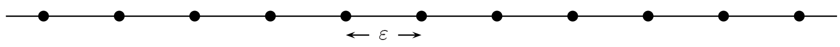
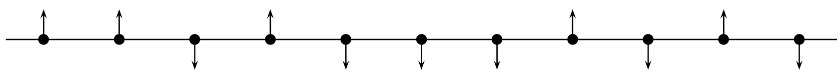
1 White noise as a scaling limit

1a A quite informal introduction

Imagine a one-dimensional array of particles situated at a small pitch ε.



Each particle has a *spin,* either 'up' or 'down'.



These 'ups and downs' are random, equiprobable (50%,50%) and independent.

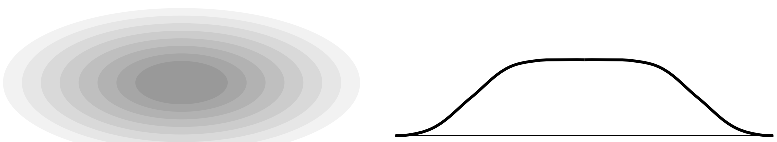
Imagine that we have spin-measuring devices, however, a single spin cannot be measured; ε is much smaller than the 'window' of a device.

infeasible feasible

Thus, a 'measurable' is not a spin $\tau(k\varepsilon)$ situated at $k\varepsilon$ but a linear combination of many spins,

$$\sum_k \varphi(k\varepsilon)\tau(k\varepsilon)\,;$$

here φ is a function that describes a measuring device (its window), called sometimes a test function.



a window its test function

1b A formalization

Given $\varepsilon \in (0,\infty)$ and $M \in (0,\infty)$, we introduce the 'configuration space'

$$\Omega_{\varepsilon,M} = \{-1,+1\}^{\varepsilon\mathbb{Z}\cap[-M,M]}\,;$$

here $\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$; $\varepsilon\mathbb{Z}$ is the infinite lattice $\{k\varepsilon : k \in \mathbb{Z}\}$; $\varepsilon\mathbb{Z}\cap[-M,M]$ is the finite portion of the lattice situated inside the interval $[-M,M]$; and $\Omega_{\varepsilon,M}$ is the set of all functions

$$\tau : \bigl(\varepsilon\mathbb{Z}\cap[-M,M]\bigr) \to \{-1,+1\}$$

called 'configurations'; clearly, the number of configurations is

$$|\Omega_{\varepsilon,M}| = 2^{|\varepsilon\mathbb{Z}\cap[-M,M]|} = 2^{1+2\,\mathrm{entier}(M/\varepsilon)}\,.$$

We introduce a probability measure $P_{\varepsilon,M}$ on $\Omega_{\varepsilon,M}$ just by letting

$$P_{\varepsilon,M}(A) = \frac{|A|}{|\Omega_{\varepsilon,M}|} \quad \text{for } A \subset \Omega_{\varepsilon,M} \, ;$$

you see, all configurations are equiprobable.

We have a probability space $(\Omega_{\varepsilon,M}, P_{\varepsilon,M})$ and random variables τ_x for $x \in \varepsilon\mathbb{Z} \cap [-M, M]$; these are i.i.d (that is, independent, identically distributed) random variables; each one takes on two values ± 1 with probabilities $0.5, 0.5$.

Given a 'test function' $\varphi : \mathbb{R} \to \mathbb{R}$, we construct a family of random variables

$$\Omega_{\varepsilon,M} \ni \tau \mapsto \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \in \mathbb{R} \, ,$$

indexed by pairs (ε, M) and defined on different probability spaces $(\Omega_{\varepsilon,M}, P_{\varepsilon,M})$. Of course, k runs over $\mathbb{Z} \cap [-M/\varepsilon, M/\varepsilon]$. The normalizing coefficient $\sqrt{\varepsilon}$ should not be unexpected, if you have at least a slight idea about limit theorems of probability theory. We could use some version of central limit theorem, but I prefer to use only such a modest argument.

1b1 Proposition. For any random variables $X_1, X_2, \dots$ the following conditions are equivalent.

(a) For every $a \in \mathbb{R}$,

$$\mathbb{P}\big(X_n \le a \big) \xrightarrow[n\to\infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} \, dx \, .$$

(b) For every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E} f(X_n) \xrightarrow[n\to\infty]{} \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \, .$$

(c) For every $\lambda \in \mathbb{R}$,

$$\mathbb{E} \exp(i\lambda X_n) \xrightarrow[n\to\infty]{} \underbrace{\int_{-\infty}^{+\infty} e^{i\lambda x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx}_{=\exp(-\frac12 \lambda^2)} \, .$$

Random variables X_n may be defined on different probability spaces. If they satisfy equivalent conditions (a)–(c), we say that X_n are *asymptotically normal* $N(0,1)$. More generally, if for some $\sigma \in (0, \infty)$ random variables $\frac{1}{\sigma} X_n$ satisfy these conditions, we say that X_n are asymptotically normal $N(0, \sigma^2)$.[1]

The random variable $X = \sqrt{\varepsilon} \sum_k \varphi(k\varepsilon)\tau(k\varepsilon)$ being a linear combination of *independent* random variables $\tau(k\varepsilon)$, we have

$$\mathbb{E} \exp(i\lambda X) = \mathbb{E} \bigg(\prod_k \exp\Big(i\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\tau(k\varepsilon) \Big) \bigg) = \\ = \prod_k \mathbb{E} \exp\Big(i\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\tau(k\varepsilon) \Big) = \prod_k \cos\Big(\lambda\sqrt{\varepsilon}\varphi(k\varepsilon) \Big)$$

[1] Still more generally, if random variables $\frac{1}{\sigma_n}(X_n - a_n)$ satisfy these conditions, we may say that X_n are asymptotically normal $N(a_n, \sigma_n^2)$.

(think, why). For small ε the approximation

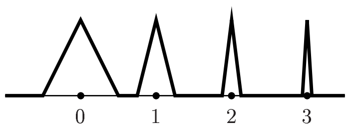
$$\cos\Big(\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\Big) \approx 1 - \frac12\varepsilon\lambda^2\varphi^2(k\varepsilon) \approx \exp\Big(-\frac12\varepsilon\lambda^2\varphi^2(k\varepsilon)\Big)\,,$$
$$\prod_k \cos\Big(\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\Big) \approx \exp\Big(-\frac12\varepsilon\lambda^2\sum_k\varphi^2(k\varepsilon)\Big)$$

suggests that

$$\prod_k \cos\Big(\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\Big) \xrightarrow[\varepsilon\to0]{} \exp\bigg(-\frac12\lambda^2\int_{-M}^{+M}\varphi^2(x)\,dx\bigg)\,;$$

that is true, if the test function φ is Riemann integrable on $[-M,M]$. That is, φ is bounded on $[-M,M]$ and continuous almost everywhere.[2] In particular, it holds for piecewise continuous functions. For such φ we conclude that $\sqrt{\varepsilon}\sum_k\varphi(k\varepsilon)\tau(k\varepsilon)$ is asymptotically normal $N(0,\sigma^2)$ where $\sigma^2=\int_{-M}^M\varphi^2(x)\,dx$.

In order to escape the finite interval $[-M,M]$ we may use the double limit $\lim_{\varepsilon\to0,M\to\infty}$, or the iterated limit $\lim_{M\to\infty}\lim_{\varepsilon\to0}$, or something like $\lim_{\varepsilon\to0,M=1/\varepsilon}$. The iterated limit gives asymptotical normality $N(0,\sigma^2)$ where $\sigma^2=\int_{-\infty}^{+\infty}\varphi^2(x)\,dx$ for every locally Riemann integrable $\varphi\in L_2(\mathbb{R})$. Other limits are more demanding to φ; for example, such a function



makes troubles (think, why).

From now on (till the end of Sect. 1) a *test function* means an element of a linear space of functions $\varphi:\mathbb{R}\to\mathbb{R}$ such that every φ of the space satisfies $\int_{-\infty}^{+\infty}\varphi^2(x)\,dx<\infty$ and

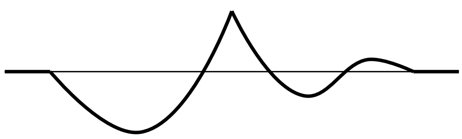
$$\operatorname{Lim}\prod_{k:k\varepsilon\in[-M,M]}\cos\big(\lambda\sqrt{\varepsilon}\varphi(k\varepsilon)\big) = \exp\bigg(-\frac12\lambda^2\int_{-\infty}^{+\infty}\varphi^2(x)\,dx\bigg)\,;$$

here and henceforth (till the end of the section) $\operatorname{Lim}(\dots)$ means one of the two limiting procedures

$$\operatorname{Lim}(\dots)\ :\quad \text{either}\quad \lim_{M\to\infty}\lim_{\varepsilon\to0}(\dots)\quad\text{or}\quad\lim_{M\to\infty,\varepsilon\to0}(\dots)\,.$$

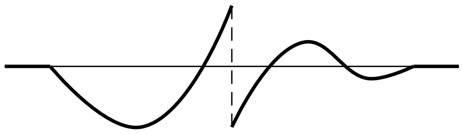
That is, we have some freedom in choosing Lim, and some freedom in choosing the space of test functions, but the two choices must be compatible.

We may restrict ourselves to compactly supported test functions. In that case there is no distinction between different choices of Lim (think, why). Continuity of φ is sufficient.

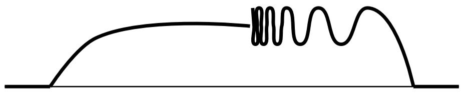


[2]In other words, points of discontinuity of f form a set of Lebesgue measure zero. Note that the indicator function of the set of rational numbers does not satisfy the condition.

Piecewise continuity is also sufficient.



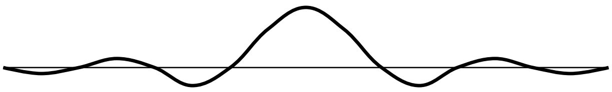
Piecewise continuity means a finite set of discontinuity points. Arbitrary discontinuities (not just jumps) are allowed, as far as φ is bounded.



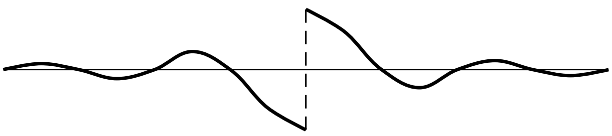
A countable set of discontinuity points is also sufficient (together with boundedness). Moreover, it is still sufficient, if the set is of Lebesgue measure zero, that is, φ is continuous almost everywhere (and bounded).

We may also consider φ with no compact support. It suffices (for every choice of Lim) if φ is continuously differentiable and

$$|\varphi(x)| = O\Big(\frac{1}{|x|}\Big) \quad \text{and} \quad |\varphi'(x)| = O\Big(\frac{1}{|x|}\Big) \quad \text{for } |x| \to \infty\,.$$



A finite set of jumps does not harm.



If we strive to a large class of test functions, we may take $\mathrm{Lim}(\dots) = \lim_{M\to\infty} \lim_{\varepsilon\to 0}(\dots)$, in which case it is sufficient if φ is continuous almost everywhere, bounded on $[-M, M]$ for all M, and $\int_{-\infty}^{+\infty} \varphi^2(x)\,dx < \infty$.

1b2 Corollary. For every test function φ such that $\int_{-\infty}^{+\infty} \varphi^2(x)\,dx = 1$, random variables $\sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon)$ are asymptotically normal $N(0,1)$ in the sense of $\mathrm{Lim}(\dots)$.

Two test functions φ_1, φ_2 determine two random variables $X_1 = \sqrt{\varepsilon}\sum_k \varphi_1(k\varepsilon)\tau(k\varepsilon)$, $X_2 = \sqrt{\varepsilon}\sum_k \varphi_2(k\varepsilon)\tau(k\varepsilon)$. Each one is asymptotically normal. However, they are dependent; what happens in the limit to their joint distribution? Here we need a multidimensional generalization of Prop. 1b1.

1b3 Proposition. For any d-dimensional random variables $X_1, X_2, \dots$ the following conditions are equivalent.

(a) For every $a \in \mathbb{R}^d$,

$$\mathbb{P}\,\big(\,X_n \le a\,\big) \xrightarrow[n\to\infty]{} (2\pi)^{-d/2} \int_{x \le a} e^{-|x|^2/2}\,dx\,;$$

here the inequalities ($X_n \le a$, $x \le a$) are treated coordinate-wise, and $|x|$ is the usual (Euclidean) norm of x.

(b) For every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\mathbb{E}\, f(X_n) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} f(x)(2\pi)^{-d/2} e^{-|x|^2/2}\,dx\,.$$

(c) For every $\lambda \in \mathbb{R}^d$,

$$\mathbb{E}\,\exp(i\langle \lambda, X_n\rangle) \xrightarrow[n\to\infty]{} \underbrace{\int_{\mathbb{R}^d} e^{i\langle\lambda,x\rangle}(2\pi)^{-d/2} e^{-|x|^2/2}\,dx}_{=\exp(-\frac12|\lambda|^2)}\,;$$

here $\langle\cdot,\cdot\rangle$ is the usual (Euclidean) scalar product.

Note that the density $(2\pi)^{-d/2}e^{-|x|^2/2}$ describes d independent normal $N(0,1)$ random variables.

If X_n satisfy these conditions we say that X_n are asymptotically multinormal $N(0,1)^{\otimes d}$.

1b4 Exercise. Let d-dimensional random variables X_n be such that for every $\lambda \in \mathbb{R}^d$, $|\lambda| = 1$, one-dimensional random variables $\langle \lambda, X_n\rangle$ are asymptotically normal $N(0,1)$. Then X_n are asymptotically multinormal $N(0,1)^{\otimes d}$.

Prove it.

Hint: look at 1b3(c).

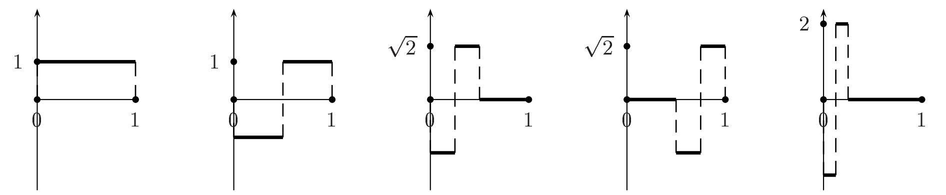
Here is a multidimensional generalization of 1b2.

1b5 Corollary. Let $\varphi_1, \dots, \varphi_d$ be orthonormal[3] test functions. Then d-dimensional random variables $X_{\varepsilon,M}$ defined by

$$(X_{\varepsilon,M})_i = \sqrt{\varepsilon}\sum_k \varphi_i(k\varepsilon)\tau(k\varepsilon)$$

are asymptotically multinormal $N(0,1)^{\otimes d}$.[4]

A convenient orthonormal basis of $L_2(0,1)$ is well-known; I mean the Haar basis:



and so on

[3]That is, $\int \varphi_k(x)\varphi_l(x)\,dx$ is equal to 0 for $k \ne l$ (orthogonality) and 1 for $k = l$ (normalization).

[4]In the sense of $\mathrm{Lim}(\dots)$.

Orthogonality is evident. Completeness follows from the fact that the first 2 Haar functions span all step functions constant on $(0,1/2)$ and $(1/2,1)$; the first 4 Haar functions span all step functions constant on $(0,1/4)$, $(1/4,1/2)$, $(1/2,3/4)$ and $(3/4,1)$; and so on.

We reproduce the Haar basis on $(k,k+1)$ for each $k \in \mathbb{Z}$ and join them all; thus we get a countable set of functions (though, with no natural order), and it evidently is an orthonormal basis of $L_2(\mathbb{R})$, consisting of Riemann integrable functions.

We want to construct a mathematical object that may be called the scaling limit,

$$\operatorname{Lim}(\Omega_{\varepsilon,M}, P_{\varepsilon,M}) \,,$$

of our discrete model.[5] It should be some probability space $(\Omega,\mathcal{F},P)$, and every test function φ should determine a random variable on $(\Omega,\mathcal{F},P)$. Roughly speaking, the random variable is

$$\operatorname{Lim}\biggl(\sqrt{\varepsilon} \sum_{k:k\varepsilon\in[-M,M]} \varphi(k\varepsilon)\tau(k\varepsilon) \biggr) \,,$$

which is quite informal, since random variables on different probability spaces cannot converge; rather, we mean convergence of their distributions. It is convenient to denote the limiting random variable by

$$\int_{-\infty}^{+\infty} \varphi(x)\, dB(x) \,,$$

but it is just a notation, not yet a definition; and $B(x)$ itself means nothing for now; we only have a vague idea that $B(x+\varepsilon) - B(x) \approx \sqrt{\varepsilon}\,\tau(k\varepsilon)$ in some sence.[6]

Consider the Haar basis $(\varphi_1,\varphi_2,\dots)$ (no matter how numbered); accordingly to Corollary 1b4, random variables $\int \varphi_i(x)\, dB(x)$ should be independent normal $N(0,1)$. Their joint distribution is a probability measure γ^∞ on the space $\mathbb{R}^\infty$ of all (infinite) sequences of real numbers; it is the product measure

$$\gamma^\infty = \gamma^1 \otimes \gamma^1 \otimes \dots$$

where γ^1 is the same as $N(0,1)$, the standard normal distribution on $\mathbb{R}$. In other words, for every $d \in \{1,2,\dots\}$ and $a_1,\dots,a_d \in \mathbb{R}$,

$$\gamma^\infty\bigl(\{ (\alpha_1,\alpha_2,\dots) : \alpha_1 \le a_1, \dots, \alpha_d \le a_d \} \bigr) = \gamma^1\bigl((-\infty,a_1] \bigr) \cdot \ldots \cdot \gamma^1\bigl((-\infty,a_d] \bigr) \,,$$
$$\gamma^1\bigl((-\infty,a] \bigr) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-u^2/2}\, du \,.$$

We may take $(\Omega,\mathcal{F},P) = (\mathbb{R}^\infty,\gamma^\infty)$; that is, $\Omega = \mathbb{R}^\infty$, $P = \gamma^\infty$, and $\mathcal{F}$ is the σ-field of all γ^∞-measurable sets (basically, Borel sets, but also all sets of γ^∞-measure zero). Coordinate functionals $\zeta_i : \Omega \to \mathbb{R}$ defined by

$$\zeta_i\bigl((\alpha_1,\alpha_2,\dots) \bigr) = \alpha_i \,,$$

are independent normal $N(0,1)$ random variables on $(\Omega,\mathcal{F},P)$. Now we *define*

$$\int \varphi_i(x)\, dB(x) = \zeta_i \,.$$

[5] It is, however, a misleading notation, for a reason to be explained in next sections.

[6] However, we'll see soon that $B(\cdot)$ is nothing but the famous Brownian motion...

1b6 Exercise. For every $d \in \{1, 2, \dots\}$ and every bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{M\to\infty} \lim_{\varepsilon\to 0} \mathbb{E}\, f\Big(\sqrt{\varepsilon} \sum_k \varphi_1(k\varepsilon)\tau(k\varepsilon), \dots, \sqrt{\varepsilon} \sum_k \varphi_d(k\varepsilon)\tau(k\varepsilon) \Big) = \\ = \mathbb{E}\, f\Big(\int \varphi_1(x)\, dB(x), \dots, \int \varphi_d(x)\, dB(x) \Big) .$$

Prove it.[7]

That is a correct interpretation of the incorrect relation

$$\operatorname{Lim} \Big(\sqrt{\varepsilon} \sum_{k: k\varepsilon\in[-M,M]} \varphi(k\varepsilon)\tau(k\varepsilon) \Big) = \int_{-\infty}^{+\infty} \varphi(x)\, dB(x) . \tag{1b7}$$

Though, these $\int \varphi(x)\, dB(x)$ are of little use for now; the equality is just

$$\operatorname{Lim} \mathbb{E}\, f\Big(\sqrt{\varepsilon} \sum_k \varphi_1(k\varepsilon)\tau(k\varepsilon), \dots, \sqrt{\varepsilon} \sum_k \varphi_d(k\varepsilon)\tau(k\varepsilon) \Big) = \mathbb{E}\, f(\zeta_1, \dots, \zeta_d) = \int_{\mathbb{R}^d} f\, d\gamma^d , \tag{1b8}$$

where $\gamma^d = \gamma^1 \otimes \dots \otimes \gamma^1$ is the standard d-dimensional normal distribution. The specific form of the Haar basis was not really used; (1b8) holds equally well for any test functions $\varphi_1, \varphi_2, \dots$ as far as they are orthonormal.

Is it reasonable, to identify ζ_i with $\int \varphi_i(x)\, dB(x)$ just for the Haar basis (φ_i) ? Well, it is not essential, but harmless. We do not need $\int \varphi_i(x)\, dB(x)$ to be ζ_i; we only need them to be orthonormal. Fortunately, we can do it at once for all orthonormal systems (φ_i).

Namely, we choose an orthonormal basis (φ_i) in $L_2(\mathbb{R})$ (you may use the Haar basis or another; moreover, here φ_i need not be Riemann integrable) and define $\int \varphi_i(x)\, dB(x) = \zeta_i$, or rather,

$$\int \Big(\sum_i c_i \varphi_i(x) \Big)\, dB(x) = \sum_i c_i \zeta_i \tag{1b9}$$

for all $c_1, c_2, \dots$ such that $c_1^2 + c_2^2 + \dots < \infty$. Thus we define $\int \varphi(x)\, dB(x)$ for all $\varphi \in L_2(\mathbb{R})$ and the following conditions are satisfied:

$$\int c\varphi(x)\, dB(x) = c \int \varphi(x)\, dB(x) ,$$

$$\int \big(\varphi(x) + \psi(x) \big)\, dB(x) = \int \varphi(x)\, dB(x) + \int \psi(x)\, dB(x) ,$$

$$\mathbb{E} \Big(\int \varphi(x)\, dB(x) \Big) = 0 ,$$

$$\operatorname{Var} \Big(\int \varphi(x)\, dB(x) \Big) = \int \varphi^2(x)\, dx ,$$

$$\operatorname{Cov} \Big(\int \varphi(x)\, dB(x), \int \psi(x)\, dB(x) \Big) = \int \varphi(x)\psi(x)\, dx .$$

[7] You may also think on more general functions f. What if f is continuous almost everywhere? What if $|f(x)| = O(|x|^2)$ for $|x| \to \infty$?

1b10 Exercise. Prove these equalities.

1b11 Exercise. Prove that 1b7 remains true for every orthonormal basis (φ_i) consisting of test functions (not just the basis used in (1b9)).

1b12 Exercise. Prove the same for arbitrary (not just orthonormal) test functions $\varphi_i \in L_2(\mathbb{R})$.

Hint: non-orthogonal functions are linear combinations of some orthogonal functions.

Till now, the symbol B was well-defined only in expressions of the form $\int \varphi(x)\,dB(x)$. Now we define $B(x)$ for $x \in \mathbb{R}$:

$$B(x) = \begin{cases} \int \mathbf{1}_{[0,x]}(y)\,dB(y) & \text{for } x > 0, \\ 0 & \text{for } x = 0, \\ -\int \mathbf{1}_{[-x,0]}(y)\,dB(y) & \text{for } x < 0. \end{cases}$$

Thus, each $B(x)$ is a random variable, distributed normally $N(0,|x|)$.

1b13 Exercise. *("Independent increments")* For every $x,y,z \in \mathbb{R}$ such that $x \le y \le z$, random variables

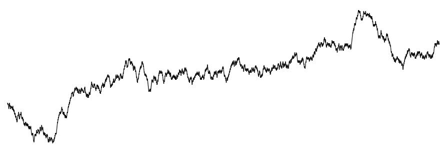
$$B(y) - B(x) \quad \text{and} \quad B(z) - B(y)$$

are independent.

Prove it. What about three or more increments?

Hint. Consider test functions $\mathbf{1}_{[x,y]}$ and $\mathbf{1}_{[y,z]}$.

We have a family $\big(B(x)\big)_{x\in\mathbb{R}}$ of random variables $B(x)$ defined on the same probability space. Such an object is called a random process.[8] It is well-known that its sample functions are continuous[9] (with probability 1) but nowhere differentiable (with probability 1); however, that is another story...



1c Two-dimensional white noise

Imagine now that we have spin-measuring devices of two types:

$$\sqrt{\varepsilon}\sum_k \varphi(k\varepsilon)\tau(k\varepsilon) \quad \text{and} \quad \sqrt{\varepsilon}\sum_k (-1)^k \varphi(k\varepsilon)\tau(k\varepsilon)\,.$$

The first type is the same as before, but the second type is new. For every given ε and φ there exists ψ such that $\psi(k\varepsilon) = (-1)^k\varphi(k\varepsilon)$ for all $k \in \mathbb{Z}$; for example, $\psi(x) = \cos(\pi x/\varepsilon)\varphi(x)$. No

[8]According to one of several *non-equivalent* definitions of a random process.

[9]More exactly: the restriction of a sample function to the dense countable set of (say) rational numbers x is uniformly continuous in x (with probability 1) and therefore may be extended by continuity.

essential distinction between the two types for a given ε. However, the distinction becomes essential in the scaling limit, as we'll see soon.

Test functions and "Lim" are the same as in 1b.

1c1 Exercise. For every test functions φ, ψ such that $\int_{-\infty}^{+\infty} \big(\varphi^2(x) + \psi^2(x)\big)\, dx = 1$, random variables $\sqrt{\varepsilon} \sum_k \big(\varphi(k\varepsilon) + (-1)^k \psi(k\varepsilon)\big) \tau(k\varepsilon)$ are asymptotically normal $N(0,1)$ in the sense of $\operatorname{Lim}(\dots)$.

Prove it.

Hint: consider separately the two sublattices of $\varepsilon\mathbb{Z}$ (even and odd).

1c2 Exercise. Let $\varphi_1, \dots, \varphi_d$ be orthonormal test functions. The $2d$-dimensional random variables formed by

$$\sqrt{\varepsilon} \sum_k \varphi_i(k\varepsilon) \tau(k\varepsilon) \,, \quad \sqrt{\varepsilon} \sum_k (-1)^k \varphi_i(k\varepsilon) \tau(k\varepsilon) \quad (i = 1, \dots, d)$$

are asymptotically multinormal $N(0,1) \otimes \dots \otimes N(0,1)$.

Prove it.

Hint: similar to 1b4.

The two-dimensional counterpart of (1b7) is such a pair of incorrect relations:

$$\operatorname{Lim} \Bigg(\sqrt{\varepsilon} \sum_{k: k\varepsilon \in [-M,M]} \varphi(k\varepsilon) \tau(k\varepsilon) \Bigg) = \int_{-\infty}^{+\infty} \varphi(x) \, dB_1(x) \,,$$
$$\operatorname{Lim} \Bigg(\sqrt{\varepsilon} \sum_{k: k\varepsilon \in [-M,M]} (-1)^k \varphi(k\varepsilon) \tau(k\varepsilon) \Bigg) = \int_{-\infty}^{+\infty} \varphi(x) \, dB_2(x) \,,$$

where $B_1(\cdot), B_2(\cdot)$ are two independent Brownian motions. In other words, the pair $\big(B_1(\cdot), B_2(\cdot)\big)$ is a 2-dimensional Brownian motion. For every α, the process $x \mapsto B_1(x) \cos\alpha + B_2(x) \sin\alpha$ is another Brownian motion, as well as the process $x \mapsto -B_1(x) \sin\alpha + B_2(x) \cos\alpha$; and these two new processes are independent (think, why).

1c3 Exercise. Both $\sqrt{\varepsilon} \sum_k \varphi(k\varepsilon) \tau(k\varepsilon)$ and $\sqrt{\varepsilon} \sum_k (-1)^k \varphi(k\varepsilon) \tau(k\varepsilon)$ are special cases of $\sqrt{\varepsilon} \sum_k e^{i\lambda k} \varphi(k\varepsilon) \tau(k\varepsilon)$ (namely, $\lambda = 0$ and $\lambda = \pi$). What about arbitrary λ? What about $\sqrt{\varepsilon} \sum_k \cos(\lambda k) \varphi(k\varepsilon) \tau(k\varepsilon)$ and $\sqrt{\varepsilon} \sum_k \sin(\lambda k) \varphi(k\varepsilon) \tau(k\varepsilon)$?