3 Level crossings

. . . the famous Rice formula, undoubtedly one of the most important results in the application of smooth stochastic processes.

R.J. Adler and J.E. Taylor[1]

3a An instructive toy model: two paradoxes

We start with a very simple random trigonometric polynomial (even simpler than 1c12):

(3a1) $$X(t) = \zeta \cos t + \eta \sin t$$

where ζ, η are independent $\mathrm{N}(0,1)$ random variables. Its distribution is the image of γ^2 under the map $(x,y) \mapsto (t \mapsto x \cos t + y \sin t)$. Time shifts of the trigonometric polynomial correspond to rotations of $\mathbb{R}^2$,

(3a2) $$\begin{aligned} X(t-\alpha) &= \zeta(\cos t \cos\alpha + \sin t \sin\alpha) + \eta(\sin t \cos\alpha - \cos t \sin\alpha) = \\ &= (\zeta\cos\alpha - \eta\sin\alpha)\cos t + (\zeta\sin\alpha + \eta\cos\alpha)\sin t\,; \end{aligned}$$

the process (3a1) is *stationary,* that is, invariant under time shifts;

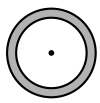
(3a3) $$\mathbb{E}\, X(t) = 0\,, \quad \mathbb{E}\, X(s)X(t) = \cos(s-t)\,.$$

The random variable

(3a4) $$M = \max_{t\in\mathbb{R}} |X(t)| = \sqrt{\zeta^2+\eta^2}$$

has the *density*

(3a5) $$f_M(u) = u\mathrm{e}^{-u^2/2} \quad \text{for } u > 0\,;$$



$$\frac{1}{2\pi}\mathrm{e}^{-u^2/2}\,\mathrm{d}u \cdot 2\pi u$$

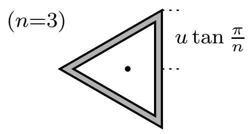
[1]See Preface (page vi) to the book "Random fields and geometry" (to appear).

it means that $\mathbb{P}\big(a < M < b\big) = \int_a^b f_M(u)\,\mathrm{d}u$. Consider also random variables

$$M_n = \max_{k\in\mathbb{Z}} X\Big(\frac{2\pi k}{n}\Big)\,; \tag{3a6}$$

clearly, $M_n \to M$ a.s. The density of M_n is

$$f_{M_n}(u) = \frac{1}{2\pi}\mathrm{e}^{-u^2/2}\cdot n\int_{-u\tan\pi/n}^{u\tan\pi/n}\mathrm{e}^{-x^2/2}\,\mathrm{d}x \quad \text{for } u>0\,. \tag{3a7}$$



We see that $f_{M_n}(u) \to f_M(u) = u\mathrm{e}^{-u^2/2}$ as $n\to\infty$. On the other hand,

$$f_{M_n}(u) \sim n\cdot\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2} \quad \text{as } u\to\infty\,. \tag{3a8}$$

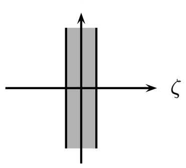
A paradox! Think, what does it mean.

Another paradox appears if we condition X to have the maximum at a given t. By stationarity we restrict ourselves to $t=0$. The condition becomes $\eta = 0$ and $\zeta > 0$; thus, $X(t) = \zeta\cos t$, and the conditional distribution of ζ is the same as the unconditional distribution of $|\zeta|$ (since ζ and η are independent). The conditional density of M is $u \mapsto \frac{2}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}$ for $u>0$. This holds for $t=0$, but also for every t; we conclude that the unconditional density of M is also $u\mapsto\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}$, in contradiction to (3a5)!

Here is another form of the same paradox. For each t the two random variables $X(t)$ and $X'(t)$ are independent (think, why), distributed $\mathrm{N}(0,1)$ each. Thus, given $X(t)=0$, the distribution of $X'(t)$ is still $\mathrm{N}(0,1)$, and the density of $|X'(t)|$ is $u\mapsto\frac{2}{\sqrt{2\pi}}\mathrm{e}^{-u^2/2}$ (for $u>0$). On the other hand, $X(\cdot)$ vanishes at two points, and $|X'(t)| = \sqrt{\zeta^2+\eta^2} = M$ at these points (think, why). This argument leads to another density, $u\mapsto u\mathrm{e}^{-u^2/2}$ (for $u>0$).

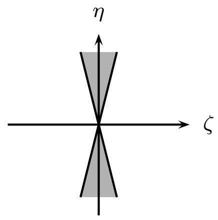
Here is an explanation. The phrase 'given that $X(0)=0$' has (at least) two interpretations, known as 'vertical window' and 'horizontal window'.

Vertical window: we condition on $|X(0)|<\varepsilon$ and take $\varepsilon\to 0$.



We get $X(t) = \eta \sin t$ with η distributed $\mathrm{N}(0,1)$ (conditionally).

Horizontal window: we require X to vanish somewhere on $(-\varepsilon, \varepsilon)$ and take $\varepsilon \to 0$.



We get $X(t) = \eta \sin t$, but now the conditional density of η is $u \mapsto \text{const} \cdot |u| \mathrm{e}^{-u^2/2}$ (and const $= 1/2$).

Think about the two interpretations of the phrase 'given that X has a maximum at 0'.

Here is still another manifestation of the paradox. Let us compare $X = \zeta \cos t + \eta \sin t$ with $Y = \zeta \cos 10t + \eta \sin 10t$. For each t the two random variables $X(t), Y(t)$ have the same density at 0 (just because both are $\mathrm{N}(0,1)$). Nevertheless, $Y(\cdot)$ has 10 times more zeros than $X(\cdot)$. Think, what does it mean in terms of the horizontal and vertical window.

3b Measures on the graph of a function

Before treating random functions we examine a single (non-random) function $f \in C^1[a,b]$; that is, $f : [a,b] \to \mathbb{R}$ is continuous, and there exists a continuous $f' : [a,b] \to \mathbb{R}$ such that $f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t$ for $x \in [a,b]$. The number $\#f^{-1}(y)$ of points $x \in [a,b]$ such that $f(x) = y$ is a function $\mathbb{R} \to \{0,1,2,\dots\} \cup \{\infty\}$.

3b1 Lemma.

$$\int_{\mathbb{R}} \#f^{-1}(y)\,\mathrm{d}y = \int_a^b |f'(x)|\,\mathrm{d}x\,.$$

3b2 Exercise. Prove 3b1 assuming that f is (a) monotone, (b) piecewise monotone.

This is enough for (say) trigonometric polynomials. In general, $f \in C^1[a,b]$ need not be piecewise monotone,[1] but 3b1 holds anyway.

Proof of Lemma 3b1 (sketch). We consider the set $C = \{x : f'(x) = 0\}$ of critical points and the set $f(C)$ of critical values; both are compact sets, and $\operatorname{mes} f(C) = 0$ by Sard's theorem (even if $\operatorname{mes} C \ne 0$). If $y \notin f(C)$ then the set

[1] Try $x^3 \sin(1/x)$.

$f^{-1}(y)$ is finite (since its accumulation point would be critical), and $f^{-1}(y+\varepsilon)$ is close to $f^{-1}(y)$ for all ε small enough; in particular, $\#f^{-1}(y+\varepsilon) = \#f^{-1}(y)$.

The sets $B_n = \{y \in \mathbb{R} \setminus f(C) : \#f^{-1}(y) = n\}$ and $A_n = f^{-1}(B_n)$ are open, $B_1 \cup B_2 \cup \dots = \mathbb{R} \setminus f(C)$, and $A_1 \cup A_2 \cup \dots = f^{-1}(\mathbb{R} \setminus f(C))$ is a full measure subset of $\mathbb{R} \setminus C$. It is sufficient to prove that

$$\int_{B_n} \#f^{-1}(y) \, \mathrm{d}y = \int_{A_n} |f'(x)| \, \mathrm{d}x$$

for each n. We consider a connected component of B_n and observe that f is monotone on each of the n corresponding intervals. $\square$

Here is an equality between two measures on $\mathbb{R}^2$ concentrated on the graph of f (below $\delta_{x,y}$ stands for the unit mass at (x, y)).

3b3 Exercise. For every $f \in C^1[a, b]$,

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} \delta_{x,y} = \int_a^b \mathrm{d}x \, |f'(x)| \delta_{x,f(x)} \, , \tag{3b4}$$

that is,

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} \mathbf{1}_A(x, y) = \int_a^b \mathrm{d}x \, |f'(x)| \mathbf{1}_A(x, f(x)) \tag{3b5}$$

for all Borel sets $A \subset \mathbb{R}^2$.

Prove it.

Hint: first, consider rectangles $A = (x_1, x_2) \times (y_1, y_2)$; second, recall the hint to 2c5.

3b6 Exercise. For every $f \in C^1[a, b]$ and every bounded Borel function $g : [a, b] \to \mathbb{R}$,

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} g(x) = \int_a^b \mathrm{d}x \, |f'(x)| g(x) \, .$$

Prove it.

Hint: first, indicators $g = \mathbf{1}_A$; second, their linear combinations.

Replacing $g(x)$ with $g(x) \operatorname{sgn} f'(x)$ we get an equivalent formula[1]

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} g(x) \operatorname{sgn} f'(x) = \int_a^b \mathrm{d}x \, f'(x) g(x) \, . \tag{3b7}$$

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[1] $\operatorname{sgn} a = \begin{cases} 1 & \text{for } a > 0, \\ 0 & \text{for } a = 0, \\ -1 & \text{for } a < 0. \end{cases}$

We have also (seemingly) more general equalities between (signed) measures,

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} g(x)\delta_{x,y} = \int_a^b \mathrm{d}x\, |f'(x)| g(x) \delta_{x,f(x)}\,; \tag{3b8}$$

$$\int_{\mathbb{R}} \mathrm{d}y \sum_{x \in f^{-1}(y)} g(x) \operatorname{sgn} f'(x) \delta_{x,y} = \int_a^b \mathrm{d}x\, f'(x) g(x) \delta_{x,f(x)}\,. \tag{3b9}$$

Taking $g(\cdot) \ge 0$ such that both sides of (3b8) are equal to 1 we get a probability measure on the graph of f. Treating it as the distribution of a pair of random variables X, Y we see that the conditional distribution of Y given $X = x$ is $\delta_{f(x)}$, and the conditional distribution of X given $Y = y$ is

$$\mathrm{const} \cdot \sum_{x \in f^{-1}(y)} g(x) \delta_x\,,$$

which does not mean that g is the unconditional density of X. Rather, the density is equal to $|f'|g$. This is another manifestation of the distinction between 'horizontal window' and 'vertical window'.

3c The same for a random function

Let μ be a probability measure on $C^1[a,b]$. Two assumption on μ are introduced below.

Given $x \in [a,b]$, we consider the joint distribution of $f(x)$ and $f'(x)$, where f is distributed μ; in other words, the image of μ under the map $f \mapsto \big(f(x), f'(x)\big)$ from $C^1[a,b]$ to $\mathbb{R}^2$. *The first assumption:* for each $x \in [a,b]$, this joint distribution is absolutely continuous, that is, has a density p_x;

$$\int \varphi(y,y') p_x(y,y')\, \mathrm{d}y\mathrm{d}y' = \int \varphi\big(f(x), f'(x)\big)\, \mu(\mathrm{d}f) \tag{3c1}$$

for every bounded Borel function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ (recall the hint to 3b6).[1] The function $(x, y, y') \mapsto p_x(y,y')$ on $[a,b] \times \mathbb{R}^2$ is (or rather, may be chosen to be) measurable, since its convolution with any continuous function of y, y' is continuous in x.

For example, μ can be an arbitrary absolutely continuous measure on the (finite-dimensional linear) space of trigonometric (or algebraic) polynomials of degree n (except for $n = 0$).

[1]The meaning of $\mathrm{d}f$ in $\mu(\mathrm{d}f)$ and $\mathrm{d}f(x)/\mathrm{d}x$ is completely different...

The second assumption:[1]

$$\iint_{[a,b]\times C^1[a,b]} |f'(x)|\,\mathrm{d}x\mu(\mathrm{d}f) < \infty\,. \tag{3c2}$$

(A simple sufficient condition: $\int \|f\|\,\mu(\mathrm{d}f) < \infty$; here $\|f\| = \max|f| + \max|f'|$ or something equivalent.)

For example, μ can be an arbitrary nondegenerate Gaussian measure on the (finite-dimensional linear) space of trigonometric (or algebraic) polynomials of degree n (except for $n = 0$).

3c3 Exercise. The function $f \mapsto \#f^{-1}(0)$ is a Borel function on $C^1[a,b]$.

Prove it.

Hint: first, $\{f : f^{-1}(0) = \emptyset\}$ is open; second, $\#f^{-1}(0) = \lim_{n\to\infty} \#\{k : f^{-1}(0) \cap [(k-1)2^{-n}, k\cdot 2^{-n}) \ne \emptyset\}$.

3c4 Exercise. The function $(y, f) \mapsto \#f^{-1}(y)$ is a Borel function on $\mathbb{R} \times C^1[a,b]$.

Prove it.

Hint: do not work hard, consider $(y, f) \mapsto f(\cdot) - y$.

For a given y we consider the expected (averaged) $\#f^{-1}(y)$,

$$\mathbb{E}\left(\#f^{-1}(y)\right) = \int_{C^1[a,b]} \#f^{-1}(y)\,\mu(\mathrm{d}f) \in [0,\infty]\,. \tag{3c5}$$

3c6 Exercise.

$$\int_{\mathbb{R}} \mathrm{d}y\,\mathbb{E}\left(\#f^{-1}(y)\right) = \int_a^b \mathrm{d}x \iint_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}y'\,p_x(y,y')|y'|\,.$$

Prove it.

Hint: 3b1 and Fubini.

Similarly, (3b5) gives

$$\int_{\mathbb{R}} \mathrm{d}y\,\mathbb{E} \sum_{x\in f^{-1}(y)} \mathbf{1}_A(x,y) = \int_a^b \mathrm{d}x \iint_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}y'\,p_x(y,y')|y'|\mathbf{1}_A(x,y) \tag{3c7}$$

for all Borel sets $A \subset \mathbb{R}^2$. More generally,[2]

$$\int_{\mathbb{R}} \mathrm{d}y\,\mathbb{E} \sum_{x\in f^{-1}(y)} g(x,y) = \int_a^b \mathrm{d}x \iint_{\mathbb{R}^2} \mathrm{d}y\mathrm{d}y'\,p_x(y,y')|y'|g(x,y) \tag{3c8}$$

[1] The function $(x, f) \mapsto f'(x)$ on $[a,b] \times C^1[a,b]$ is continuous, therefore, Borel.

[2] In 3b6 we use $g(x)$, but $g(x,y)$ can be used equally well (which does not increase generality).

for every bounded Borel function $g : \mathbb{R}^2 \to \mathbb{R}$. Substituting $g(x)h(y)$ for $g(x, y)$ we get

$$\int_{\mathbb{R}} \mathrm{d}y\, h(y)\, \mathbb{E} \sum_{x \in f^{-1}(y)} g(x) = \int_{\mathbb{R}} \mathrm{d}y\, h(y) \int_a^b \mathrm{d}x\, g(x) \int_{\mathbb{R}} \mathrm{d}y'\, p_x(y, y')|y'| \tag{3c9}$$

for all bounded Borel functions $g : [a, b] \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$. Therefore

$$\mathbb{E} \sum_{x \in f^{-1}(y)} g(x) = \int_a^b \mathrm{d}x\, g(x) \int_{\mathbb{R}} \mathrm{d}y'\, p_x(y, y')|y'| \tag{3c10}$$

for almost all $y \in \mathbb{R}$. Especially,

$$\mathbb{E}\left(\# f^{-1}(y)\right) = \int_a^b \mathrm{d}x \int_{\mathbb{R}} \mathrm{d}y'\, p_x(y, y')|y'| \tag{3c11}$$

for almost all $y \in \mathbb{R}$. Some additional assumptions could ensure (continuity in y and therefore) the equality for *every* y.

3d Gaussian case: Rice's formula

Let γ be a (centered) Gaussian measure on $C^1[a, b]$ such that for every $x \in [a, b]$

$$\int_{C^1[a,b]} |f(x)|^2\, \gamma(\mathrm{d}f) = 1\,, \tag{3d1}$$

$$\int_{C^1[a,b]} |f'(x)|^2\, \gamma(\mathrm{d}f) = \sigma^2(x) > 0 \tag{3d2}$$

for some $\sigma : [a, b] \to (0, \infty)$.[1]

Each $x \in [a, b]$ leads to two measurable (in fact, continuous) linear functionals

$$f \mapsto f(x) \quad \text{and} \quad f \mapsto f'(x)$$

on $(C^1[a, b], \gamma)$. The former is distributed $\mathrm{N}(0, 1)$ by (3d1); the latter is distributed $\mathrm{N}(0, \sigma^2(x))$ by (3d2).

3d3 Exercise. The function $\sigma(\cdot)$ is continuous on $[a, b]$.

Prove it.

Hint: $f'(x+\varepsilon) - f'(x) \to 0$ (as $\varepsilon \to 0$) almost sure, therefore in probability, therefore (using normality!) in $L_2(\gamma)$.

[1] See also 3d14.

The joint distribution of $f(x)$ and $f'(x)$ is a Gaussian measure on $\mathbb{R}^2$. It appears to be the product measure, $\mathrm{N}(0,1) \times \mathrm{N}(0, \sigma^2(x))$; in other words, the two random variables $f(x)$ and $f'(x)$ are independent. It is sufficient to prove that they are orthogonal,

$$\int_{C^1[a,b]} f(x) f'(x) \, \gamma(\mathrm{d}f) = 0 \, . \tag{3d4}$$

Proof: by (3d1),

$$\begin{aligned} 0 &= \int \frac{|f(x+\varepsilon)|^2 - |f(x)|^2}{2\varepsilon} \, \gamma(\mathrm{d}f) = \\ &= \int \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \cdot \frac{f(x+\varepsilon) + f(x)}{2} \, \gamma(\mathrm{d}f) \xrightarrow[\varepsilon \to 0]{} \int f(x) f'(x) \, \gamma(\mathrm{d}f) \, . \end{aligned}$$

(Once again, convergence almost sure implies convergence in $L_2(\gamma)$...)

We see that γ satisfies (3c1) with

$$p_x(y, y') = \frac{1}{2\pi\sigma(x)} \exp\left(-\frac{y^2}{2} - \frac{y'^2}{2\sigma^2(x)} \right) . \tag{3d5}$$

Condition (3c2) boils down to $\sigma(\cdot) \in L_1[a,b]$, which is ensured by (3d3). Thus, we may use the theory of 3c.

3d6 Exercise. (Rice's formula)[1] For almost all[2] $y \in \mathbb{R}$,

$$\mathbb{E}\left(\# f^{-1}(y) \right) = \frac{1}{\pi} \mathrm{e}^{-y^2/2} \int_a^b \sigma(x) \, \mathrm{d}x \, .$$

Prove it.

Hint: (3c11) and (3d5).

Let us try it on the toy model (3a1). Here $[a,b] = [0, 2\pi]$, $\sigma(\cdot) = 1$, and we get $\mathbb{E}\left(\# f^{-1}(y) \right) = 2\mathrm{e}^{-y^2/2}$. In fact, $\# f^{-1}(0) = 2$ a.s., and $\# f^{-1}(y) = 2$ if $M > |y|$, otherwise 0; therefore $\mathbb{E}\left(\# f^{-1}(y) \right) = 2\mathbb{P}\big(M > y \big) = 2 \int_y^\infty f_M(u) \, \mathrm{d}u = 2\mathrm{e}^{-y^2/2}$ by (3a5).

3d7 Exercise. Calculate $\mathbb{E}\left(\# f^{-1}(y) \right)$ for the random trigonometric polynomial of 1c12,

$$X(t) = \zeta_1 \cos t + \eta_1 \sin t + \frac{1}{2} \zeta_2 \cos 2t + \frac{1}{2} \eta_2 \sin 2t \, .$$

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[1] Kac 1943, Rice 1945, Bunimovich 1951, Grenander and Rosenblatt 1957, Ivanov 1960, Bulinskaya 1961, Itô 1964, Ylvisaker 1965 et al. See [1, Sect. 10.3].

[2] In fact, for all y, see 3d12.

Integrating Rice's formula in y we get

$$\int_{\mathbb{R}} \mathrm{d}y \, \mathbb{E}\left(\# f^{-1}(y)\right) = \sqrt{\frac{2}{\pi}} \int_a^b \sigma(x) \, \mathrm{d}x \, , \tag{3d8}$$

which can be obtained simpler, by averaging (in f) the equality 3b1 (and using Fubini's theorem). Basically, Rice's formula states that $\mathbb{E}\left(\# f^{-1}(y)\right) = \mathrm{const} \cdot \mathrm{e}^{-y^2/2}$, where the coefficient does not depend on y; its value follows easily from 3b1. We may rewrite Rice's formula as

$$\mathbb{E}\left(\# f^{-1}(y)\right) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2} \, \mathbb{E} \int_a^b |f'(x)| \, \mathrm{d}x \tag{3d9}$$

for almost all $y \in \mathbb{R}$; note that $\mathbb{E} \int_a^b |f'(x)| \, \mathrm{d}x$ is the expected total variation of f. For example, on the toy model (3a1), $\# f^{-1}(0) = 2$ a.s., the total variation is equal to $4M$, and $\mathbb{E}\, M = \int_0^\infty u f_M(u) \, \mathrm{d}u = \sqrt{\pi/2}$.

The right-hand side of Rice's formula is continuous (in y); in order to get it for all (rather than almost all) y we will prove that the left-hand side is also continuous (in y). First, we do it for a one-dimensional non-centered Gaussian measure.

3d10 Exercise. Let $g, h \in C^1[a,b]$ and $h(x) \ne 0$ for all $x \in [a,b]$. Then the function

$$y \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}u \, \mathrm{e}^{-u^2/2} \, \# f_u^{-1}(y) \, ,$$

where $f_u(\cdot) = g(\cdot) + u h(\cdot)$, is continuous on $\mathbb{R}$.

Prove it.

Hint: the integral is equal to the total variation of the function $x \mapsto \Phi\big(\frac{y-g(x)}{h(x)}\big)$ where $\Phi(u) = \gamma^1\big((-\infty, u]\big)$.

3d11 Lemma. The function $y \mapsto \mathbb{E}\left(\# f^{-1}(y)\right)$ is continuous on $\mathbb{R}$.

Proof (sketch). It is sufficient to prove it on small subintervals of $[a,b]$, due to additivity. Assume that γ is infinite-dimensional (finite dimension is similar but simpler). We have $\gamma = V(\gamma^\infty)$ for some $V : S_1 \to E$, $S_1 \subset \mathbb{R}^\infty$ being a linear subspace of full measure. Thus, $f = g + uh$, where $u \sim \mathrm{N}(0,1)$, $h = V((1,0,0,\dots))$ and $g = V((0,\cdot,\cdot,\dots))$. Conditionally, given g, we may apply 3d10 provided that h does not vanish. Otherwise we do it on a neighborhood of any given point, picking up an appropriate coordinate of $\mathbb{R}^\infty$. $\square$

3d12 Corollary. Formulas 3d6, (3d8), (3d9) hold for all $y \in \mathbb{R}$ (not just almost all).

Especially,

$$\mathbb{E}\left(\# f^{-1}(0)\right) = \frac{1}{\sqrt{2\pi}} \mathbb{E} \int_a^b |f'(x)| \, \mathrm{d}x \, . \tag{3d13}$$

3d14 Exercise. Formulas 3d6, (3d8), (3d9), (3d13) still hold if $\sigma(\cdot)$ may vanish.

Prove it.

Hint: pass to the new variable $x_{\text{new}} = \int_0^x \sigma(x_1) \, \mathrm{d}x_1$.

3e Some integral geometry

We consider a curve on $S^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}$ parameterized by some $[a, b]$;

$$Z \in C^1\big([a,b], \mathbb{R}^n\big) \, , \quad Z\big([a,b]\big) \subset S^{n-1} \, .$$

It leads to a Gaussian random vector in $C^1[a, b]$,

$$f(x) = \langle Z(x), \xi \rangle \, ,$$

where ξ is distributed γ^n. (Thus, $f(x) = \zeta_1 Z_1(x) + \dots + \zeta_n Z_n(x)$ where $\zeta_1, \dots, \zeta_n$ are independent random variables distributed $\mathrm{N}(0, 1)$ each, and $Z_1, \dots, Z_n \in C^1[a, b]$.)

The function $f(\cdot)$ vanishes when the curve $Z(\cdot)$ intersects the hyperplane $\{z \in \mathbb{R}^n : \langle z, \xi \rangle = 0\}$. The latter is just a random hyperplane distributed uniformly, since $\xi/|\xi|$ is distributed uniformly of S^{n-1}. Thus, $\mathbb{E}\left(\# f^{-1}(0)\right)$ is the mean number of intersections.

On the other hand,

$$\sigma^2(x) = \mathbb{E} \, |f'(x)|^2 = \mathbb{E} \, |\langle Z'(x), \xi \rangle|^2 = |Z'(x)|^2 \, ,$$

thus, $\int_a^b \sigma(x) \, \mathrm{d}x$ is nothing but the length of the given curve. By Rice's formula (for $y = 0$),

$$\frac{\text{the mean number of intersections}}{\text{the length of the curve}} = \frac{1}{\pi} \, . \tag{3e1}$$

For example, the toy model (3a1) corresponds to $Z : [0, 2\pi] \to \mathcal{S}^1$, $Z(t) = (\cos t, \sin t)$. The curve is the unit circle, of length 2π. The number of intersections is equal to 2 always.

References

[1] H. Cramér, M.R. Leadbetter, *Stationary and related stochastic processes,* Wiley 1967.

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