$

is a martingale. We have

$$\begin{aligned}
\psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big) - M_{\delta,\varepsilon}((k+1)\varepsilon) &= \varepsilon \sum_{i=0}^{k} \varphi_{\delta,\varepsilon}\big(X_\varepsilon(i\varepsilon)\big) = \\
&= \varepsilon \sum_{i:X_\varepsilon(i\varepsilon)=0} \bigg(-\frac{1}{\sqrt{\varepsilon}} + o\Big(\frac{1}{\sqrt{\varepsilon}}\Big)\bigg) + \varepsilon \sum_{i:X_\varepsilon(i\varepsilon)>0} \big(\varphi_\delta(X_\varepsilon(i\varepsilon)) + o(1)\big) = \\
&= \varepsilon \sum_{i:X_\varepsilon(i\varepsilon)=0} \bigg(-\frac{1}{\sqrt{\varepsilon}} + o\Big(\frac{1}{\sqrt{\varepsilon}}\Big) + O(1)\bigg) + \varepsilon \sum_{i=0}^{k} \big(\varphi_\delta(X_\varepsilon(i\varepsilon) + o(1)\big) = \\
&= -\underbrace{\sqrt{\varepsilon} \cdot \#\{i : X_\varepsilon(i\varepsilon) = 0\}}_{L_\varepsilon(k\varepsilon)} \cdot \big(1 + o(1)\big) + \underbrace{\varepsilon \sum_{i=0}^{k} \varphi_\delta\big(X_\varepsilon(i\varepsilon)\big)}_{L_{\varphi_\delta,\varepsilon}(k\varepsilon)} + o(1)\,,
\end{aligned}$$

thus

$$M_{\delta,\varepsilon}((k+1)\varepsilon) - L_\varepsilon(k\varepsilon)\big(1 + o(1)\big) + L_{\varphi_\delta,\varepsilon}(k\varepsilon) = \underbrace{\psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big)}_{\in[0,\delta]} + o(1)$$

for $\varepsilon \to 0$. We see that the difference $L_\varepsilon - L_{\varphi_\delta,\varepsilon}$ is close to the martingale $M_{\delta,\varepsilon}$. Is it small?

Martingale differences are orthogonal, therefore

$$\|M_{\delta,\varepsilon}((k+1)\varepsilon)\|^2 = \sum_{i=0}^{k} \|M_{\delta,\varepsilon}((i+1)\varepsilon) - M_{\delta,\varepsilon}(i\varepsilon)\|^2\,.$$

However, $M_{\delta,\varepsilon}((k+1)\varepsilon) - M_{\delta,\varepsilon}(k\varepsilon)$ is equal to $\psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big) - \psi_\delta\big(X_\varepsilon(k\varepsilon)\big)$ minus its conditional expectation; it follows that

$$\|M_{\delta,\varepsilon}((k+1)\varepsilon) - M_{\delta,\varepsilon}(k\varepsilon)\|^2 \le \|\psi_\delta\big(X_\varepsilon((k+1)\varepsilon)\big) - \psi_\delta\big(X_\varepsilon(k\varepsilon)\big)\|^2 = \varepsilon \|\psi'_\delta\big(X(k\varepsilon)\|^2 + o(\varepsilon)\,,$$

thus (assuming $k = O(1/\varepsilon)$),

$$\|M_{\delta,\varepsilon}((k+1)\varepsilon)\|^2 \le \varepsilon \sum_{i=0}^{k} \|\psi'_\delta\big(X(k\varepsilon)\big)\|^2 + o(1)\,.$$

Taking into account that $|\psi'_\delta(x)| \le 2$ for $x \in [0, \delta]$ and $\psi'_\delta(x) = 0$ for other x, we guess that the right-hand side is small. There are several ways to prove it; here is one. We note that our ψ_δ satisfies $|\psi'_\delta(x)|^2 \le 2\delta\varphi_\delta(x)$ for all x. Thus,

$$\varepsilon \sum_{i=0}^{k} \|\psi'_\delta\big(X(k\varepsilon)\big)\|^2 \le 2\delta\,\mathbb{E}\,\underbrace{\varepsilon \sum_{i=0}^{k} \varphi_\delta\big(X(i\varepsilon)\big)}_{L_{\varphi_\delta,\varepsilon}(k\varepsilon)}\,,$$

$$\|M_{\delta,\varepsilon}((k+1)\varepsilon)\|^2 \le 2\delta\,\mathbb{E}\,L_{\varphi_\delta,\varepsilon}(k\varepsilon) + o(1)\,,$$

which gives us

$$\|L_\varepsilon(k\varepsilon)\big(1+o(1)\big) - L_{\varphi_\delta,\varepsilon}(k\varepsilon)\| \le \sqrt{2\delta\,\mathbb{E}\,L_{\varphi_\delta,\varepsilon}(k\varepsilon) + o(1)} + \delta + o(1)\,;$$

$$\|\big(1+o(1)\big)L_\varepsilon(t) - L_{\varphi_\delta,\varepsilon}\| \le \sqrt{2\delta\,\mathbb{E}\,L_{\varphi_\delta,\varepsilon}(t) + o(1)} + \delta + o(1)\,;$$

these "$o(1)$" (for $\varepsilon \to 0$) are uniform in t on bounded intervals (but not in δ). Taking into account that $\sup_\varepsilon \|L_\varepsilon(t)\| < \infty$ (due to (6d6) and 6d5), we get

$$\|L_{\varphi_\delta,\varepsilon}(t)\| \le O(1) + \sqrt{\|L_{\varphi_\delta,\varepsilon}(t)\| + O(1)}\,,$$

thus $\sup_\varepsilon \|L_{\varphi_\delta,\varepsilon}(t)\| < \infty$, and so,

$$\limsup_{\varepsilon\to 0} \|L_\varepsilon(t) - L_{\varphi_\delta,\varepsilon}(t)\| \le \text{const} \cdot \sqrt{\delta}\,.$$

In the scaling limit we get[14] $\|L(t) - L_{\varphi_\delta}(t)\| \le \text{const} \cdot \sqrt{\delta}$, and finally,

$$L_{\varphi_\delta}(t) \to L(t) \quad \text{in } L_2(\Omega) \quad \text{for } \delta \to 0\,;$$

here $L(t)$ is the local time (just the same as $Y(t)$). So, the local time is a function of the reflecting Brownian motion.

[14] In general, if $Z_n \to Z$ in distribution, then $\|Z\| \le \limsup_n \|Z_n\|$ (think, why).