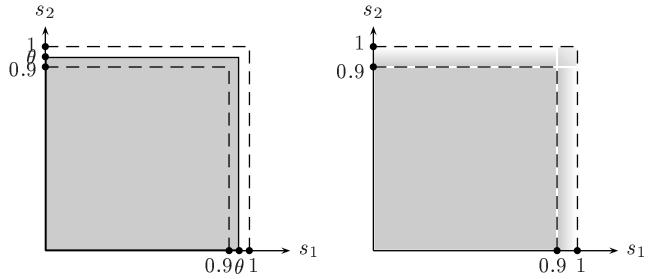
9 The end of monotone equilibria

9a Entry cost *and* unknown distributions

We introduce an entry cost c into our 'not-so-simple auction' (recall 8a), that is, a first price, private value, single unit auction with two players, having such a distribution of (correlated) signals:



Formally, it is a symmetric game described by

$$\mathcal{S}_1 = \mathcal{S}_2 = \mathcal{S} = \mathbb{R}\,; \qquad \mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A} = [0,\infty)\,;$$
$$\Theta = \mathbb{R}\,; \quad P_\Theta = \mathrm{U}(0.9, 1)\,;$$
$$P_{S_1|\theta} = P_{S_2|\theta} = P_{S|\theta} = \mathrm{U}(0,\theta)\,;$$
$$\mathbf{\Pi}_1 = \mathbf{\Pi}_2 = \mathbf{\Pi} \text{ is the function defined by}$$
$$\mathbf{\Pi}(a_1, s_1; a_2, s_2) = \mathbf{G}(a_1, s_1; a_2) - \mathbf{L}(a_1; a_2)\,,$$

$$\mathbf{G}(a_1, s_1; a_2) = \begin{cases} 0 & \text{if } a_1 < a_2, \\ 0 & \text{if } a_1 = a_2 = 0, \\ \frac{1}{2} s_1 & \text{if } a_1 = a_2 > 0, \\ s_1 & \text{if } a_1 > a_2; \end{cases} \tag{9a1}$$

$$\mathbf{L}(a_1; a_2) = \begin{cases} 0 & \text{if } a_1 = 0, \\ c & \text{if } 0 < a_1 < a_2, \\ c + \frac{1}{2} a_1 & \text{if } 0 < a_1 = a_2, \\ c + a_1 & \text{if } a_1 > a_2 \end{cases}$$

(recall (3a1), (3e1), (3g1); the reserve price is zero). Hopefully, the game has a symmetric monotone equilibrium[1] (like 8c) with a participation threshold (like 3g):

$$\begin{aligned} &A = \varphi(S)\,; \\ &\varphi(s) = 0 \quad \text{for } s < s_0\,; \\ &\varphi(s) > 0 \quad \text{for } s > s_0\,; \\ &\varphi \text{ is continuous and strictly increasing on } (s_0, 1)\,. \end{aligned} \tag{9a2}$$

Inequality (8c2) must hold for all $s, t \in (s_0, 1)$, since the entry cost, added to both sides, may be canceled. Therefore (assuming smoothness) we get the differential equation (8c4) as

[1] What about other equilibria (non-monotone and/or non-symmetric)? I do not know.

before;[2] now, however, it holds on $(s_0, 1)$ rather than $(0, 1)$.

We introduce the *associated* auction (as in 8c) with independent signals distributed $\tilde{F}$; the function $\tilde{F}$, defined by (8c6), appeared to describe just the uniform distribution, $\mathrm{U}(0, 1)$. We ascribe an entry cost $\tilde{c}$ to the associated auction; note that $\tilde{c}$ need not be equal to c. The equilibrium strategy function of the associated auction is known to us (recall 7a):

$$\varphi^{\text{assoc}}(s) = \max\left(0, \frac{1}{2}s\Big(1 - \frac{\tilde{c}}{s^2}\Big)\right).$$

The function φ^{assoc} satisfies the same differential equation (8c4). Having the free parameter $\tilde{c}$, we have a one-parameter family of solutions; no need to solve the differential equation. Hopefully $\varphi = \varphi^{\text{assoc}}$ if $\tilde{c}$ is chosen appropriately.

Let us find an equation for s_0. We have two conditions:

- the action $\varphi(s)$ must be (optimal, therefore) better[3] for s than the action 0 (quit), whenever $s \in (s_0, 1)$;[4]
- the action 0 must be better for s than any positive action, whenever $s \in (0, s_0)$.

The first condition means (recall (8c1)) $0 \le (s - \varphi(s)) F_{\varphi(S_2)|S_1=s}(\varphi(s)) - c$. However, $F_{\varphi(S_2)|S_1=s}(\varphi(s)) = F_{S_2|S_1=s}(s) = F_s(s)$. Thus,

$$\big(s - \varphi(s)\big) F_s(s) \ge c \quad \text{for } s \in (s_0, 1)\,, \tag{9a3}$$

therefore (by continuity) $s_0 F_{s_0}(s_0) \ge c$.

The second condition means

$$(s - a) F_{\varphi(S_2)|S_1=s}(a) - c \le 0 \quad \text{for all } a > 0, s \in (0, s_0)\,, \tag{9a4}$$

hence (by continuity; take $a \to 0+$) $s_0 F_{s_0}(s_0) \le c$. Therefore, s_0 must satisfy

$$s_0 F_{s_0}(s_0) = c\,. \tag{9a5}$$

Also, the first condition implies $(s_0 - \varphi(s_0+)) F_{s_0}(s_0) \ge c$, hence

$$\varphi(s_0+) = 0$$

(in other words, φ is continuous on the whole $(0, 1)$). It means that $\varphi = \varphi^{\text{assoc}}$ if $\tilde{c}$ is chosen to be s_0^2. So,

$$\begin{gathered} \varphi(s) = \max\left(0, \frac{1}{2}s\Big(1 - \frac{s_0^2}{s^2}\Big)\right); \\ \tilde{c} = s_0^2\,; \\ c = s_0 F_{s_0}(s_0)\,. \end{gathered} \tag{9a6}$$

[2] It does not mean the same φ as before, since the differential equation has a (one-parameter) family of solutions.

[3] I did not say *strictly* better.

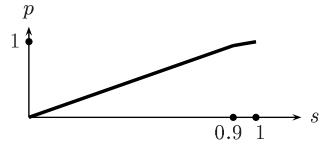
[4] For almost all $s \in (s_0, 1)$; however, everything is continuous here ...

For now we do not claim that such φ describes an equilibrium. Rather, we claim that no other φ (satisfying (9a2)) can do so.

Being a solution of the differential equation (8c4), φ satisfies also (8c3) and (8c2) on $(s_0, 1)$ due to the superadditivity argument (recall 8d). It remains to check the two conditions about participation.

In order to check (9a3) it suffices to show that the function $s \mapsto (s - \varphi(s))F_s(s)$ increases on $(s_0, 1)$. In fact, both factors $s - \varphi(s)$ and $F_s(s)$ increase. Indeed, $s - \varphi(s) = s - \frac{1}{2}s\bigl(1 - \frac{s_0^2}{s^2}\bigr) = \frac{s_0}{2}\bigl(\frac{s}{s_0} + \frac{s_0}{s}\bigr)$ is minimal at $s = s_0$ and increases for $s \in (s_0, \infty)$. Also,

$$F_s(s) = \begin{cases} \frac{1}{9\ln(10/9)} s & \text{for } s \in (0, 0.9), \\ \frac{1-s}{\ln(1/s)} & \text{for } s \in (0.9, 1) \end{cases}$$



(check it by integrating the conditional density written out on page 103, or adapt formulas of page 105; see also (9a8)), an increasing function on $(0, 1)$. So, (9a3) holds.

In order to check (9a4) it is worth thinking first, which $a \in (0, 1)$ maximizes $(s - a)F_{\varphi(S_2)|S_1=s}(a)$ for a given $s \in (0, s_0)$. In other words: which action is optimal for the first player having the signal s, if he is forced to participate, or equivalently, released from the entry cost. It is a question about the best response to the given strategy $A_2 = \varphi(S_2)$ of the second player. The argument of 8b (recall 8b2) shows that the best response of $s \in (0, s_0)$ must be less than that of every signal of $(s_0, 1)$; it must be $0+$. I mean, the best response probably does not exist, but anyway, $\sup_a (s - a)F_{\varphi(S_2)|S_1=s}(a) = \lim_{a \to 0+} (s - a)F_{\varphi(S_2)|S_1=s}(a) = s\mathbb{P}\bigl(\varphi(S_2) = 0 \bigm| S_1 = s\bigr) = sF_s(s_0)$. Thus (9a4) is equivalent to

$$sF_s(s_0) \le c \quad \text{for all } s \in (0, s_0)\,. \tag{9a7}$$

In order to check (9a7) it suffices to show that the function $s \mapsto sF_s(s_0)$ increases on $(0, s_0)$. We have

$$F_s(t) = \begin{cases} \frac{t}{9\ln(10/9)} & \text{for } s \le t \le 0.9; \\ 1 + \frac{1-t-\ln(1/t)}{\ln(10/9)} & \text{for } s \le 0.9 \le t; \\ \frac{1-t+\ln(t/s)}{\ln(1/s)} & \text{for } 0.9 \le s \le t. \end{cases} \tag{9a8}$$

Note that $F_s(s_0)$ does not depend on s as far as $s \in (0, 0.9)$. The question is, whether or not the function $s \mapsto sF_s(s_0)$ increases on $(0.9, s_0)$, when $s_0 > 0.9$. Here we have

$$\begin{gathered} sF_s(s_0) = \frac{1 - s_0 + \ln(s_0/s)}{\ln(1/s)} s = \biggl(1 - \frac{s_0 - 1 + \ln(1/s_0)}{\ln(1/s)}\biggr) s\,; \\ \frac{d}{ds}\bigl(sF_s(s_0)\bigr) = 1 - \bigl(s_0 - 1 + \ln(1/s_0)\bigr)\biggl(\frac{1}{\ln^2 s} - \frac{1}{\ln s}\biggr)\,; \end{gathered}$$

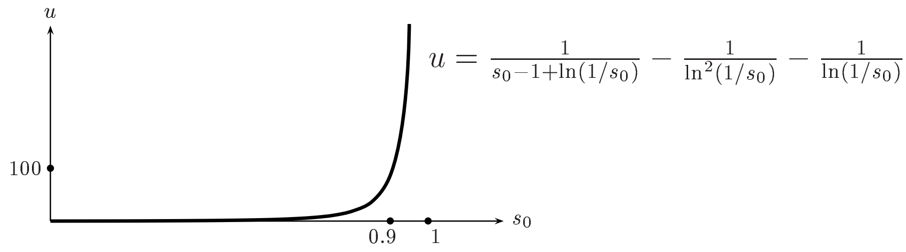
we need

$$\frac{1}{\ln^2(1/s)} + \frac{1}{\ln(1/s)} \le \frac{1}{s_0 - 1 + \ln(1/s_0)} \quad \text{for } s \in (0.9, s_0)\,,$$

which is enough to check for $s = s_0$ only:

$$\frac{1}{\ln^2(1/s_0)} + \frac{1}{\ln(1/s_0)} \le \frac{1}{s_0 - 1 + \ln(1/s_0)} \, ;$$

the latter happens to be true for all $s_0 \in (0.9, 1)$ (in fact, for all $s_0 \in (0,1)$):



So, for every $c \in (0,1)$ the game (9a1) has one and only one symmetric monotone equilibrium of the form (9a2), namely (9a6).

9b Many players *and* entry cost *and* unknown distributions

We consider the symmetric game of n players, described by $(\mathcal{S}, \mathcal{A}, \varTheta, P_\Theta, (P_{S|\theta}), \mathbf{\Pi}, n)$ (recall 5a), where $\mathcal{S}, \mathcal{A}, \varTheta, P_\Theta, (P_{S|\theta}), \mathbf{\Pi}$ are defined by (9a1), while n is now arbitrary. Recalling the case of n players but independent signals (considered in 5d, page 61) and the case of correlated signals but two players (considered in 9a) we may hope for a symmetric monotone equilibrium of the form (9a2):

$$\begin{aligned}
&A = \varphi(S)\, ; \\
&\varphi(s) = 0 \quad \text{for } s < s_0\, ; \\
&\varphi(s) > 0 \quad \text{for } s > s_0\, ; \\
&\varphi \text{ is continuous and strictly increasing on } (s_0, 1)\, .
\end{aligned}$$

Everything should be similar to 9a, but $X = \max(S_2, \dots, S_n)$ is used instead of S_2.

The associated auction has n independent signals distributed $\tilde F$,

$$\tilde F(s) = \exp\biggl(-\frac{1}{n-1} \int_s^1 \frac{f_{S_1,X}(t,t)}{\int_0^t f_{S_1,X}(t,u)\,du}\,dt \biggr) = \exp\biggl(-\frac{1}{n-1} \int_s^1 \frac{f_{X|S_1=t}(t)}{F_{X|S_1=t}(t)}\,dt \biggr) .$$

Fortunately, we do not need the whole conditional distribution of X given $S_1 = s$; rather, we need its restriction to $(0,s)$, which is easy to calculate due to a special property of our distribution (recall page 104):

$$\mathbb{P}\bigl(X \le x \,\big|\, \Theta = \theta \bigr) = \mathbb{P}\bigl(S_2 \le x, \dots, S_n \le x \,\big|\, \Theta = \theta \bigr) = \min\biggl(\Bigl(\frac{x}{\theta}\Bigr)^{n-1}, 1 \biggr) ;$$

$$S_1 \longleftarrow \Theta \longrightarrow X$$

$$\begin{aligned}
\mathbb{P}\bigl(X \le x \,\big|\, S_1 \bigr) &= \mathbb{E}\bigl(\mathbb{P}\bigl(X \le x \,\big|\, S_1, \Theta \bigr) \,\big|\, S_1 \bigr) = \\
&= \mathbb{E}\bigl(\mathbb{P}\bigl(X \le x \,\big|\, \Theta \bigr) \,\big|\, S_1 \bigr) = \mathbb{E}\bigl(\min\bigl((x/\Theta)^{n-1}, 1\bigr) \,\big|\, S_1 \bigr) \, ;
\end{aligned}$$

for $x \le s$ we get (taking into account that $\Theta \ge S_1$)

$$\mathbb{P}\left(X \le x \,\middle|\, S_1 = s \right) = \mathbb{E}\left(\min\bigl((x/\Theta)^{n-1}, 1\bigr) \,\middle|\, S_1 = s \right) = \mathbb{E}\left((x/\Theta)^{n-1} \,\middle|\, S_1 = s \right) ,$$

thus, $\mathbb{P}\left(X \le x \,\middle|\, S_1 = s \right) = \operatorname{const}(s) \cdot x^{n-1}$, where $\operatorname{const}(s) = \mathbb{E}\left(1/\Theta^{n-1} \,\middle|\, S_1 = s \right)$; therefore

$$\begin{gathered}
F_{X|S_1=s}(x) = \operatorname{const}(s) \cdot x^{n-1} \quad \text{for } x \in [0,s]\,; \\
f_{X|S_1=s}(x) = \operatorname{const}(s) \cdot (n-1)x^{n-2} \quad \text{for } x \in [0,s]\,; \\
\frac{f_{X|S_1=s}(s)}{F_{X|S_1=s}(s)} = \frac{n-1}{s}\,; \\
\tilde{F}(s) = \exp\biggl(-\frac{1}{n-1} \int_s^1 \frac{n-1}{t}\, dt \biggr) = s\,;
\end{gathered}$$

still the uniform distribution, $\mathrm{U}(0,1)$.

The equilibrium strategy of the associated auction was calculated in 7a:

$$\varphi^{\text{assoc}}(s) = \max\biggl(0, \frac{n-1}{n} s \Bigl(1 - \frac{\tilde{c}}{s^n} \Bigr) \biggr) .$$

The first condition for the participation threshold s_0 (recall 9a, page 114) is $0 \le \bigl(s - \varphi(s)\bigr) F_{\varphi(X)|S_1=s}\bigl(\varphi(s)\bigr) - c$; now (9a3) becomes

$$\bigl(s - \varphi(s)\bigr) F_{X|S_1=s}(s) \ge c \quad \text{for } s \in (s_0, 1) \tag{9b1}$$

and implies $s_0 F_{X|S_1=s_0}(s_0) \ge c$.

The second condition for s_0 is (recall 9a4)

$$(s-a) F_{\varphi(X)|S_1=s}(a) - c \le 0 \quad \text{for all } a > 0,\ s \in (0, s_0)\,, \tag{9b2}$$

and implies $s_0 F_{X|S_1=s_0}(s_0) \le c$. Instead of (9a5) we get $s_0 F_{X|S_1=s_0}(s_0) = c$, and also $\varphi(s_0+) = 0$, as before; (9a6) turns into

$$\begin{gathered}
\varphi(s) = \max\biggl(0, \frac{n-1}{n} s \Bigl(1 - \frac{s_0^n}{s^n} \Bigr) \biggr)\,; \\
\tilde{c} = s_0^n\,; \\
c = s_0 F_{X|S_1=s_0}(s_0)\,.
\end{gathered} \tag{9b3}$$

Either it gives an equilibrium of the form (9a2), or there is no such equilibrium at all.

The left-hand side of (9b1) is equal to c when $s = s_0$. Let us calculate its derivative in s at $s = s_0$. We have

$$F_{X|S_1=s}(s) = s^{n-1} \mathbb{E}\left(1/\Theta^{n-1} \,\middle|\, S_1 = s \right)\,;$$

Bayes formula (for densities)

$$f_{\Theta|S_1=s}(\theta) = \frac{f_{S_1||\Theta=\theta}(s) f_\Theta(\theta)}{f_{S_1}(s)}$$

gives for $s \in (0.9, 1)$

$$f_{\Theta|S_1=s}(\theta) = \begin{cases} \frac{1}{\theta \ln(1/s)} & \text{when } \theta \in (s,1), \\ 0 & \text{when } \theta \in (0.9, s). \end{cases}$$

Hence

$$\mathbb{E}\left(1/\Theta^{n-1} \,\middle|\, S_1 = s \right) = \int_{0.9}^{1} \frac{1}{\theta^{n-1}} f_{\Theta|S_1=s}(\theta)\, d\theta =$$

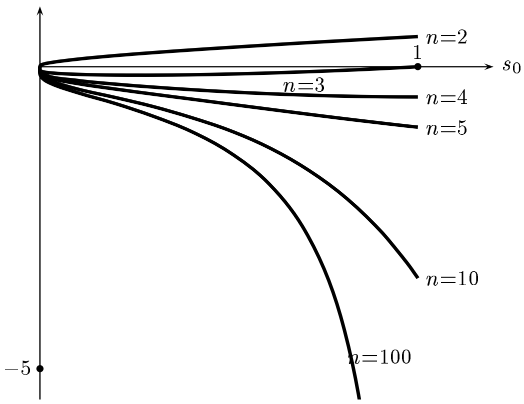
$$= \frac{1}{\ln(1/s)} \int_s^1 \frac{1}{\theta^n}\, d\theta = \frac{1}{\ln(1/s)} \frac{1}{n-1} \left(\frac{1}{s^{n-1}} - 1 \right)$$

and

(9b4) $$F_{X|S_1=s}(s) = \frac{1}{n-1} \frac{1-s^{n-1}}{\ln(1/s)} \quad \text{for } s \in [0.9, 1)\,.$$

An elementary but somewhat tedious calculation gives

$$\frac{d}{ds}\bigg|_{s=s_0} \Big((s - \varphi(s)) F_{X|S_1=s}(s) \Big) = \frac{1}{n-1} \frac{1-s_0^{n-1}}{\ln^2(1/s_0)} - \frac{1}{n-1} \frac{n-2+s_0^{n-1}}{\ln(1/s_0)}\,.$$

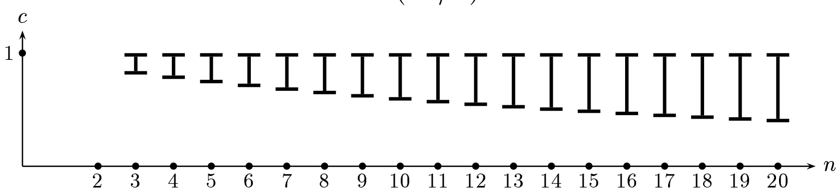


What a surprise! The case $n = 2$ is not just the simplest case, it is an exceptional case! For all other n, the derivative is negative for all $s_0 \in (0.9, 1)$.[5] It means nonexistence of equilibria of the form (9a2) for $n > 2$, if the entry cost c satisfies

(9b5) $$0.9 F_{X|S_1=0.9}(0.9) < c < 1$$

which corresponds to $0.9 < s_0 < 1$. Using (9b4) we see that the nonexistence is ensured for

$$0.9 \frac{1}{n-1} \frac{1-0.9^{n-1}}{\ln(10/9)} < c < 1\,.$$



[5]In fact, the expression is negative for all $s_0 \in (0,1)$, which becomes relevant, if we replace $\Theta \sim \mathrm{U}(0.9, 1)$ with $\Theta \sim \mathrm{U}(\theta^{\min}, 1)$ for any $\theta^{\min} \in (0,1)$.

It does not mean existence for *all* other c, since (a) a positive derivative at s_0 does not ensure the inequality on the whole $(s_0, 1)$, and (b) the second condition could exclude more cases. In order to examine a given c (and n), we have to check (9b1) for all $s \in (s_0, 1)$, and (9b2) for all $s \in (0, s_0)$; however, in (9b2) we may take $a \to 0+$ (similarly to 9a):

$$sF_{X|S_1=s}(s_0) \le c \quad \text{for all } s \in (0, s_0) \tag{9b6}$$

(recall (9a7)).

Assume that $s_0 < 0.9$ (otherwise we know the answer already). If $s \in (0, 0.9)$ then $F_{X|S_1=s}$ does not depend on s (a special property of our distribution), which makes the second condition (9b6) evidently satisfied:

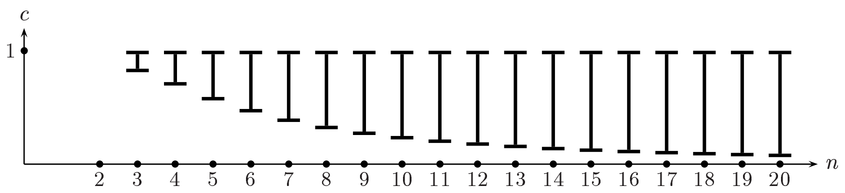
$$sF_{X|S_1=s}(s_0) = sF_{X|S_1=s_0}(s_0) \le s_0 F_{X|S_1=s_0}(s_0) = c\,.$$

For the first condition, the case $s \in (0, 0.9)$ is also easy:

$$\begin{aligned}\big(s - \varphi(s)\big)F_{X|S_1=s}(s) &= \big(s - \varphi(s)\big)F_{X|S_1=0.9}(s) = \big(s - \varphi(s)\big)\cdot \text{const}\cdot s^{n-1} = \\ &= \text{const}\cdot s^{n-1}\bigg(s - \frac{n-1}{n} s\Big(1 - \frac{s_0^n}{s^n}\Big)\bigg) = \text{const}\cdot\Big(\frac{1}{n}s^n + \frac{n-1}{n}s_0^n\Big)\,,\end{aligned}$$

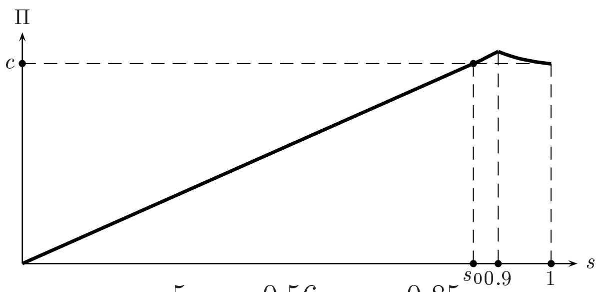
which evidently increases in s. It remains to check the first condition for $s \in (0.9, 1)$:

$$\big(s - \varphi(s)\big)\frac{1}{n-1}\frac{1 - s^{n-1}}{\ln(1/s)} \ge c \quad \text{for } s \in (0.9, 1)\,. \tag{9b7}$$

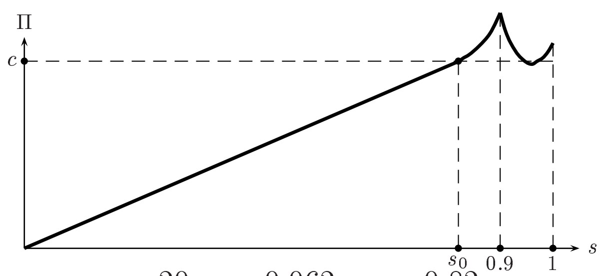


You see, the nonexistence holds unless c is small or n is not large. Say, for $n = 20$ it holds for $c > 0.062$.

It is instructive to see the expected profit of, say, the first player *released from the entry cost* and playing the best response against others that still play the strategy (9b3), especially for the critical value of the entry cost; the expected profit is a function of the signal.



$n = 5,\ c = 0.56,\ s_0 = 0.85$



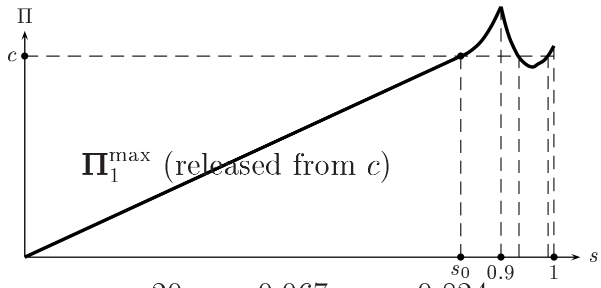
$n = 20,\ c = 0.062,\ s_0 = 0.82$

You see, for $s \in (0, 0.9)$ it is a convex function, just like the case of independent signals, and no wonder: here s is effectively s^{int} only, since s^{ext} is effectively constant (say, 0.9) due to a special property of our distribution. The interval $(0, s_0)$ is a bunch (corresponding to the 'virtual action' 0+).

After 0.9 the situation changes dramatically. Here s is both s^{int} and s^{ext}; a higher s^{int} means good news (for the player), however, a higher s^{ext} means bad news. The latter leads to a decrease of the profit. If it lowers under the entry cost c then participation is not optimal for corresponding signals. That is the failure of the participation threshold.

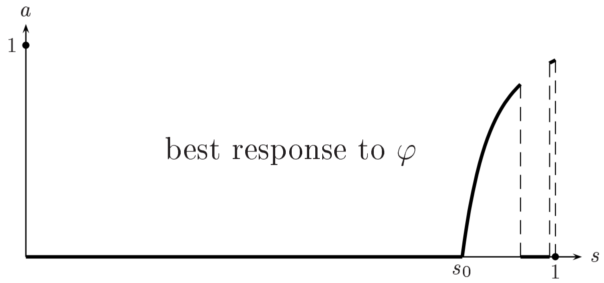
If players $2, \dots, n$ play the strategy φ of (9b3) and the first player is released from the entry cost then his best response is φ; his expected payoff function can cross c more than once.

$n = 20$, $c = 0.067$, $s_0 = 0.824$



$n = 20$, $c = 0.067$, $s_0 = 0.824$

Returning to the normal situation (the entry cost is incurred) we get a *non-monotone best response* to the monotone strategy (even though signals are affiliated):



$n = 20$, $c = 0.067$, $s_0 = 0.824$

You see, in the absence of entry cost, a more aggressive bidding of others makes the best response more aggressive. However, in presence of an entry cost, a more aggressive bidding of others can prevent participation.

9c To burst or not to burst[6]

The nonexistence of monotone equilibria, pointed out in 9b, is of quite general nature, as we'll see soon. After all, monotone equilibria fail because they cannot give a satisfactory answer to the question, to burst or not to burst, as explained below.

Return for a while to the joint distribution of n signals, used in 9b:

$$\mathbb{P}\left(S_1 \le s_1, \dots, S_n \le s_n \,\middle|\, \Theta = \theta \right) = F_\theta(s_1) \dots F_\theta(s_n) = \min\left(\frac{s_1}{\theta}, 1\right) \dots \min\left(\frac{s_n}{\theta}, 1\right),$$
$$\Theta \sim \mathrm{U}(0.9, 1)\,.$$

[6]See also: M. Landsberger and B. Tsirelson, "Correlated signals against monotone equilibria", April 2000, Preprint SSRN 222308.

Assume existence of a monotone equilibrium for each n. (We already know from 9b that the assumption is false; however, we want to find a more general argument, why.) Then we have a participation threshold $t_n \in (0,1)$ for each n. The number of participants is a random variable

$$K_n = \mathbf{1}_{(t_n,\infty)}(S_1) + \dots + \mathbf{1}_{(t_n,\infty)}(S_n)\,;$$

the mean number of participants is

$$\mathbb{E}\,K_n = np_n\,,$$

where p_n is the participation probability,

$$p_n = \mathbb{P}\big(\,S_1 > t_n\,\big) = \mathbb{E}\mathbb{P}\big(\,S_1 > t_n\,\big|\,\Theta\,\big) = 1 - F_{S_1}(t_n)\,.$$

What happens for $n \to \infty$? Taking into account entry cost, and a single unit to be sold, we may expect $p_n \to 0$, and moreover, $p_n = O(1/n)$, that is, boundedness of np_n. Indeed, the total entry cost paid by all players is $np_n \cdot c$ in the mean, while the total gain of all players never exceeds $\max(S_1,\dots,S_n)$. Voluntary participation implies

$$np_nc \le \mathbb{E}\,\max(S_1,\dots,S_n)\,.$$

If signals have a compact support, $\mathbb{P}\big(\,S_1 \le s^{\max}\,\big) = 1$, then we get $np_nc \le s^{\max}$, thus

(9c1) $$p_n \le \frac{s^{\max}}{c}\frac{1}{n}\,.$$

If signals have a non-compact support, we may do similarly to 7d: $\mathbb{E}\,\max(S_1,\dots,S_n) \le n\int_{1-\frac1n}^1 S^*(p)\,dp$ (think, why); assuming (recall 7d5)

$$\int_{1-\varepsilon}^1 S^*(p)\,dp \le M\varepsilon^\alpha \quad \text{for all } \varepsilon \in (0,1)\,,$$

where M and α do not depend on n, we get $p_n = O(1/n^\alpha)$, namely,

(9c2) $$p_n \le \frac{M}{c}\frac{1}{n^\alpha}\,.$$

In any case $p_n \to 0$; therefore

$$t_n \to 1\,.$$

It is quite natural, isn't it? On one hand, it really is; only few players are willing to pay the entry cost in the hope of winning the single unit. On the other hand, however, consequences are terrible: *in most cases, the auction is empty!* Recall, $\Theta \sim \mathrm{U}(0.9,1)$, thus $t_n \to 1$ implies $\mathbb{P}\big(\,\Theta < t_n\,\big) \to 1$. However, $S_1,\dots,S_n \le \Theta$, hence

$$\mathbb{P}\big(\,K_n = 0\,\big) = \mathbb{P}\big(\,S_1 < t_n,\dots,S_n < t_n\,\big) \ge \mathbb{P}\big(\,\Theta < t_n\,\big) \xrightarrow[n\to\infty]{} 1\,.$$

It is strange; it is a pity for the auctioneer; but above all, it simply cannot happen in equilibrium. If a player knows that, very probably, he has no competitors, then he definitely wants to participate. *An auction cannot be empty too often!*

So, once again, what is happening to K_n for $n \to \infty$? The expectation $\mathbb{E} K_n = np_n = n\big(1 - F_{S_1}(t_n)\big)$ is able to behave nicely, in particular, tend to a given number (neither 0 nor ∞). However, the *mean* value $\mathbb{E} K_n$ of K_n is not its *typical* value. In most cases $K_n = 0$. In rare cases $K_n > 0$, and here K_n is typically large, much greater than $\mathbb{E} K_n$. That is a burst-like behavior.

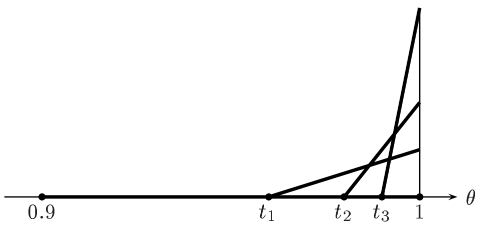
Specifically, we have (recall p. 100) $f_{S_1}(s) = 10 \min\big(\ln(1/s), \ln(10/9)\big) \sim \text{const} \cdot (1-s)$ for $s \to 1-$, thus $1 - F_{S_1}(s) \sim \text{const} \cdot (1-s)^2$ for $s \to 1-$. The case $1 - F_{S_1}(t_n) = p_n \sim \text{const}/n$ appears when $1 - t_n \sim \text{const}/\sqrt{n}$. In most cases, $\Theta < t_n$. In rare cases, $\Theta > t_n$; then typically $\Theta - t_n \sim \text{const}/\sqrt{n}$ and $K_n \sim n \cdot \text{const}/\sqrt{n} \sim \text{const} \cdot \sqrt{n}$. We have, roughly,

$$
\begin{aligned}
&K_n = 0 \quad \text{with probability } 1 - \text{const}/\sqrt{n}\,;\\
&K_n \sim \text{const} \cdot \sqrt{n} \quad \text{with probability const}/\sqrt{n}\,.
\end{aligned}
$$

Thus, $\mathbb{E} K_n \sim \text{const}$, however, K_n is either large or zero; that is the burst.

Finding a threshold t (for a given n) means finding a participation ray (t, ∞).[7] In general, a pure strategy $A = \varphi(S)$ determines its *participation set* $\{s : \varphi(s) \ne 0\}$, not just a ray. However, if φ is a *monotone* (increasing) strategy, then its participation set is necessarily a ray. Alas, rays appear to be inappropriate to be participation sets for large n. *Rays are a burst collection* in the following sense:

- If $E_1, E_2, \dots$ are rays (of the form (t, ∞) each) such that $\mathbb{P}\big(S_1 \in E_n\big) > 0$ and $\mathbb{P}\big(S_1 \in E_n\big) \xrightarrow[n\to\infty]{} 0$ then $\dfrac{\mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta = \theta\big)}{\mathbb{P}\big(S_1 \in E_n\big)} \xrightarrow[n\to\infty]{} 0$ for all $\theta \in (0.9, 1)$.



Note that

$$
\frac{\mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta = \theta\big)}{\mathbb{P}\big(S_1 \in E_n\big)} = \frac{P_{S|\theta}(E_n)}{\int P_{S|\theta}(E_n)\, dP_\Theta(\theta)}\,;
$$

the random variable $\mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta\big) / \mathbb{P}\big(S_1 \in E_n\big)$ is of expectation 1 for every n. Nevertheless, for $n \to \infty$ it may tend to 0[8] almost surely (which happens in our special case), or (more generally) in probability,[9] which is stipulated by the following definition.

[7] Which boils down to $(t, 1)$ for our example.

[8] Neither bounded convergence theorem nor dominated convergence theorem (nor monotone convergence theorem) can be applied here.

[9] Recall the definition: $X_n \to 0$ in probability, if $\mathbb{P}\big(|X_n| \le \varepsilon\big) \to 1$ for every $\varepsilon > 0$. Convergence a.s. implies convergence in probability; the converse is wrong.

9c3. Definition. A collection $\mathcal{E}$ of subsets of the signal space $\mathcal{S}$ is called a *burst* collection (with respect to $(P_{S|\theta}), P_\Theta$), if for all $E_1, E_2, \dots \in \mathcal{E}$ such that $\mathbb{P}\big(S_1 \in E_n \big) > 0$ and $\mathbb{P}\big(S_1 \in E_n \big) \to 0$, the following sequence of random variables[10] converges to 0 in probability (when $n \to \infty$):

$$\frac{\mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta \big)}{\mathbb{P}\big(S_1 \in E_n \big)} \, .$$

The following event is clearly related to emptiness of an auction:

$$S_1 \notin E_n, \dots, S_n \notin E_n \, .$$

Its probability is

$$\mathbb{P}\big(S_1 \notin E_n, \dots, S_n \notin E_n \big) = \mathbb{E}\big(\mathbb{P}\big(S_1 \notin E_n, \dots, S_n \notin E_n \,\big|\, \Theta \big) \big) = \\ \mathbb{E}\big(\mathbb{P}\big(S_1 \notin E_n \,\big|\, \Theta \big) \dots \mathbb{P}\big(S_n \notin E_n \,\big|\, \Theta \big) \big) = \mathbb{E}\big(1 - \mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta \big) \big)^n \, .$$

This is why the following lemma is relevant.

9c4. Lemma. A burst collection cannot contain $E_1, E_2, \dots$ such that

$$\limsup_n \big(n\mathbb{P}\big(S_1 \in E_n \big) \big) < \infty \, , \tag{9c5}$$

$$\limsup_n \mathbb{E}\big(1 - \mathbb{P}\big(S_1 \in E_n \,\big|\, \Theta \big) \big)^n < 1 \, . \tag{9c6}$$

(For a proof see the preprint cited on page 120.)

On the other hand, let E_n be the participation set of a strategy supporting a symmetric equilibrium (for n players). Then (9c5) is satisfied due to (9c1), as far as a positive entry cost (not depending on n) is incurred, and signals have a compact support (not depending on n). Also (9c6) is satisfied; the emptiness probability must be bounded away from 1, unless players are utterly repelled by a high entry cost. So, under conditions mentioned, *participation sets cannot be chosen from a burst collection.*

9d Burst collections

An example of a burst collection involves not just a collection $\mathcal{E}$ of subsets E of a signal space $\mathcal{S}$, but also a parametric space Θ, a probability distribution P_Θ on Θ, and a family $(P_{S|\theta})$ of probability distributions on S, indexed by $\theta \in \Theta$.[11]

Our first example was

$$\Theta = (0.9, 1), \quad P_\Theta = \mathrm{U}(0.9, 1), \quad P_{S|\theta} = \mathrm{U}(0, \theta), \\ \mathcal{S} = \mathbb{R}, \quad \mathcal{E} = \{ (t, \infty) : t \in \mathbb{R} \} \, .$$

Here the right endpoint $s^{\max}(\theta)$ of the support of $P_{S|\theta}$ has a nonatomic distribution; indeed, $s^{\max}(\Theta) = \Theta \sim \mathrm{U}(0.9, 1)$. That is the only relevant feature of $(P_{S|\theta})$. The parameter space Θ may be multidimensional.

[10] They are functions of Θ.

[11] One may also eliminate Θ by considering a probability measure on the (infinite dimensional) space of all probability measures on $\mathcal{S}$.

9d1. Exercise. Let Θ be two-dimensional, $\Theta = (\Theta_1, \Theta_2) \sim \mathrm{U}(0, 0.1) \otimes \mathrm{U}(0.9, 1)$,[12] and let $P_{S|\theta} = \mathrm{U}(\theta_1, \theta_2)$. Then the collection

$$\mathcal{E} = \{(t, \infty) : t \in \mathbb{R}\} \cup \{[t, \infty) : t \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\} \tag{9d2}$$

is a burst collection.

Prove it.

Hint: if $t_n \to 1-$ and $E_n = (t_n, \infty)$ then for every θ, $P_{S|\theta}(E_n)$ vanishes for n large enough.

The collection (9d2) consists of all *increasing sets*[13] $E \subset \mathbb{R}$, that is, sets satisfying the condition

$$\forall x, y \in \mathbb{R} \quad \bigl(x \le y \;\&\; x \in E \quad \Longrightarrow \quad y \in E \bigr) \,; \tag{9d3}$$

it just means that the indicator $\mathbf{1}_E(\cdot)$ is an increasing function on $\mathbb{R}$.

If $E = (t, \infty)$ then $P_{S|\theta}(E) = 1 - F_\theta(t)$, where F_θ is the (cumulative) distribution function of $P_{S|\theta}$. The mixture P_S of all $P_{S|\theta}$ has its distribution function

$$F(t) = \int F_\theta(t) \, dP_\Theta(\theta) \,,$$

and the right endpoint of its support, $s^{\max} = \sup\{s : F(s) < 1\} \in (-\infty, +\infty]$. The quotient

$$\frac{1 - F_\theta(t)}{1 - F(t)} = \frac{P_{S|\theta}(E)}{P_S(E)}$$

tends to 0 when $t \to s^{\max}-$ (or $t \to \infty$, if $s^{\max} = \infty$) in an extravagant way: it just vanishes in a neighborhood of $s^{\max}$. In the next example the support of F_θ does not depend on θ, and the quotient converges to 0 without vanishing.

9d4. Exercise. Let P_Θ be some nonatomic distribution on $\Theta = (0, \infty)$, and

$$P_{S|\theta} = \mathrm{Exp}(\theta) \,,$$

the exponential distribution on $\mathcal{S} = \mathbb{R}$; in other words, $\mathbb{P}\bigl(S > s \,\big|\, \Theta = \theta \bigr) = \exp(-s/\theta)$. Then increasing sets are a burst collection.

Prove it.

Hint: for every θ we may take $\theta_1 > \theta$ such that

$$\begin{aligned} 1 - F(t) = \int_0^\infty \exp\Bigl(-\frac{s}{\theta} \Bigr) \, dF_\Theta(\theta) \ge \int_{\theta_1}^\infty \exp\Bigl(-\frac{s}{\theta} \Bigr) \, dF_\Theta(\theta) \ge \\ \ge \mathrm{const} \cdot \exp\Bigl(-\frac{s}{\theta_1} \Bigr) = o\biggl(\exp\Bigl(-\frac{s}{\theta} \Bigr) \biggr) \quad \text{for } s \to \infty \,. \end{aligned}$$

Sometimes, however, the burst does not appear.

[12] That is, Θ_1, Θ_2 are independent, $\Theta_1 \sim \mathrm{U}(0, 0.1)$, $\Theta_2 \sim \mathrm{U}(0.9, 1)$.

[13] Known also under the name 'upper layers'.

9d5. Exercise. Let P_Θ be some nonatomic distribution on $\Theta = \mathbb{R}$, and $P_{S|\theta}$ be the shifted exponential distribution on $\mathcal{S} = \mathbb{R}$:

$$\mathbb{P}\left(S > s \,\middle|\, \Theta = \theta \right) = \exp(\theta - s) \quad \text{for } s \ge \theta\,.$$

Then increasing sets are not a burst collection.

Prove it.

Hint: $1 - F_\theta(s) \sim \text{const} \cdot e^{-s}$ for $s \to \infty$, irrespective of θ; also $1 - F(s) \sim \text{const} \cdot e^{-s}$.

9d6. Lemma. Assume that for every θ the distribution $P_{S|\theta}$ has a density f_θ, and

$$\frac{f_\theta(s)}{f(s)} \to 0 \quad \text{for } s \to +\infty\,;$$

here $f(x) = \int_\Theta f_\theta(s)\, dP_\Theta(\theta)$ is the density of P_S, assumed to be non-zero almost everywhere. Then increasing sets are a burst collection.

(For a proof see the preprint cited on page 120.)

An important example is the multinormal (that is, multidimensional normal) distribution. Here f and f_θ are normal densities; they differ both in mean values and in variances. The variance of f_θ is strictly less than that of f, hence $f_\theta(x)/f(x) \to 0$ irrespective of mean values.

Lemma 9d6 still holds for a multidimensional signal space $\mathcal{S} = \mathbb{R}^d$ (using $|s| \to \infty$ rather than $s \to +\infty$).[14] Thus, the burst argument excludes monotone equilibria also for multidimensional signals.

Small deviations from monotonicity do not invalidate the argument. Instead of a 'sharp threshold' t one may consider a 'fuzzy threshold' $(t-\varepsilon, t+\varepsilon)$ such that $S > t+\varepsilon$ ensures participation, and $S < t - \varepsilon$ ensures non-participation. The burst argument can be generalized accordingly.

What about a compact support? Inequality (9c1) needs a finite $s^{\max}$, the upper bound of signals. The burst argument can be used for unbounded (in particular, normally distributed) signals, provided that *valuations* are bounded. Otherwise, we turn to (9c2) which, however, needs a modification to Definition 9c3. Namely, a δ-burst collection is defined similarly to 9c3, using

$$\frac{\left(\mathbb{P}\left(S_1 \in E_n \,\middle|\, \Theta \right)\right)^{1/\delta}}{\mathbb{P}\left(S_1 \in E \right)}$$

instead of the quotient used in 9c3. (The case $\delta = 1$ returns us to 9c3.) The burst argument works whenever $\alpha \ge 1/\delta$ (see (9c2) for α). The most strong burst, $\delta = \infty$, is produced by a 'floating support', as in 9d1; here, monotone equilibria are excluded for all α. A weaker burst is produced by the multinormal distribution; here δ depends on the correlation between signals.[15]

[14] Note that increasing sets in $\mathbb{R}^d$ are not so simple as in $\mathbb{R}$.

[15] Namely, if the correlation coefficient between signals is ρ (equivalently, the correlation coefficient between Θ and S_1 is $\sqrt{\rho}$), then the δ-burst happens for any $\delta < 1/(1-\rho)$.

The burst argument is quite insensitive to auction rules. Non-private value auctions, all-pay auctions, and many others are included. The number of units to be sold need not be just 1;[16] it is enough if it is kept fixed when $n \to \infty$.

Here are possible reasons for monotone equilibria to appear:

- The number of players[17] is small.
- The number of units to be sold is not small as compared to the number of players.
- Players are well informed about the distribution of signals, or at least, its right tail.[18]
- The entry cost is *very* small (or reimbursed).

All that is about *symmetric* auctions (and symmetric equilibria). An asymmetric auction game with 100 players, 2 strong and 98 weak, can easily have a monotone equilibrium such that the two strong players always participate while others always quit.

[16]For a multi-unit auction, the action space may be multi-dimensional, which may invalidate the notion of a monotone bidding strategy. However, the notion of a monotone *participation* strategy still works.

[17]That is, *potential* bidders.

[18]That is, about its rate of convergence to 1 for high signals. In the case of a compact support, knowing its right endpoint is necessary (but not sufficient).