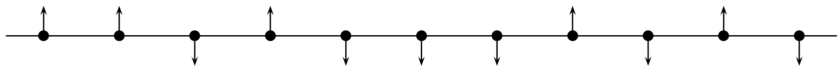
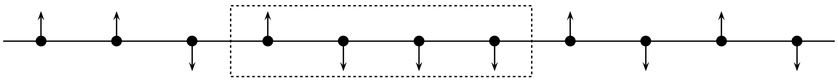
2 Poisson noise as a scaling limit

2a A quite informal introduction

Similarly to 1a, we imagine a one-dimensional array of random spins ('ups' and 'downs')

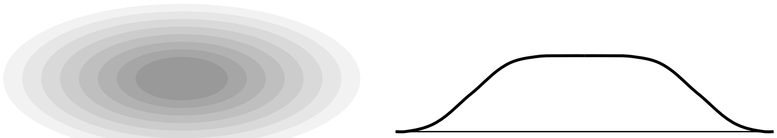


In contrast to 1a, assume that our measuring devices are sensitive not to single spins but to combinations of one 'up' and $m-1$ 'downs' (in that order, left-to-right); m is a parameter.



(here $m = 4$)

And, similarly to 1a, a device has a 'window' described by a test function.



a window its test function

2b A formalization

Similarly to 1b, we have i.i.d. random variables $\tau(k\varepsilon)$ on the probability space $(\Omega_{\varepsilon,M}, P_{\varepsilon,M})$. Given a 'test function' $\varphi : \mathbb{R} \to \mathbb{R}$, we construct random variables

$$X_{\varepsilon,M,\varphi} = \sum_k \varphi(k\varepsilon) \frac{1+\tau(k\varepsilon)}{2} \frac{1-\tau\big((k+1)\varepsilon\big)}{2} \dots \frac{1-\tau\big((k+m-1)\varepsilon\big)}{2} \, ;$$

you see, the product vanishes unless we have the desired combination

$$\tau(k\varepsilon) = +1 \, , \quad \tau\big((k+1)\varepsilon\big) = -1 \, , \quad \dots , \quad \tau\big((k+m-1)\varepsilon\big) = -1 \, .$$

Naturally, k runs over all integers satisfying $[k\varepsilon, (k+m-1)\varepsilon] \subset [-M, M]$. Note that, unlike 1b, no small coefficient (like $\sqrt{\varepsilon}$) is stipulated before the sum. Instead, we take limits for

$$\varepsilon \to 0 \, , \quad m \to \infty \quad 2^m \varepsilon \to 1 \, ,$$

or just

$$m \to \infty \, , \quad \varepsilon = \frac{1}{2^m} \, .$$

Consider events

$$A_k = \{ \tau \in \Omega_{\varepsilon,M} : \tau(k\varepsilon) = +1, \tau\big((k+1)\varepsilon\big) = -1, \dots , \tau\big((k+m-1)\varepsilon\big) = -1 \} \, .$$

Clearly, $\mathbb{P}\left(A_k \right) = 2^{-m}$. Also,

$$\mathbb{P}\left(A_k \cap A_l \right) = \begin{cases} 2^{-m} & \text{if } k = l, \\ 2^{-2m} & \text{if } |k-l| \ge m, \\ 0 & \text{otherwise} \end{cases}$$

(think, why).

2b1 Exercise. $\operatorname{Lim}\mathbb{E}\,X_{\varepsilon,M,\varphi} = \int_{-M}^{M}\varphi(x)\,dx$ for every $\varphi:\mathbb{R}\to\mathbb{R}$, Riemann integrable on $(-M,M)$. Here "Lim" means the limit for $m\to\infty$, $\varepsilon=2^{-m}$.

Prove it.

2b2 Exercise. $\operatorname{Lim}\mathbb{E}\big(X_{\varepsilon,M,\varphi}X_{\varepsilon,M,\psi}\big) = \int_{-M}^{M}\varphi(x)\psi(x)\,dx + \big(\int_{-M}^{M}\varphi(x)\,dx\big)\big(\int_{-M}^{M}\psi(x)\,dx\big)$ for every φ,ψ Riemann integrable on $(-M,M)$.

Prove it.

If you are acquainted with Poisson processes, you probably guess that our scaling limit should be described by a random number k of random points $x_1,\dots,x_k\in(-M,M)$, namely, each k has its probability

$$\frac{(2M)^k}{k!}e^{-2M}$$

according to the Poisson distribution and, given k, the random points $x_1,\dots,x_k$ are indepe-dent, uniformly distributed on $(-M,M)$. Let us try it.

2b3 Exercise.

$$\sum_{k=0}^{\infty}\frac{(2M)^k}{k!}e^{-2M}\cdot\int\cdots\int\big(\varphi(x_1)+\dots+\varphi(x_k)\big)\,\frac{dx_1}{2M}\dots\frac{dx_k}{2M} = \int_{-M}^{M}\varphi(x)\,dx\,,$$

$$\sum_{k=0}^{\infty}\frac{(2M)^k}{k!}e^{-2M}\cdot\int\cdots\int\big(\varphi(x_1)+\dots+\varphi(x_k)\big)\big(\psi(x_1)+\dots+\psi(x_k)\big)\,\frac{dx_1}{2M}\dots\frac{dx_k}{2M} =$$
$$= \int_{-M}^{M}\varphi(x)\psi(x)\,dx + \bigg(\int_{-M}^{M}\varphi(x)\,dx\bigg)\bigg(\int_{-M}^{M}\psi(x)\,dx\bigg)$$

for all bounded measurable functions φ,ψ on $(-M,M)$.

Prove it.

In general, we get the sum over all partitions of $\{1,\dots,n\}$. Say, $\operatorname{Lim}\mathbb{E}\big(X_{\varepsilon,M,\varphi_1}X_{\varepsilon,M,\varphi_2}X_{\varepsilon,M,\varphi_3}X_{\varepsilon,M,\varphi_4}X_{\varepsilon,M,\varphi_5}\big)$ contains $\big(\int\varphi_1\big)\big(\int\varphi_2\big)\big(\int\varphi_3\big)\big(\int\varphi_4\big)\big(\int\varphi_5\big)$, and $\int\varphi_1\varphi_2\varphi_3\varphi_4\varphi_5$, and $\big(\int\varphi_1\varphi_2\varphi_3\big)\big(\int\varphi_4\varphi_5\big)$, and $\big(\int\varphi_1\varphi_2\big)\big(\int\varphi_3\varphi_4\big)\big(\int\varphi_5\big)$, etc. And the same holds for the Poisson process! That is,

$$\operatorname{Lim}\mathbb{E}\prod_{i=1}^{n}X_{\varepsilon,m,\varphi_i} = \sum_{k=0}^{\infty}\frac{(2M)^k}{k!}e^{-2M}\cdot\int\cdots\int\prod_{i=1}^{n}\big(\varphi_i(x_1)+\dots+\varphi_i(x_k)\big)\,\frac{dx_1}{2M}\dots\frac{dx_k}{2M}$$

for all n and all $\varphi_1,\dots,\varphi_n$ Riemann integrable on $(-M,M)$.

In particular, if $\varphi_1=\dots=\varphi_n=1$ on $(-M,M)$, we get

$$\operatorname{Lim}\mathbb{E}\,X^n_{\varepsilon,M,1} = \sum_{k=0}^{\infty}\frac{(2M)^k}{k!}e^{-2M}\cdot k^n = \mathbb{E}\,\nu^n_{2M}\,,$$

where ν_{2M} is a random variable distributed Poisson$(2M)$. Does it imply that

$$\operatorname{Lim}\mathbb{E}\,f(X_{\varepsilon,M,1}) = \mathbb{E}\,f(\nu_{2M}) \tag{2b4}$$

for every bounded continuous $f:\mathbb{R}\to\mathbb{R}$? Yes, it does, though it is not evident. Here are the relevant general results.

2b5 Proposition. (Carleman) Let X be a random variable having all moments (that is, $\mathbb{E}|X|^n < \infty$ for all n). If

$$\sum_n \frac{1}{\sqrt[2n]{\mathbb{E}\,X^{2n}}} = \infty$$

then every random variable Y such that $\mathbb{E}\,Y^n = \mathbb{E}\,X^n$ for all n, has the same distribution as X.

You see, such a distribution is uniquely determined by its moments.[1] For the Poisson distribution, $X \sim \mathrm{Poisson}(\lambda)$, we have

$$\sum_{n=0}^{\infty} \frac{\mathbb{E}\,X^n}{n!} t^n = \mathbb{E}\,e^{tX} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \cdot e^{tk} = e^{\lambda(e^t - 1)}$$

for all t, therefore $\mathbb{E}\,X^n = O(n!) = O(n^n)$, thus $\sqrt[2n]{\mathbb{E}\,X^{2n}} = O(n)$, and Carleman's condition is satisfied.[2]

2b6 Proposition. Let X be a random variable having all moments, such that the distribution of X is uniquely determined by its moments. Let X_k be random variables such that

$$\mathbb{E}\,X_k^n \xrightarrow[k\to\infty]{} \mathbb{E}\,X^n \quad \text{for } n = 1, 2, \dots$$

Then

$$\mathbb{E}\,f(X_k) \xrightarrow[k\to\infty]{} \mathbb{E}\,f(X)$$

for all bounded continuous functions $f : \mathbb{R} \to \mathbb{R}$.

Now (2b4) is checked; $X_{\varepsilon,M,1}$ converge in distribution to ν_{2M}. We may construct the limiting model as follows:

$$\Omega = \biguplus_{k=0}^{\infty} [-M, M]^k \,,$$

$$\mathbb{P}\,(\,A\,) = \sum_{k=0}^{\infty} \frac{(2M)^k}{k!} e^{-2M} \cdot \int \cdots \int \mathbf{1}_A(x_1, \dots, x_k) \, \frac{dx_1}{2M} \cdots \frac{dx_k}{2M}$$

for $A \subset \Omega$ (measurable). We *define*

$$\int_{-M}^{M} \varphi(x)\, d\Pi(x) = \oplus_{k=0}^{\infty} \big(\varphi(x_1) + \dots + \varphi(x_k) \big) \,;$$

that is, $\int \varphi(x)\, d\Pi(x)$ is a random variable $\Omega \to \mathbb{R}$, whose restriction to $[-M, M]^k$ is $(x_1, \dots, x_k) \mapsto \varphi(x_1) + \dots + \varphi(x_k)$. For now, $\Pi(\cdot)$ is defined only in $\int \varphi(x)\, d\Pi(x)$. However, we define

$$\Pi(x) = \begin{cases} \int \mathbf{1}_{[0,x]}(y)\, d\Pi(y) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\int \mathbf{1}_{[x,0]}(y)\, d\Pi(y) & \text{for } x < 0, \end{cases}$$

[1] In general, two different distributions can have tha same (finite) moments.

[2] Sufficiency of the stronger condition $\sqrt[n]{\mathbb{E}|X|^n} = O(n)$ is easier to prove; see Feller, vol. 2, chap. 15, sect. 4.

and so, each $\Pi(x)$ is a random variable.

2b7 Exercise.

$$\mathbb{E} \exp \Big(\int_{-M}^{M} \varphi(x)\, d\Pi(x) \Big) = \exp \Big(\int_{-M}^{M} \big(e^{\varphi(x)} - 1 \big)\, dx\,.$$

Prove it.

Hint: just calculate the sum (over $k = 0, 1, \dots$) of integrals (in $x_1, \dots, x_k$).

2b8 Exercise. The distribution of $\Pi(y) - \Pi(x)$ is Poissonian; namely,

$$\mathbb{P}\,\big(\, \Pi(y) - \Pi(x) = k \,\big) = \frac{(y-x)^k}{k!} e^{-(y-x)}$$

for $-M \le x \le y \le M$ and $0 \le k < \infty$.

Prove it.

Hint: use 2b7 for $\varphi = a\mathbf{1}_{[x,y]}$. Or just calculate...

2b9 Exercise. *("Independent increments")* For every $x, y, z \in [-M, M]$ such that $x \le y \le z$, random variables

$$\Pi(y) - \Pi(x) \quad \text{and} \quad \Pi(z) - \Pi(y)$$

are independent.

Prove it. What about three or more increments?

Hint. Use 2b7 for $\varphi = a\mathbf{1}_{[x,y]} + b\mathbf{1}_{[y,z]}$. (Or just calculate.)

Till now, M was fixed. Now we are in position to compare such constructions for $M = M_1$ and $M = M_2$, $M_1 < M_2$. If $\varphi(\cdot)$ vanishes outside of $[-M_1, M_1]$ then $\int \varphi(x)\, d\Pi(x)$ is defined twice, using M_1 and using M_2; but it is the same (in distribution). In fact, we have a measure preserving map $\Omega_{M_2} \to \Omega_{M_1}$.

Thus, we may forget any M and consider the Poisson process $\Pi(\cdot)$ on the whole $\mathbb{R}$.[3] The random variable $\int \varphi(x)\, d\Pi(x)$ is well-defined for every Riemann integrable (therefore, compactly supported) function $\varphi : \mathbb{R} \to \mathbb{R}$.

Similarly to 1b6 and 1b12 we have

$$\operatorname{Lim} \mathbb{E} f(X_{\varepsilon,M,\varphi_1}, \dots, X_{\varepsilon,M,\varphi_d} = \mathbb{E} f\Big(\int \varphi_1(x)\, d\Pi(x), \dots, \int \varphi_d(x)\, d\Pi(x) \Big)$$

for every $d \in \{1, 2, \dots\}$, every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, and every Riemann integrable $\varphi_1, \dots, \varphi_d : [-M, M] \to \mathbb{R}$. It follows from (2b4) (which is $d = 1$), and such a generalization of 1b3(b,c).

2b10 Proposition. For any d-dimensional random variables $X, X_1, X_2, \dots$ the following conditions are equivalent.

[3] Maybe, the simplest way to $\Pi(\cdot)$ is, to take *independent* Poisson processes on $(k, k+1)$ for all $k \in \mathbb{Z}$ and combine them appropriately. (Did you understand, how?)

(a) For every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E} f(X_n) \xrightarrow[n\to\infty]{} \mathbb{E} f(X) \, .$$

(b) For every $\lambda \in \mathbb{R}^d$,

$$\mathbb{E} \exp\bigl(i\langle \lambda, X_n \rangle\bigr) \xrightarrow[n\to\infty]{} \mathbb{E} \exp\bigl(i\langle \lambda, X \rangle\bigr) \, .$$

What about test functions with no compact support? Well, every bounded, integrable, and locally Riemann integrable function may be used under $\lim_{M\to\infty} \lim_{\varepsilon\to 0}(\dots)$.