4 Distributions (generalized functions)

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Waiving sharp values $f(t)$ and retaining only smoothed averages $\int f(t)\varphi(t)\,\mathrm{d}t$ we get a generalized function, called a distribution. Differentiation becomes always possible. Fourier transform becomes more transparent.

4a Introduction: how to measure $f(t)$?

A usual medical thermometer cannot measure the temperature at a given millisecond, but only the temperature averaged over many seconds. Faster thermometers exist, but still, each thermometer averages the temperature over some time interval. The same holds for any other physical variable; no measuring device can measure an instant value.

Similarly, a voltmeter averages the voltage over some time. An AC (alternating current) voltage can be measured by a rectifier and a voltmeter. But also the rectifier averages over time.[1] Such behavior is typical for nonlinear devices. In this sense,

$$\sin \omega t \to 0 \quad \text{as } \omega \to \infty\,;$$

a high-frequency input produces a weak output. How to adapt mathematical analysis to this physical idea?

To this end we stop treating $\sin \omega t$ as a family of instant values $(\sin \omega t)_{t\in\mathbb{R}}$ and start treating it as a family of averaged values

$$\Big(\int \varphi(t) \sin \omega t \,\mathrm{d}t \Big)_{\varphi\in\Phi}$$

where Φ is a class of functions (to be specified later). Each $\varphi \in \Phi$ (so-called test function) describes a possible measuring device. We get

$$\int \varphi(t) \sin \omega t \,\mathrm{d}t \to 0 \quad \text{as } \omega \to \infty$$

[1] A high-speed rectifier able to work on the infrared frequencies (terahertz range) was proposed in 2007.

for each $\varphi \in \Phi$ (provided that Φ is chosen appropriately).

4b Continuous test functions

First derivative of a step function appears.

ON A CIRCLE

We start with the Banach space $C(\mathbb{T} \to \mathbb{R})$ of all real-valued continuous functions on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (or equivalently, 1-periodic continuous functions $\mathbb{R} \to \mathbb{R}$) with the norm $\|\varphi\| = \max |\varphi(\cdot)|$.

The dual Banach space $C(\mathbb{T} \to \mathbb{R})^*$ consists, by definition, of bounded linear functionals

$$T : C(\mathbb{T} \to \mathbb{R}) \to \mathbb{R}\,, \quad \langle T, \varphi \rangle \le \|T\| \cdot \|\varphi\|\,;$$

$\langle T, \varphi \rangle$ is just a convenient notation for $T(\varphi)$.

4b1 Example. Given $f \in L_1(\mathbb{T} \to \mathbb{R})$, we introduce $T \in C(\mathbb{T} \to \mathbb{R})^*$ by

$$\langle T, \varphi \rangle = \int_0^1 f(x)\varphi(x)\,\mathrm{d}x\,.$$

4b2 Example. ("Dirac delta-function") Given $x \in \mathbb{T}$, we introduce $\delta_x \in C(\mathbb{T} \to \mathbb{R})^*$ by

$$\langle \delta_x, \varphi \rangle = \varphi(x)\,.$$

4b3 Example. ("Positive Radon measure") Given a bounded increasing function $F : [0, 1) \to \mathbb{R}$, we introduce $T \in C(\mathbb{T} \to \mathbb{R})^*$ by the Riemann-Stieltjes integral

$$\langle T, \varphi \rangle = \int_{[0,1)} \varphi(x)\,\mathrm{d}F(x)\,.$$

Note that 4b2 is a special case of 4b3. Also 4b1 is a special case of 4b3 if $f(\cdot) \ge 0$.

In fact, the general form of $T \in C(\mathbb{T} \to \mathbb{R})^*$ is given by 4b3 with a function $F : [0, 1) \to \mathbb{R}$ of bounded variation (equivalently: the difference of two bounded increasing functions); these T are called real (or "signed") Radon measures on $\mathbb{T}$.

The complex-valued case is similar: either $T : C(\mathbb{T} \to \mathbb{R}) \to \mathbb{C}$ is linear over $\mathbb{R}$, or equivalently, $T : C(\mathbb{T} \to \mathbb{C}) \to \mathbb{C}$ is linear over $\mathbb{C}$; in both cases we get the space $C(\mathbb{T})^* = C(\mathbb{T} \to \mathbb{C})^*$ of complex Radon measures on $\mathbb{T}$.

We treat 4b1 as a linear embedding of $L_1(\mathbb{T})$ into $C(\mathbb{T})^*$,

$$\langle f, \varphi \rangle = \int_0^1 f(x)\varphi(x)\,\mathrm{d}x \quad \text{for } f \in L_1(\mathbb{T}) \subset C(\mathbb{T})^*\,;$$

thus,

$$C(\mathbb{T}) \subset L_1(\mathbb{T}) \subset C(\mathbb{T})^*\,.$$

In fact, these embeddings are one-to-one.

Note that we use here *bilinear* forms $\langle \cdot, \cdot \rangle$, while in Sections 1–3 $\langle \cdot, \cdot \rangle$ stands for the *Hermitian* form $\langle f, g \rangle = \int f(x)\overline{g(x)}\,\mathrm{d}x$.

Functions $n1\!\!1_{(0,1/n)} \in L_1(\mathbb{T})$ are not a Cauchy sequence in the L_1 metric. However,

$$\langle n1\!\!1_{(0,1/n)}, \varphi \rangle \to \langle \delta_0, \varphi \rangle \quad \text{as } n \to \infty$$

for every $\varphi \in C(\mathbb{T})$.

4b4 Definition. Let $T_n, T \in C(\mathbb{T})^*$. We say that $T_n \to T$ in $C(\mathbb{T})^*$ if

$$\forall \varphi \in C(\mathbb{T}) \quad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle\,.$$

We see that

$$n1\!\!1_{(0,1/n)} \to \delta_0 \quad \text{in } C(\mathbb{T})^*\,.$$

Also,

$$x_n \to x \quad \text{implies} \quad \delta_{x_n} \to \delta_x\,.$$

The embedding of $L_1(\mathbb{T})$ into $C(\mathbb{T})^*$ is continuous:

$$\text{if } f_n \to f \text{ in } L_1(\mathbb{T}) \quad \text{then } f_n \to f \text{ in } C(\mathbb{T})^*$$

(think, why). In fact, $L_1(\mathbb{T})$ is dense in $C(\mathbb{T})^*$, and $C(\mathbb{T})$ is dense in $L_1(\mathbb{T})$ and in $C(\mathbb{T})^*$.

The group $\mathbb{T}$ acts by shifts on $L_1(\mathbb{T})$, as well as $C(\mathbb{T})$,

$$(S_x f)(y) = f(x+y) \quad \text{for } f \in L_1(\mathbb{T})\,; \quad (S_x\varphi)(y) = \varphi(x+y) \quad \text{for } \varphi \in C(\mathbb{T})\,.$$

Can we extend $S_x : L_1(\mathbb{T}) \to L_1(\mathbb{T}) \subset C(\mathbb{T})^*$ continuously to $C(\mathbb{T})^* \to C(\mathbb{T})^*$? If $L_1(\mathbb{T}) \ni f_n \to T \in C(\mathbb{T})^*$ in $C(\mathbb{T})^*$ then for all $\varphi \in C(\mathbb{T})$ we have

$$\langle S_x f_n, \varphi \rangle = \langle f_n, S_{-x}\varphi \rangle \to \langle T, S_x\varphi \rangle\,.$$

Defining $S_x T$ by

$$\langle S_x T, \varphi \rangle = \langle T, S_{-x}\varphi \rangle \quad \text{for } \varphi \in C(\mathbb{T})$$

we observe that $S_x : C(\mathbb{T})^* \to C(\mathbb{T})^*$ is continuous, that is,

$$T_n \to T \quad \text{implies} \quad S_x T_n \to S_x T \,,$$

and extends $S_x : L_1(\mathbb{T}) \to L_1(\mathbb{T})$:

$$\begin{tikzcd} L_1(\mathbb{T}) \arrow[r, hook] \arrow[d, "S_x"] & C(\mathbb{T})^* \arrow[d, "S_x"] \\ L_1(\mathbb{T}) \arrow[r, hook] & C(\mathbb{T})^* \end{tikzcd}$$

In fact it is the (unique) extension by continuity, since $L_1(\mathbb{T})$ is dense in $C(\mathbb{T})^*$.

4b5 Exercise. Check that $S_x \delta_y = \delta_{y-x}$.

If $f \in C^1(\mathbb{T}) \subset L_1(\mathbb{T}) \subset C(\mathbb{T})^*$ (a smooth f), then

$$\frac{S_\varepsilon f - f}{\varepsilon} \to f' \quad \text{as } \varepsilon \to 0$$

(check it).

We say that $T \in C(\mathbb{T})^*$ is differentiable, if the following limit exists:

$$T' = \lim_{\varepsilon \to 0} \frac{S_\varepsilon T - T}{\varepsilon} \,.$$

4b6 Exercise. The indicator function $\mathbb{1}_{(0,0.5)}$, treated as an element of $C(\mathbb{T})^*$, is differentiable, and

$$\mathbb{1}'_{(0,0.5)} = \delta_0 - \delta_{0.5} \,.$$

(a) Prove it.
(b) decide, whether

$$\left\| \frac{S_\varepsilon T - T}{\varepsilon} - \big(\delta_0 - \delta_{0.5} \big) \right\|_{C(\mathbb{T})^*} \to 0 \quad \text{as } \varepsilon \to 0 \,,$$

or not?

4b7 Exercise. δ_0 is not differentiable in $C(\mathbb{T})^*$.

Prove it.

4b8 Exercise. ("Integration by parts") If $T \in C(\mathbb{T})^*$ is differentiable then

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \quad \text{for all } \varphi \in C^1(\mathbb{T}) \subset C(\mathbb{T}) \,.$$

Prove it.

In fact, every bounded increasing function $F : [0,1) \to \mathbb{R}$, treated as an element of ($L_1(\mathbb{T})$ and therefore of) $C(\mathbb{T})^*$, is differentiable, and

$$\langle F', \varphi \rangle = \int_{[0,1)} \varphi(x) \, \mathrm{d}F(x) + \big(F(0) - F(1-) \big) \varphi(0) \, .$$

The same holds for functions F of bounded variation.

ON AN OPEN INTERVAL

Denote by $D_0(0,1)$ the linear (mot Banach) space of all continuous functions $(0,1) \to \mathbb{C}$ with compact supports inside $(0,1)$. Define convergence in $D_0(0,1)$ as follows: for $\varphi, \varphi_n \in D_0(0,1)$,

$$\begin{array}{ccc} \varphi_n \to \varphi & \text{iff} & \varphi_n \to \varphi \text{ uniformly, and the supports of all } \varphi_n \\ \text{in } D_0(0,1) & & \text{are contained in a single compact subset of } (0,1). \end{array}$$

Accordingly, $D_0'(0,1)$ consists of all linear functionals $T : D_0(0,1) \to \mathbb{C}$ such that

$$\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \quad \text{whenever} \quad \varphi_n \to \varphi \text{ in } D_0(0,1) \, .$$

4b9 Example. Let $f : (0,1) \to \mathbb{C}$ be (measurable and) locally integrable, that is, $\int_a^b |f(x)| \, \mathrm{d}x < \infty$ whenever $0 < a < b < 1$. We let

$$\langle f, \varphi \rangle = \int_0^1 f(x) \varphi(x) \, \mathrm{d}x$$

and get an element of $D_0'(0,1)$ (denoted by f, still).

The space of locally integrable functions is thus embedded,

$$L_1^{\mathrm{loc}}(0,1) \subset D_0'(0,1) \, .$$

In fact, the embedding is one-to-one.

4b10 Example. ("Dirac delta-function") As before, for $x \in (0,1)$,

$$\langle \delta_x, \varphi \rangle = \varphi(x) \, .$$

In fact, the general form of $T \in D_0'(0,1)$ is $\langle T, \varphi \rangle = \int_0^1 \varphi(x) \, \mathrm{d}F(x)$ where $F : (0,1) \to \mathbb{C}$ is of locally finite variation.

Convergence in $D_0'(0,1)$ is defined as before,

$$T_n \to T \text{ in } D_0'(0,1) \quad \text{iff} \quad \forall \varphi \in D_0(0,1) \;\; \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \, .$$

Once again,

$$n\mathbb{1}_{(x,x+1/n)} \to \delta_x \,. \tag{4b11}$$

Given a locally integrable function $f : (0,1) \to \mathbb{C}$, we may construct another locally integrable function $g : (0,1) \to \mathbb{C}$ by $g(x) = f(x/2)$. However, what about, say, $x \mapsto \delta_{0.3}(x/2)$? Do you guess it is $\delta_{0.6}$?

We try to extend the map

$$f \mapsto \big(\tilde f : x \mapsto f(x/2)\big)$$

continuously to $D_0'(0,1)$. If $L_1^{\text{loc}}(0,1) \ni f_n \to T \in D_0'(0,1)$ in $D_0'(0,1)$ and $\tilde f_n : x \mapsto f_n(x/2)$ then

$$\begin{aligned}\langle \tilde f_n, \varphi\rangle = \int_0^1 \tilde f_n(x)\varphi(x)\,\mathrm{d}x = \int_0^1 f_n\Big(\frac{x}{2}\Big)\varphi(x)\,\mathrm{d}x = \\ = 2\int_0^{0.5} f_n(x)\varphi(2x)\,\mathrm{d}x = \langle f_n, \psi\rangle \to \langle T, \psi\rangle\end{aligned}$$

where $\psi \in D_0(0,1)$ is defined by $\psi(x) = 2\varphi(2x)$ for $x \in (0, 0.5)$, otherwise 0. Defining $\tilde T$ by

$$\langle \tilde T, \varphi\rangle = \langle T, \tilde\psi\rangle$$

we observe that

$$T_n \to T \quad \text{implies} \quad \tilde T_n \to \tilde T$$

and the map $T \mapsto \tilde T$ extends the map $f \mapsto \tilde f$. For example, if $T = \delta_{0.3}$ then $\tilde T = 2\delta_{0.6}$. In this sense

$$\delta_{0.3}(x/2) = 2\delta_{0.6}(x)\,.$$

Compare it with (4b11). Similarly, for any continuously differentiable $\alpha : (0,1) \to (0,1)$ such that $\alpha'(\cdot) > 0$ one defines

$$\langle T(\alpha(\cdot)), \varphi\rangle = \langle T, (\alpha^{-1})' \cdot \varphi(\alpha^{-1}(\cdot))\rangle\,,$$

thus extending the map $f \mapsto f(\alpha(\cdot))$ by continuity. In particular,

$$\delta_x(\alpha(\cdot)) = (\alpha^{-1})'(x)\delta_{\alpha^{-1}(x)}\,.$$

Shifts do not act on the interval, but anyway, we may define differentiation in $D_0'(0,1)$ via 'integration by parts' (recall 4b8):

$$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle \tag{4b12}$$

for all continuously differentiable $\varphi \in D_0(0,1)$ (these being dense in $D_0(0,1)$). If such T' exists, we say that T is differentiable in $D_0'(0,1)$.

Once again,

$$\mathbb{1}'_{(}a,b) = \delta_a - \delta_b$$

and more generally,

$$\langle F', \varphi \rangle = \int_0^1 \varphi(x)\,\mathrm{d}F(x)$$

for F of locally finite variation. And of course, δ_x are not differentiable in $D'_0(0,1)$.

4c Smooth test functions

Higher derivatives of a step function appear.

SMOOTHNESS OF FINITE ORDER

Denote by $D_m(0,1)$ the linear space of all m times continuously differentiable functions $(0,1) \to \mathbb{C}$ with compact supports inside $(0,1)$.

An example:

$$\varphi(x) = \begin{cases} (x-0.1)^{m+1}(0.9-x)^{m+1} & \text{for } x \in (0.1, 0.9), \\ 0 & \text{otherwise.} \end{cases}$$

Define convergence in $D_m(0,1)$ as follows: for $\varphi, \varphi_n \in D_m(0,1)$,

$$\begin{array}{c} \varphi_n \to \varphi \\ \text{in } D_m(0,1) \end{array} \quad \text{iff} \quad \begin{array}{c} \varphi_n^{(k)} \to \varphi^{(k)} \text{ uniformly } (k = 0, 1, \dots, n), \text{ and the supports of} \\ \text{all } \varphi_n \text{ are contained in a single compact subset of } (0,1). \end{array}$$

Accordingly, $D'_m(0,1)$ consists of all linear functionals $T : D_m(0,1) \to \mathbb{C}$ such that

$$\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \quad \text{whenever} \quad \varphi_n \to \varphi \text{ in } D_m(0,1)\,.$$

Elements of $D'_m(0,1)$ are called *distributions of order m* on $(0,1)$.

Note that convergence in $D_{m+1}(0,1)$ implies convergence in $D_m(0,1)$ (the converse being wrong). Thus, $D'_m(0,1)$ is embedded into $D'_{m+1}(0,1)$ as a linear subspace. The embedding is one-to-one, since $D_{m+1}(0,1)$ is in fact dense in $D_m(0,1)$.

4c1 Example.

$$\langle \delta'_x, \varphi \rangle = -\varphi'(x) \quad \text{for } x \in (0,1)\,;$$

δ'_x belongs to $D'_1(0,1)$ but not $D'_0(0,1)$. More generally,

$$\langle \delta_x^{(k)}, \varphi \rangle = (-1)^k \varphi^{(k)}(x) \quad \text{for } k = 0, 1, \dots, m \text{ and } x \in (0,1)\,;$$

$\delta_x^{(m)}$ belongs to $D'_m(0,1)$ but not $D'_{m-1}(0,1)$.

In the spirit of (4b12) we define the derivative $T' \in D'_{m+1}(0,1)$ for every $T \in D'_m(0,1)$ by

$$\forall \varphi \in D_{m+1}(0,1) \quad \langle T', \varphi \rangle = -\langle T, \varphi' \rangle \,.$$

Similarly, $T'' \in D'_{m+2}(0,1)$, and so on.

Thus, $(\delta_x^{(k)})' = \delta_x^{(k+1)}$.

4c2 Definition. Let $T_n, T \in D'_m(0,1)$. We say that $T_n \to T$ in $D'_m(0,1)$ if

$$\forall \varphi \in D_m(0,1) \quad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \,.$$

4c3 Exercise.

$$\frac{\delta_{x+\varepsilon} - \delta_x}{\varepsilon} \to \delta'_x \quad \text{in } D'_1(0,1) \text{ as } \varepsilon \to 0 \,.$$

Prove it.

Any other interval may be used in place of $(0,1)$, of course.

4c4 Exercise. The following limit exists in $D'_1(-1,1)$

$$T = \lim_{\varepsilon \to 0+} \Big(x \mapsto \frac{1}{x} \Big) \cdot 1\!\!1_{(-1,-\varepsilon)\cup(\varepsilon,1)}$$

and satisfies

$$\langle T, \varphi \rangle = \int_{-1}^{+1} \frac{\varphi(x) - \varphi(0)}{x} \, \mathrm{d}x \quad \text{for all } \varphi \in D_1(-1,1) \,.$$

Prove it.

This T is called the principal value of $1/x$ and denoted by $\operatorname{pv}(x \mapsto 1/x)$. In fact, its derivative $T' \in D'_2(-1,1)$ is

$$\langle T', \varphi \rangle = \int_{-1}^{+1} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} \, \mathrm{d}x \quad \text{for all } \varphi \in D_2(-1,1) \,.$$

4c5 Exercise. Prove that $\operatorname{pv}(x \mapsto 1/x)$ is the derivative of the integrable function $x \mapsto \ln |x|$ treated as a distribution of order 0.

The set $D_m(0,1)$ is not only a linear space but also an algebra: $\varphi, \psi \in D_m(0,1) \implies \varphi \cdot \psi \in D_m(0,1)$.

4c6 Definition. The product $\varphi \cdot T \in D'_m(0,1)$ of $\varphi \in D_m(0,1)$ and $T \in D'_m(0,1)$ is defined by

$$\langle \varphi \cdot T, \psi \rangle = \langle T, \varphi \cdot \psi \rangle \quad \text{for } \psi \in D_m(0,1) \,.$$

As before, the map $D'_m(0,1) \ni T \mapsto \varphi \cdot T \in D'_m(0,1)$ is a continuous extension of the map $D_m(0,1) \ni \psi \mapsto \psi \cdot \psi \in D_m(0,1) \subset D'_m(0,1)$.

4c7 Exercise.

$$(x \mapsto x) \cdot \mathrm{pv}(x \mapsto 1/x) = 1 \, .$$

Prove it.

However, distributions are not an algebra. The product of distributions is generally undefined.

SMOOTHNESS OF INFINITE ORDER

The space $D(0,1) = \cap_m D_m(0,1)$ consists of all infinitely differentiable functions $(0,1) \to \mathbb{C}$ with compact supports inside $(0,1)$.

An example:

$$\varphi(x) = \begin{cases} \exp\Big(-\frac{1}{(x-0.1)(0.9-x)} \Big) & \text{for } x \in (0.1, 0.9), \\ 0 & \text{otherwise.} \end{cases}$$

Convergence in $D(0,1)$ is defined as follows: for $\varphi, \varphi_n \in D(0,1)$,

$\varphi_n \to \varphi$ in $D(0,1)$	iff	for each k, $\varphi_n^{(k)} \to \varphi^{(k)}$ uniformly, and the supports of all φ_n are contained in a single compact subset of $(0,1)$.

Accordingly, $D'(0,1)$ consists of all linear functionals $T : D(0,1) \to \mathbb{C}$ such that

$$\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \quad \text{whenever} \quad \varphi_n \to \varphi \text{ in } D(0,1) \, .$$

Convergence in $D(0,1)$ implies convergence in $D_m(0,1)$ for each m. Thus,

$$D'_0(0,1) \subset D'_1(0,1) \subset \dots \subset D'(0,1) \, .$$

Elements of $D'(0,1)$ are called *distributions* on $(0,1)$. They need not be of finite order.

4c8 Example. $T = \sum_k \delta_{1/k}^{(k)} \in D'(0,1)$; that is,

$$\langle T, \varphi \rangle = \sum_{k=2}^{\infty} \varphi^{(k)} \Big(\frac{1}{k} \Big)$$

(a finite sum...).

In fact, every $T \in D'(0,1)$ is of finite order (that is, continuous on D_m) within $(\varepsilon, 1-\varepsilon)$, $\varepsilon > 0$.

Now we can differentiate everything!

4c9 Definition. The derivative $T' \in D'(0,1)$ of $T \in D'(0,1)$ is defined by

$$\forall \varphi \in D_(0,1) \quad \langle T', \varphi \rangle = -\langle T, \varphi' \rangle \, .$$

Similarly, $T'' \in D'(0,1)$, and so on.

4c10 Definition. Let $T_n, T \in D'(0,1)$. We say that $T_n \to T$ in $D'(0,1)$ if

$$\forall \varphi \in D(0,1) \quad \langle T_n, \varphi \rangle \to \langle T, \varphi \rangle \, .$$

4c11 Exercise. The differentiation operator is continuous. That is,

$$\text{if } T_n \to T \text{ in } D'(0,1) \quad \text{then} \quad T'_n \to T' \text{ in } D'(0,1) \, .$$

Prove it.

Any other interval may be used in place of $(0,1)$, of course.

4c12 Exercise. For every $m = 0, 1, 2, \dots$

$$\biggl(x \mapsto \frac{1}{\sqrt{2\pi}\sigma} \frac{\mathrm{d}^m}{\mathrm{d}x^m} \exp\Bigl(-\frac{x^2}{2\sigma^2} \Bigr) \biggr) \to \delta_0^{(m)} \quad \text{in } D'(0,1) \text{ as } \sigma \to 0+ \, .$$

Prove it.

4d Fourier transform

Fourier transform of a plane wave appears.

On the whole $\mathbb{R}$ we can introduce $D(\mathbb{R})$ and $D'(\mathbb{R})$ as before. However, $D'(\mathbb{R})$ is too large for Fourier transform because of arbitrarily fast growth on infinity allowed. We increase the space of test functions as follows (thus decreasing the space of distributions).

4d1 Definition. The space $S(\mathbb{R})$ consists of all infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{C}$ such that

$$\forall k, m \quad \sup_{x \in \mathbb{R}} |x^k \varphi^{(m)}(x)| < \infty \, .$$

Elements of $S(\mathbb{R})$ are called *rapidly decreasing* test functions, and $S(\mathbb{R})$ is called the Schwartz space. Note that $x^k \varphi^{(m)}(x) \to 0$ as $|x| \to \infty$.

4d2 Example. $\varphi(x) = \exp(-x^2)$.

4d3 Exercise. If $\varphi \in S(\mathbb{R})$ then $\varphi' \in S(\mathbb{R})$ and $(x \mapsto x\varphi(x)) \in S(\mathbb{R})$.
Prove it.

Convergence in $S(\mathbb{R})$ is defined as follows: for $\varphi, \varphi_n \in S(\mathbb{R})$,

$$\begin{matrix} \varphi_n \to \varphi \\ \text{in } S(\mathbb{R}) \end{matrix} \qquad \text{iff} \qquad \forall k, m \;\; \sup_{x\in\mathbb{R}} |x^k \varphi_n^{(m)}(x)| \xrightarrow[n\to\infty]{} 0\,.$$

Accordingly, $S'(\mathbb{R})$ consists of all linear functionals $T : S(\mathbb{R}) \to \mathbb{C}$ such that

$$\langle T, \varphi_n \rangle \to \langle T, \varphi \rangle \quad \text{whenever} \quad \varphi_n \to \varphi \text{ in } S(\mathbb{R})\,.$$

Elements of $S'(\mathbb{R})$ are called *temperedtempered distribution* distributions on $\mathbb{R}$.

Note that $D(\mathbb{R}) \subset S(\mathbb{R})$ and $\varphi_n \to 0$ in $S(\mathbb{R})$ whenever $\varphi_n \to 0$ in $D(\mathbb{R})$. Thus,

$$S'(\mathbb{R}) \subset D'(\mathbb{R})\,.$$

The embedding is one-to-one, since $D(\mathbb{R})$ is in fact dense in $S(\mathbb{R})$.

4d4 Example. $\delta_x \in S'(\mathbb{R})$.

4d5 Example. Every polynomial P, treated as a distribution, belongs to $S'(\mathbb{R})$:

$$\langle P, \varphi \rangle = \int_{-\infty}^{+\infty} P(x)\varphi(x)\,\mathrm{d}x\,.$$

In contrast, $(x \mapsto \mathrm{e}^x) \notin S'(\mathbb{R})$.

Recall the operators Q, P (Sect. 2); $\mathcal{F}P = Q\mathcal{F}$ and $P\mathcal{F} = -\mathcal{F}Q$.

4d6 Lemma. Let $f \in \mathrm{Domain}(P)$, $\mathrm{i}Pf = g$, then f can be corrected on a null set getting

$$f(b) - f(a) = \int_a^b g(x)\,\mathrm{d}x \quad \text{whenever } a < b\,.$$

Proof. Theorem 2c3 gives us f_n such that $f_n \to f$ in L_2 and $f_n' \to g$ in L_2. Thus, $f_n(b) - f_n(a) = \int_a^b f_n'(x)\,\mathrm{d}x \to \int_a^b g(x)\,\mathrm{d}x$. Also, $\int_0^1 f_n(x)\,\mathrm{d}x - f_n(0) = \int_0^1 (1-x) f_n'(x)\,\mathrm{d}x \to \int_0^1 (1-x)g(x)\,\mathrm{d}x$, therefore $\lim_n f_n(0)$ exists. It follows that f_n converge pointwise; their limit must be equal to f almost everywhere. It remains to correct f as follows: $f(x) = \lim_n f_n(x)$ for all x. $\square$

Continuity of f follows. If g is continuous then f is continuously differentiable and $f' = g$.

4d7 Exercise. The following three conditions on $\varphi \in L_2(\mathbb{R})$ are equivalent:
(a) $\varphi \in S(\mathbb{R})$;
(b) $Q^n P^m \varphi \in L_\infty(\mathbb{R})$ for all m, n;
(c) $P^m Q^n \varphi \in L_\infty(\mathbb{R})$ for all m, n.
Prove it.
Hint: $QP - PQ = \mathrm{i}\mathbb{1}$.

4d8 Exercise. The following three conditions on $\varphi_1, \varphi_2, \dots \in L_2(\mathbb{R})$ are equivalent:
(a) $\varphi_k \to 0$ in $S(\mathbb{R})$;
(b) $Q^n P^m \varphi_k \to 0$ in the norm of $L_\infty(\mathbb{R})$ for all m, n;
(c) $P^m Q^n \varphi_k \to 0$ in the norm of $L_\infty(\mathbb{R})$ for all m, n.
Prove it.

4d9 Proposition. (a) If a function φ belongs to $S(\mathbb{R})$ then its Fourier transform $\mathcal{F}\varphi$ also belongs to $S(\mathbb{R})$;
(b) if $\varphi_n \to \varphi$ in $S(\mathbb{R})$ then $\mathcal{F}\varphi_n \to \mathcal{F}\varphi$ in $S(\mathbb{R})$.

Proof. (a) Let $\varphi \in S(\mathbb{R})$, then $\varphi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, therefore $\mathcal{F}\varphi = \psi \in L_2(\mathbb{R}) \cap L_\infty(\mathbb{R})$.

By 4d3, $Q^n P^m \varphi \in S(\mathbb{R})$, therefore $P^n Q^m \psi = P^n Q^m \mathcal{F}\varphi = P^n \mathcal{F} P^m \varphi = \pm \mathcal{F} Q^n P^m \varphi \in L_\infty$ for all m, n. By 4d7, $\psi \in S(\mathbb{R})$.

(b) For every $\varphi \in S(\mathbb{R})$,

$$\|\varphi\|_1 = \int |\varphi(x)| \,\mathrm{d}x \le \biggl(\int \frac{\mathrm{d}x}{1+|x|} \biggr) \sup_x \bigl((1+|x|)|\varphi(x)| \bigr) \le \\ \le \text{const} \cdot (\|\varphi\|_\infty + \|Q\varphi\|_\infty) \,,$$

therefore $\|\mathcal{F}\varphi\|_\infty \le \text{const} \cdot (\|\varphi\|_\infty + \|Q\varphi\|_\infty)$. Applying it to $Q^n P^m (\varphi_k - \varphi)$ and using 4d8 we get $\|\mathcal{F} Q^n P^m (\varphi_k - \varphi)\|_\infty \to 0$ as $k \to \infty$. As before it follows that $\|P^n Q^m (\mathcal{F}\varphi_k - \mathcal{F}\varphi)\|_\infty \to 0$. By 4d8 again, $\mathcal{F}\varphi_k - \mathcal{F}\varphi \to 0$ in $S(\mathbb{R})$. $\square$

4d10 Lemma.

$$\langle \mathcal{F}f, g \rangle = \langle f, \mathcal{F}g \rangle \quad \text{for } f, g \in L_2(\mathbb{R}) \,. \quad \text{(Beware: not the scalar product!)}$$

Proof. In terms of the scalar product $\langle f, g \rangle_{L_2} = \int f(x) \overline{g(x)} \,\mathrm{d}x$, unitarity of $\mathcal{F}$ gives $\langle \mathcal{F}f, \mathcal{F}g \rangle_{L_2} = \langle f, g \rangle_{L_2}$. In terms of the bilinear form $\langle f, g \rangle = \int f(x) g(x) \,\mathrm{d}x$ it becomes $\langle \mathcal{F}f, \overline{\mathcal{F}g} \rangle = \langle f, \overline{g} \rangle$. However,

$$\overline{\mathcal{F}g} = \mathcal{F}^{-1} \overline{g} \quad \text{for } g \in L_2(\mathbb{R}) \,,$$

since for all g of the dense set $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ we have

$$\overline{\mathcal{F}g} : t \mapsto \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{\mathrm{i}st} \overline{g(s)} \, \mathrm{d}s \,,$$
$$\mathcal{F}^{-1}\overline{g} = J\mathcal{F}\overline{g} : t \mapsto (\mathcal{F}\overline{g})(-t) = \frac{1}{\sqrt{2\pi}} \int \mathrm{e}^{\mathrm{i}st} \overline{g(s)} \, \mathrm{d}s \,.$$

Thus, $\langle \mathcal{F}f, \mathcal{F}^{-1}\overline{g} \rangle = \langle f, \overline{g} \rangle$; $\langle \mathcal{F}f, \mathcal{F}^{-1}g \rangle = \langle f, g \rangle$; and finally $\langle \mathcal{F}f, g \rangle = \langle f, \mathcal{F}g \rangle$. $\square$

Let $f_n \in L_2(\mathbb{R}) \subset S'(\mathbb{R})$, $f_n \to T$ in $S'(\mathbb{R})$. Then for every $\varphi \in S(\mathbb{R})$,

$$\langle \mathcal{F}f_n, \varphi \rangle = \langle f_n, \mathcal{F}\varphi \rangle \to \langle T, \mathcal{F}\varphi \rangle \,.$$

Thus we get a continuous extension of $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R}) \subset S'(\mathbb{R})$ to $S'(\mathbb{R}) \to S'(\mathbb{R})$.

4d11 Definition. The Fourier transform $\mathcal{F}T \in S'(\mathbb{R})$ of $T \in S'(\mathbb{R})$ is defined by

$$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle \quad \text{for } \varphi \in S(\mathbb{R}) \,.$$

It is the (unique) extension by continuity, since $L_2(\mathbb{R})$ is in fact dense in $S'(\mathbb{R})$.

4d12 Exercise. Extend the operator J of Sect. 1 from $L_2(\mathbb{R})$ to $S'(\mathbb{R})$ and prove that the formulas

$$\mathcal{F}^{-1} = \mathcal{F}J = J\mathcal{F}$$

hold for operators on $S'(\mathbb{R})$.

4d13 Exercise. Prove that

$$\mathcal{F}\delta_0 = (2\pi)^{-1/2} \cdot \mathbb{1} \quad \text{and} \quad \mathcal{F}\mathbb{1} = (2\pi)^{1/2}\delta_0 \,,$$

moreover,

$$\mathcal{F}\delta_x : y \mapsto (2\pi)^{-1/2}\mathrm{e}^{-\mathrm{i}xy} \quad \text{and} \quad \mathcal{F}\big(y \mapsto \mathrm{e}^{-\mathrm{i}xy}\big) = (2\pi)^{1/2}\delta_x \,.$$

4d14 Definition. For $T \in S'(\mathbb{R})$ we define $T', PT, QT \in S'(\mathbb{R})$ by

$$\begin{aligned}
\langle T', \varphi \rangle &= -\langle T, \varphi' \rangle \,, \\
\langle PT, \varphi \rangle &= -\langle T, P\varphi \rangle = \mathrm{i}\langle T, \varphi' \rangle \,, \\
\langle QT, \varphi \rangle &= \langle T, Q\varphi \rangle = \langle T, (x \mapsto x\varphi(x)) \rangle \,.
\end{aligned}$$

Of course, $\mathrm{i}PT = T'$.

4d15 Exercise. Prove the formulas

$$\mathcal{F}P = Q\mathcal{F} \quad \text{and} \quad P\mathcal{F} = -\mathcal{F}Q$$

for operators on $S'(\mathbb{R})$.

4d16 Exercise. Prove that

$$\mathcal{F}\delta_0' : x \mapsto \mathrm{i}(2\pi)^{-1/2}x \quad \text{and} \quad \mathcal{F}(x \mapsto x) = \mathrm{i}(2\pi)^{1/2}\delta_0' \,,$$

moreover,

$$\mathcal{F}\delta_0^{(m)} : x \mapsto \mathrm{i}^m(2\pi)^{-1/2}x^m \quad \text{and} \quad \mathcal{F}\big(x \mapsto x^m\big) = \mathrm{i}^m(2\pi)^{1/2}\delta_0^{(m)} \,;$$

still more generally,

$$\mathcal{F}\delta_x^{(m)} : y \mapsto \mathrm{i}^m(2\pi)^{-1/2}y^m\mathrm{e}^{-\mathrm{i}xy} \quad \text{and} \quad \mathcal{F}\big(y \mapsto y^m\mathrm{e}^{-\mathrm{i}xy}\big) = \mathrm{i}^m(2\pi)^{1/2}\delta_x^{(m)}$$

for $m = 0, 1, 2, \dots$ and $x \in \mathbb{R}$.

Can we use the formula

$$\mathcal{F}(f * g) = (2\pi)^{1/2}(\mathcal{F}f) \cdot (\mathcal{F}g)$$

for defining the convolution of distributions? Generally not, since the product of distributions is generally undefined. However, we may define $f \cdot T \in S'(\mathbb{R})$ for $T \in S'(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ an infinitely differentiable function such that $f, f', f'', \dots$ are functions of polynomial growth, that is,

$$\forall m \;\; \exists n \quad \frac{|f^{(m)}(x)|}{|x|^n} \to 0 \text{ as } |x| \to \infty \,;$$

such f are called multipliers in $S(\mathbb{R})$, since $\forall \varphi \in S(\mathbb{R}) \;\; f \cdot \varphi \in S(\mathbb{R})$ (think, why). For such f we define (recall 4c6)

$$\langle f \cdot T, \varphi \rangle = \langle T, f \cdot \varphi \rangle \quad \text{for all } \varphi \in S(\mathbb{R}) \,.$$

Given a multiplier f in $S(\mathbb{R})$, we define the convolution $T*(\mathcal{F}^{-1}f) \in S'(\mathbb{R})$ for all $T \in S'(\mathbb{R})$ by

$$T * (\mathcal{F}^{-1}f) = (2\pi)^{1/2}\mathcal{F}^{-1}(f \cdot \mathcal{F}T) \,.$$

All polynomials are multipliers, of course. Taking $f = \mathcal{F}\delta_0'$ we have $(2\pi)^{1/2}f(x) = \mathrm{i}x$;

$$T * \delta_0' = (2\pi)^{1/2}\mathcal{F}^{-1}(f \cdot \mathcal{F}T) = \mathcal{F}^{-1}(\mathrm{i}Q\mathcal{F}T) = \mathrm{i}PT = T' \,.$$

Similarly,

$$T * \delta_0^{(m)} = T^{(m)} \quad \text{for } m = 0, 1, 2, \dots \text{ and } T \in S'(\mathbb{R}) \, .$$

In particular,

$$\delta_0^{(m)} * \delta_0^{(n)} = \delta_0^{(m+n)} \, .$$

In fact, $\mathcal{F}T$ is a multiplier in $S(\mathbb{R})$ whenever T is *compactly supported,* that is,

$$\exists C \ \ \forall \varphi \quad \big(\varphi(\cdot) = 0 \text{ outside } [-C, C] \quad \text{implies} \quad \langle T, \varphi \rangle = 0 \big) \, .$$

Thus, $T_1 * T_2$ is well-defined if at least one of T_1, T_2 is compactly supported.

4e List of formulas

T is a distribution, φ is a test function.

(4e1) $$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle \, ;$$

(4e2) $$\langle \delta_x, \varphi \rangle = \varphi(x) \, ;$$

(4e3) $$\big(1\!\!1_{(a,b)} \big)' = \delta_a - \delta_b \, ;$$

(4e4) $$\big(\delta_x^{(m)} \big)' = \delta_x^{(m+1)} \, ;$$

(4e5) $$\langle \delta_x^{(m)}, \varphi \rangle = (-1)^m \varphi^{(m)}(x) \, ;$$

(4e6) $$\delta_x(\alpha(\cdot)) = (\alpha^{-1})'(x) \delta_{\alpha^{-1}(x)} \, .$$

$\mathcal{F}$ is the Fourier transform, "$*$" means convolution.

(4e7) $$\langle \mathcal{F}T, \varphi \rangle = \langle T, \mathcal{F}\varphi \rangle \, ;$$

(4e8) $$\mathcal{F}P = Q\mathcal{F} \, ; \qquad P\mathcal{F} = -\mathcal{F}Q \, ;$$

(4e9) $$\mathcal{F}\delta_0 = (2\pi)^{-1/2} \cdot 1\!\!1 \, ; \qquad \mathcal{F}1\!\!1 = (2\pi)^{1/2} \delta_0 \, ;$$

(4e10) $$\mathcal{F}\delta_x : y \mapsto (2\pi)^{-1/2} \mathrm{e}^{-\mathrm{i}xy} \, ; \qquad \mathcal{F}\big(y \mapsto \mathrm{e}^{-\mathrm{i}xy} \big) = (2\pi)^{1/2} \delta_x \, ;$$

(4e11) $$\mathcal{F}\delta_0' : x \mapsto \mathrm{i}(2\pi)^{-1/2} x \, ; \qquad \mathcal{F}(x \mapsto x) = \mathrm{i}(2\pi)^{1/2} \delta_0' \, ;$$

(4e12) $$\mathcal{F}\delta_0^{(m)} : x \mapsto \mathrm{i}^m (2\pi)^{-1/2} x^m \, ; \qquad \mathcal{F}\big(x \mapsto x^m \big) = \mathrm{i}^m (2\pi)^{1/2} \delta_0^{(m)} \, ;$$

(4e13)
$$\mathcal{F}\delta_x^{(m)} : y \mapsto \mathrm{i}^m (2\pi)^{-1/2} y^m \mathrm{e}^{-\mathrm{i}xy} \, ; \qquad \mathcal{F}\big(y \mapsto y^m \mathrm{e}^{-\mathrm{i}xy} \big) = \mathrm{i}^m (2\pi)^{1/2} \delta_x^{(m)} \, ;$$

(4e14) $$T * \delta_0' = T' \, ;$$

(4e15) $$T * \delta_0^{(m)} = T^{(m)} \, ;$$

(4e16) $$\delta_0^{(m)} * \delta_0^{(n)} = \delta_0^{(m+n)} \, .$$

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