Exam of 26.07.2007 — Solutions

1

The function $h(i) = P_i(V_{10} < V_{-1})$ satisfies

(*) $$h(i) = \frac{1}{2}h(i-1) + \frac{1}{2}h(i+1) \quad \text{for } i = 1, \dots, 9\,;$$

(**) $$h(0) = \frac{1}{11}h(-1) + \frac{10}{11}h(1)\,;$$

(***) $$h(-1) = 0\,, \quad h(10) = 1\,.$$

By (*), h is linear on $\{0, \dots, 10\}$ (since $h(i) - h(i-1) = h(i+1) - h(i)$ for $i = 1, \dots, 9$); therefore $10\big(h(1) - h(0)\big) = h(10) - h(0)$. By (**), $h(0) - h(-1) = 10\big(h(1) - h(0)\big)$. Therefore $h(0) - h(-1) = h(10) - h(0)$ and $h(0) = \frac{1}{2}h(-1) + \frac{1}{2}h(10) = \frac{1}{2}$. This is the answer.

2

Let M_n be the total strength of all A-monsters after n fights (or after the last fight, if n exceeds the total number of fights). We have $M_0 = a_1 + \dots + a_{10}$ and

$$M_1 - M_0 = \begin{cases} b_1 & \text{with probability } \frac{a_1}{a_1+b_1}\,, \\ -a_1 & \text{with probability } \frac{b_1}{a_1+b_1}\,, \end{cases}$$

therefore $\mathbb{E}\,(M_1 - M_0) = 0$. After n fights (if the game is not over yet), given the past, a fight between an A-monster of some strength a and a B-monster of some strength b (these a, b depending on the past) leads to

$$M_{n+1} - M_n = \begin{cases} b & \text{with (conditiional) probability } \frac{a}{a+b}\,, \\ -a & \text{with (conditiional) probability } \frac{b}{a+b}\,, \end{cases}$$

therefore $\mathbb{E}\,\big(\, M_{n+1} - M_n \,\big|\, \text{the past}\,\big) = 0$, which means that the process M_n is a martingale.

For n large enough (in fact, for $n \ge 19$) we have either $M_n = 0$ or $M_n = a_1 + \dots + a_{10} + b_1 + \dots + b_{10}$ (since the game is over). Then

$$a_1 + \dots + a_{10} = \mathbb{E}\, M_0 = \mathbb{E}\, M_n = (a_1 + \dots + a_{10} + b_1 + \dots + b_{10}) \cdot \mathbb{P}\,\big(\, M_n \ne 0\,\big)\,,$$

therefore

$$\mathbb{P}\,\big(\, M_n = 0\,\big) = \frac{b_1 + \dots + b_{10}}{a_1 + \dots + a_{10} + b_1 + \dots + b_{10}}\,.$$

The probability does not depend on the order of the A-monsters.

3

3a

Random variables X_1 and $X_2 - X_1$ are independent and identically distributed, therefore $\mathbb{P}\left(X_1 < X_2 - X_1 \right) = \mathbb{P}\left(X_1 > X_2 - X_1 \right)$. Taking into account that $\mathbb{P}\left(X_1 = X_2 - X_1 \right) = 0$ (since the distribution is nonatomic) we get $\mathbb{P}\left(X_1 < X_2 - X_1 \right) = 0.5$. Thus, $\mathbb{P}\left(X_1 < \frac{1}{2} X_2 \right) = 0.5$.

Alternatively (if you prefer), $\mathbb{P}\left(X_1 < X_2 - X_1 \right) = \frac{\lambda}{\lambda+\lambda} = 0.5$ by the 'Exponential race' formula (see the textbook, (1.8) on page 128).

3b

According to the 'Poisson race' approach (see the textbook, Example 3.4 on page 141) we treat $X_1, X_2, \dots$ as 'red arrivals' with a rate λ, and $2Y_1, 2Y_2, \dots$ as 'green arrivals' with the rate $\mu = \lambda/2$. Now, $\mathbb{P}\left(Y_1 < \frac{1}{2} X_2 \right)$ is the probability that we will get the first green arrival before the second red arrival. That is, at least one green arrival in the first two. Each of these two arrivals is red with probability $\frac{\lambda}{\lambda+\mu} = \frac{2}{3}$. Thus,

$$\mathbb{P}\left(Y_1 < \frac{1}{2} X_2 \right) = 1 - \mathbb{P}\left(\text{both are red} \right) = 1 - \frac{2}{3} \cdot \frac{2}{3} = \frac{5}{9} \,.$$

4

We have two independent $M/M/1$ queues (see the textbook, Example 4.1 on page 176) with $\mu = 2\lambda$. The stationary distribution is

$$\pi(n) = \frac{1}{2} \cdot \left(\frac{1}{2} \right)^n \quad \text{for } n = 0, 1, 2, \dots$$

(for each queue). Therefore

$$\mathbb{P}\left(X = 2 \right) = \pi(0)\pi(2) + \pi(1)\pi(1) + \pi(2)\pi(0) = \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{2} = \frac{3}{16} \,.$$

5

We have an $M/M/s$ queue (see the textbook, Example 4.3 on page 179) with $s = 2$, input rate 2λ, and $\mu = 2\lambda$. Detailed balance:

$$\pi(0) = \pi(1), \; \pi(1) = 2\pi(2), \; \pi(2) = 2\pi(3), \; \dots$$

The stationary distribution is

$$\pi(0) = \frac{1}{3}, \; \pi(1) = \frac{1}{3}, \; \pi(2) = \frac{1}{6}, \; \dots$$

Therefore

$$\mathbb{P}\left(X = 2 \right) = \pi(2) = \frac{1}{6} \,.$$