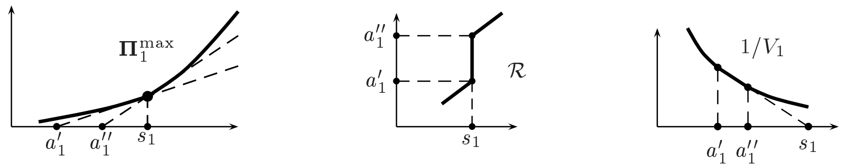
Addendum: An important argument of Bernard Lebrun

Assume that the action A_1 has a gap (a_1', a_1''),[1] while the signal S_1 has no gaps. In terms of the convex function $\mathbf{\Pi}_1^{\max}$ it means a jump of the derivative. In terms of the weakly increasing relation $\mathcal{R}$ (between s_1 and a_1, recall (6d7)) it means a vertical segment ($s_1 = \text{const}$). And in terms of the convex function $1/V_1$ it means a linear segment.



Anyway, the gap (a_1', a_1'') corresponds to some s_1; both a_1' and a_1'' are optimal for s_1. Note that[2] $V_1(a_1') = W_1(a_1')$ and $V_1(a_1'') = W_1(a_1'')$, but $V_1(a) \ge W_1(a)$ for $a \in (a_1', a_1'')$. We have

$$(s_1 - a_1')W_1(a_1') = (s_1 - a_1'')W_1(a_1'') \ge (s_1 - a)W_1(a) \quad \text{for } a \in (a_1', a_1''),$$

that is,

$$\frac{s_1 - a_1'}{s_1 - a_1''} = \frac{W_1(a_1'')}{W_1(a_1')} \quad \text{and} \quad \frac{s_1 - a}{s_1 - a_1'} \le \frac{W_1(a_1')}{W_1(a)} \quad \text{for } a \in (a_1', a_1'').$$

On the other hand, the function $W_1/W_2 = F_{A_2}/F_{A_1}$ increases on $[a_1', a_1'']$, since F_{A_1} is constant here. Hence

$$\frac{W_1(a_1')}{W_2(a_1')} \le \frac{W_1(a)}{W_2(a)}, \quad \text{that is,} \quad \frac{W_1(a_1')}{W_1(a)} \le \frac{W_2(a_1')}{W_2(a)},$$

and we get

$$\frac{s_1 - a}{s_1 - a_1'} \le \frac{W_2(a_1')}{W_2(a)} \quad \text{for all } a \in (a_1', a_1'').$$

Assume in addition that some point a_2 of (a_1', a_1'') belongs to the support of A_2; then a_2 is optimal for (the second player having) some signal s_2:

$$(s_2 - a_2)W_2(a_2) \ge (s_2 - a)W_2(a) \quad \text{for all } a;$$
$$\frac{s_2 - a_2}{s_2 - a} \ge \frac{W_2(a)}{W_2(a_2)} \quad \text{for all } a.$$

We have

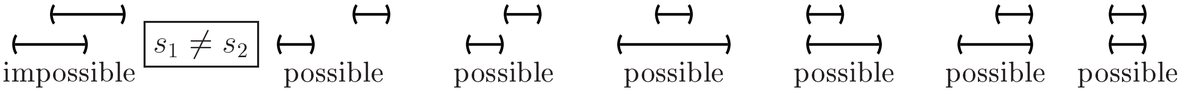
$$\frac{s_1 - a_2}{s_1 - a_1'} \le \frac{W_2(a_1')}{W_2(a_2)} \le \frac{s_2 - a_2}{s_2 - a_1'},$$

[1]That is, the support of (the distribution of) the action A_1 (of the first player) contains a_1' and a_1'' but no one point of the open interval (a_1', a_1'').

[2]It was noted before Lemma 6d13 that V_1 and W_1 may differ at a point of the support. However, it cannot happen when W_1 is continuous.

therefore $s_1 \le s_2$, since the function $s \mapsto \frac{s-a_2}{s-a_1'}$ increases strictly on (a_2, ∞).

If A_2 also has a gap (a_2', a_2'') corresponding to s_2, and $s_2 < s_1$, then the two gaps (a_1', a_1'') and (a_2', a_2'') cannot overlap. Indeed, points a_2' and a_2'' belong to the support of A_2 and are optimal for s_2; therefore they cannot belong to (a_1', a_1''). The opposite case $s_2 > s_1$ is similar: a_1', a_1'' cannot belong to (a_2', a_2''), thus the two gaps still cannot overlap.



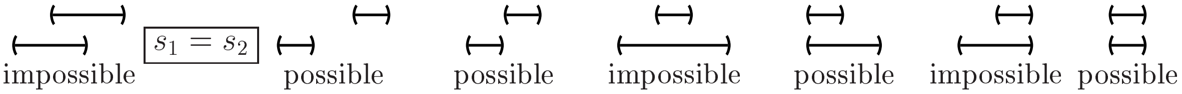
We see that any two gaps are either disjoint (that is, $(a_1', a_1'') \cap (a_2', a_2'') = \emptyset$) or linearly ordered by inclusion (that is, $(a_1', a_1'') \subset (a_2', a_2'')$ or $(a_1', a_1'') \supset (a_2', a_2'')$[3]) provided, however, that these gaps correspond to different signals ($s_1 \ne s_2$). The remaining case ($s_1 = s_2$) will be considered later.

We return for a while to a gap (a_1', a_1'') of A_1 and a point $a_2 \in (a_1', a_1'')$ belonging to the support of A_2. We know that the corresponding signals s_1, s_2 satisfy $s_1 \le s_2$. Can it happen that $s_1 = s_2$? Recall that

$$\frac{s_1 - a_2}{s_1 - a_1'} \le \frac{W_1(a_1')}{W_1(a_2)} \le \frac{W_2(a_1')}{W_2(a_2)} \le \frac{s_2 - a_2}{s_2 - a_1'} \, ;$$

for $s_1 = s_2$ all these inequalities must turn into equalities. In particular, the increasing function $W_1/W_2 = F_{A_2}/F_{A_1} = \mathrm{const} \cdot F_{A_2}$ must be constant on (a_1', a_2), which means $\mathbb{P}\big(a_1' < A_2 < a_2\big) = 0$.

Now we are in position to consider two gaps (a_1', a_1'') and (a_2', a_2'') corresponding to the same signal $s_1 = s_2$. If $a_2' \in (a_1', a_1'')$ then (recall that a_2' belongs to the support of A_2) $\mathbb{P}\big(a_1' < A_2 < a_2'\big) = 0$, and a_2' is an isolated point of the support of A_2, in contradiction to nonatomicity (recall Sect. 6c). So, a_2' cannot belong to (a_1', a_1''). Similarly, a_1' cannot belong to (a_2', a_2'').



The principal conclusion remains true (be s_1 and s_2 equal or not):

$$\boxed{\text{Any two gaps are either disjoint or linearly ordered}}$$

Any chain (I mean, a set linearly ordered by inclusion) of gaps is necessarily finite (in fact, not longer that the number of players). Therefore, if there exists at least one gap then there exists a minimal gap (containing no other gap). Such a minimal gap (a_k', a_k'') is a cell (in the sense of Sect. 6f), and the function $1/W_k$ is strictly convex on (a_k', a_k'') by 6g8, which leads to a contradiction. It means that there are no gaps at all!

So, uniqueness of the equilibrium is proven in full generality (without assuming (6f3)), which is an important result of Bernard Lebrun. Arguments of Sect. 6h are not needed. Also, arguments of Sect. 6g may be replaced with corresponding arguments of Lebrun.

I thank Bernard Lebrun for sending me his working paper.

[1] Bernard Lebrun, "First price auctions in the asymmetric n bidder case." Les Cahiers de Recherche du GREEN-Department d'Economique de l'Universite Laval (working paper series) #97-03, 1997.

[3]Or both: $(a_1', a_1'') = (a_2', a_2'')$.