

The Containment Game: between the Firefighter Problem and Conway’s Angel Problem

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Abstract

The containment game is a full information game for two players, initialised with a set of occupied vertices in an infinite connected graph G . On the t -th turn, the first player, called Spreader, extends the occupied set to $g(t)$ adjacent vertices, and then the second player, called Container, removes q unoccupied vertices from the graph. If the spreading process continues perpetually – Spreader wins, and otherwise – Container wins. For $g = \infty$ this game reduces to a solitaire game for Container, known as the Firefighter Problem. On \mathbb{Z}^2 , for $q = 1/k$ and $g \equiv 1$ it is equivalent to Conway’s angel problem.

We introduce the game and study it on the strongly connected two dimensional integer lattice $\mathbb{Z} \boxtimes \mathbb{Z}$. Writing $q(G, g)$ for the set of q values for which Container wins against a given $g(t)$, we study the minimal asymptotics of $g(t)$ for which $q(G, g) = q(G, \infty)$, i.e. for which defeating Spreader is as hard as winning the firefighter problem solitaire. We show, by providing explicit winning strategies, a sub-linear upper bound $g(t) = O(t^{6/7})$ and a lower bound of $g(t) = \Omega(t^{1/2})$.

Keywords— Conway’s angel problem, Firefighter problem, combinatorial games, pursuit-evasion games.

1 Introduction

The (G, q) -*firefighter problem* is a single real parameter solitaire combinatorial game played on a graph G . In this game, introduced by Hartnell [24], given a finite starting set of *burning* vertices $\mathcal{B}_0 \subset V$, at every turn $t \in \{1, 2, \dots\}$, the *firefighter* player chooses an arbitrary collection of at most $\lfloor tq \rfloor - \lfloor (t-1)q \rfloor$ non-burning vertices and deletes them from the graph. Then, neighbours of burning vertices in the remainder graph become burning. If no new burning vertices are generated, we say that the fire is *contained*. The firefighter player *wins* (G, q) if they are able to contain any any finite initial burning set. We denote

$$q_G := \{q : (G, q) \text{ is firefighter win for every finite } \mathcal{B}_0\}.$$

Since winning is monotone in q this set always forms an infinite open or closed ray.

The problem has been extensively studied on the integer lattice with nearest neighbour adjacency, either with respect to L^∞ or to L^1 , graphs which are denoted by $\mathbb{Z} \boxtimes \mathbb{Z}$ and $\mathbb{Z} \square \mathbb{Z}$, respectively. Following the work of Fogarty [19], Hod and the first author [16] showed that $q_{\mathbb{Z} \boxtimes \mathbb{Z}} = (3, \infty)$ while $q_{\mathbb{Z} \square \mathbb{Z}} = (3/2, \infty)$.

On these graphs, the firefighter problem is strongly related to the celebrated Conway’s Angel problem, a two-player game played between an angel of speed k and a devil (first appearing in [6, Section 19]). In this game, an angel is located at the origin of the graph $G = \mathbb{Z} \boxtimes \mathbb{Z}$. In every turn the angel player may reposition the angel to any unblocked vertex at distance k or less from its current location. The devil player then deletes any single vertex from

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the graph, except for the one occupied by the angel. The devil player wins if the angel is eventually unable to move; otherwise – the angel wins. Both the $\mathbb{Z} \boxtimes \mathbb{Z}$ and the $\mathbb{Z} \square \mathbb{Z}$ variants have been considered [10, 33].

The firefighter problem could be interpreted as a “non-deterministic angel” variant of the angel problem where the devil player gets to play q turns on average between subsequent angel player’s turns. In this analogy the firefighter player take the role of a devil player which is oblivious to the location of the angel and must nevertheless contain it with absolute certainty.

This gives rise to the following informal question, which motivates our study:

How much non-determinism is required for the angel to be as hard to contain as the fire?

To pose this formally, we introduce the following generalisation of both games, which we call the *containment game*, played between two players which we call *Spreader* and *Container*. This is also a generalisation of the k -firefighter game suggested by Devlin and Hartke [14] and studied on finite graphs by Bonato, Messinger and Pralat [9].

Given a *spreading* function $g : \mathbb{N} \rightarrow \mathbb{N}$ we define the $(G = (V, E), q, g)$ *containment game* as the following two player combinatorial game. Given a finite initial set \mathcal{B}_0 of occupied vertices, at every turn Container deletes a collection of at most $\lfloor tq \rfloor - \lfloor (t-1)q \rfloor$ non-occupied vertices from the graph. Then Spreader selects $g(t)$ vertices at distance at most one from \mathcal{B}_{t-1} in the remainder graph and occupies them by adding them to \mathcal{B}_{t-1} to form \mathcal{B}_t . If after some point in time no new vertices are occupied – Container wins, and otherwise – Spreader wins. We denote the set of deleted vertices up to time t by \mathcal{F}_t and the induced graph on $V \setminus \mathcal{F}_t$ by G_t . Given g we define

$$q(G, g) := \{q : (G, q, g) \text{ is Container win for every finite } \mathcal{B}_0\},$$

observing that this is, again, an infinite open or closed ray, and that $q(G, g') \subset q(G, g)$ for all $g' > g$. We write (G, q, ∞) for the containment game without any restriction on the number of vertices selected by Spreader each turn. It is not difficult to observe that in this case the optimal Spreader strategy is to always extend \mathcal{B}_t to all of its available neighbours, and the game played by Container is non other than the firefighter problem solitaire. On the other hand, for $g \equiv 1$ and $q = \frac{1}{k}$, the game reduces to Conway’s angel problem of speed k .

Our question of interest could be now phrased as:

“Which is the minimal $g(t)$ asymptotics for which $q(G, g) = q(G, \infty)$?”

For brevity and conciseness we show our results only for $G = \mathbb{Z} \boxtimes \mathbb{Z}$, although analogs for $G = \mathbb{Z} \square \mathbb{Z}$ seem to follow from the same arguments. We also study the game on the following simpler graph $G = \mathbb{Z}_{\triangleright}^2$, which is the sub-graph of $\mathbb{Z} \boxtimes \mathbb{Z}$ restricted to $\{(x, y) : 0 \leq x \leq y\}$ equipped with the edge set $\{((x, y), (x + i, y + 1)) : 0 \leq x \leq y, i \in \{0, 1\}\}$. This graph captures the essence of our methods and serves as a stepping stone for the study of $\mathbb{Z} \boxtimes \mathbb{Z}$.

Our main result is that a sub-linear g can achieve $q(\mathbb{Z} \boxtimes \mathbb{Z}, g) = q(\mathbb{Z} \boxtimes \mathbb{Z}, \infty)$.

Theorem 1. *Let $G \in \{\mathbb{Z}_{\triangleright}^2, \mathbb{Z} \boxtimes \mathbb{Z}\}$. For all sufficiently large $C > 0$ we have $q(G, Ct^{6/7}) = q(G, \infty)$.*

Remark 1. *Obtaining concrete sub-linear bounds in Theorem 1 involves certain complications throughout the proof. On first reading we therefore recommend the reader to consider a linear analog, asserting that for $G \in \{\mathbb{Z}_{\triangleright}^2, \mathbb{Z} \boxtimes \mathbb{Z}\}$ and all $\alpha > 0$ we have $q(G, \alpha t) = q(G, \infty)$. Remarks regarding the components of the proof that are unnecessary for this case are scattered throughout the paper.*

Remark 2. *Theorem 1 is obtained via an explicit Spreader strategy. In this strategy vertices occupied at turn t are always adjacent to vertices occupied at turn $t - 1$. Hence, our result is valid also for a variant of the game, which we call the Container-Avoider game, where the occupied vertices move and split, rather than accumulate.*

We complement this theorem with the following lower-bound.

Theorem 2. *Let $G \in \{\mathbb{Z}_{\triangleright}^2, \mathbb{Z} \boxtimes \mathbb{Z}\}$. For all $c < \frac{1}{6}$ we have $q(G, ct^{1/2}) \subsetneq q(G, \infty)$.*

Accumulating firefighters. Note that rather than asking the number of deleted vertices up to time t to be at most qt as per [16], we pose the stronger restriction that the number of vertices deleted at turn t is at most $\lfloor tq \rfloor - \lfloor (t-1)q \rfloor$. In the firefighter problem, where the fire is deterministic, the two models are equivalent. In the containment game, however, a Spreader with a sub-linear spread function cannot win against an accumulating Container of any strength q even on $\mathbb{Z}_{\triangleright}^2$ (using a proof similar to that of Theorem 2).

1.1 Related work

The firefighter problem. Ever since its introduction in 1995 [24], Hartnell’s firefighter problem has been the subject of study in diverse contexts. The case of infinite G studied here has been studied for the triangular grid \mathbb{Z}_Δ^2 , where Dean et al. [13] improved upon a result of Gavenčiak, Kratochvíl, and Prałat [22], showing that a single firefighter per turn plus one additional firefighter at some turn are sufficient to contain a single fire source. It has been conjectured that this bound is tight and $q_{\mathbb{Z}_\Delta^2} = (1, \infty)$, but currently no rigorous proof is available. Indeed, understanding the problem on the triangular and hexagonal lattices remain important open problems in this topic.

The problem on a slab $G = \mathbb{Z} \boxtimes \mathbb{Z} \boxtimes [k]$ has been studied by Deutsch, Hod and the first author, who showed $q_G = (3k, \infty)$. On \mathbb{Z}^d , rough bounds were obtained by Develin and Hartke [14]. The problem has also been studied on Cayley graphs. Dyer, Martinez-Pedroza and Thorne [15] showed that the critical growth rate is roughly invariant under quasi-isometrics. The same authors have also obtained bounds on the critical growth for the number of firefighters required to contain any finite starting fire on Cayley graphs of polynomial growth in [13]. This was later improved by Amir, Baldasso, and Kozma [4], to obtain bounds that are tight up to a constant. The problem has also been studied for Cayley graphs of exponentially growing groups [30] and on groups of intermediate growth [5]. Other questions relating the game to group theory have been studied [3, 32].

The problem has been widely considered also on finite graphs where one wishes either to reduce the number of burning vertices when the process terminates, or to minimise the time it takes to contain the fire. This has been studied mainly from an algorithmic point of view, showing that the problems are NP-hard [17, 27, 31], but could be approximated up to a constant factor in polynomial time [2, 11, 23, 25, 26]. These results are surveyed in [18].

Conway’s Angel problem. The origin of Conway’s Angel problem is somewhat obscure. Variants are mentioned by Martin Gardner as early as ’74 [21], where credits are given to D. Silverman and R. Epstein. In its current transformation, the problem first appeared in the classical monograph by Berlkamp, Conway and Guy [6, Section 19]. Conway [12] later showed that an angel of speed 1 loses. Several years after, Kutz [29] showed that an angel of speed $2 - \epsilon$ loses for every $\epsilon > 0$. In the meantime, the problem of showing that an angel of high enough speed wins acquired some notoriety, until finally in ’07 four independent papers by Bowditch [10], Gács [20], Kloster [28] and Máthé [33] established this fact and solved the original problem. The last two of these show that, in fact, speed 2 is sufficient.

Precursors of current work. The idea of studying a restricted fire model arose in the context of the original firefighter problem on finite graphs. After being suggested by Devlin and Hartke [14], the constant g and finite G case of the containment game suggested here was studied by Bonato, Messinger and Prałat [9]. There the quantity of interest was the expected percentage of surviving vertices under optimal play, when the initial occupied set is a single uniformly chosen vertex, and $q = 1$.

Other pursuit games on graphs. Many other variants of pursuit games on graphs have been studied, both for applicative reasons and as a method for understanding graph connectivity. We provide references to several prominent examples which are somewhat related to the games studied here. The Burning Number of a Graph [8] is a solitaire initialised with an empty burning set, where at every time-step, neighbours of burning vertices are added to the burning set along with a single additional vertex chosen by the player. The goal here is to reduce the number of rounds required to burn all vertices in the graph. Cops and Robbers is another two-player pursuit game, where Container is moving a set of blocked vertices (cops) along the graph’s edges trying to catch a finite number of Avoider (robbers) vertices (c.f. Remark 2). See [7] for a monograph on this game. Finally, Invisible Rabbits [1] is an oblivious variant of the cops and robbers game, where the cops’ movements are not restricted by the edges of the graph, but the Container player is oblivious to the location of the robbers, and must nevertheless catch them.

1.2 Open problems

We pose several open problems concerning the containment game and its variants.

Firstly, we would be most interested in the following problem.

Problem 1. *Close the gap between Theorem 1 and Theorem 2.*

We conjecture that the lower bound in Theorem 2 is tight, so that $q(G, t^{1/2+\epsilon}) = q(G, \infty)$ for all $\epsilon > 0$.

In addition, the proof of Theorem 2 merely establish the fact that $1 \in q(\mathbb{Z}_{\nu}^2, \infty) \setminus q(\mathbb{Z}_{\nu}^2, Ct^{1/2})$ and $3 \in q(\mathbb{Z} \boxtimes \mathbb{Z}, \infty) \setminus q(\mathbb{Z} \boxtimes \mathbb{Z}, Ct^{1/2})$. This leaves us with the following.

Problem 2. *Is $\inf q(G, t^{1/2-\epsilon}) < \inf q(G, \infty)$ for $G \in \{\mathbb{Z}_{\nu}^2, \mathbb{Z} \boxtimes \mathbb{Z}\}$ and every $\epsilon > 0$?*

Constant expansion. Here we have analysed the parameter range for which the containment game on a graph G satisfies $q(G, g) = q(G, \infty)$. It would be interesting to consider the more general question of recovering the dependence of $q(G, g)$ on g . The case of constant expansion, i.e. $g \equiv k$, is of particular interest, both for its tighter relations with Conway's angel problem and for its elegance, pitting a constant power Spreader against a constant power Container. It can be shown, by means similar to the proof of Theorem 2, that Container wins the game $(\mathbb{Z} \boxtimes \mathbb{Z}, k, 3 - \frac{c}{k})$ for some c independent of k . As this bound tends to 3 as k tends to infinity, the following question stands.

Problem 3. *What is $\liminf_{k \rightarrow \infty} q(\mathbb{Z} \boxtimes \mathbb{Z}, k)$? is it strictly less than 3?*

High dimensions. By [4] we know that the critical number of firefighters needed to contain the fire in the strongly connected \mathbb{Z}^d is $\Theta(t^{d-2})$, and recovering the exact constant seems within reach (using techniques from [16]). One can ask what is the critical spreading function that would be equivalent to unrestricted spreading in this case and is it asymptotically smaller, namely:

Problem 4. *Is there $g(t) = o(t^{d-1})$ which is equivalent to unrestricted spreading in the strongly connected \mathbb{Z}^d graph?*

In general, we suspect that for all groups of sub-exponential growth, some $g(t)$ which grows asymptotically slower than the isoperimetric profile should be able to imitate unrestricted fire spreading.

Probabilistic variant. It should also be mentioned that the setting of probabilistic spread where the fire spreads to a neighbour with constant probability at every turn, appears not to have been studied so far. The realistic application of this setting, along with natural limiting shape questions and relations to classical models in particle systems, make this a rather appealing variant to study.

1.3 Proof highlights

The main novel ingredients in our theorems are already well expressed in the case of \mathbb{Z}_{ν}^2 . Much like the fire in the lower bound of [19] for the firefighter problem, the occupied vertices in our Spreader strategy in \mathbb{Z}_{ν}^2 are always placed on a single row $\mathbb{Z} \times \{t\}$, progressing northwards at every time-step. The essential difference is that here most of this row is populated by a sparse array of occupied vertices, at distance h apart. Each vertex (x, t) among these is viewed as representing the fire in the horizontal segment $(h\lfloor x/h \rfloor, t+h) + \{(0, 0), \dots, (0, h-1)\}$, to which we refer as the *simulated fire*.

Interestingly, it is possible to show that as long as Container refrains from deleting at time t vertices in the band $\mathbb{Z} \times [t, t+h-1]$, the evolution potential inequalities of [16] concerning the fire could still be made valid with respect to this simulated fire. Hence, the core difficulty is to handle situations in which Container plays at close proximity to the the occupied vertices. Such moves have the potential of eliminating a large number of simulated fire vertices by blocking the simulating occupied vertex directly by deleting only a handful of vertices. We call these events *disruptions* and compensate for them by taking advantage of the fact that that it is highly inefficient for Container to create cavities in the front (observe e.g. that isolated vertices spread to many more neighbours than densely packed ones). To do so we *alert* many vertices surrounding each disruption. Alerted vertices start spreading to all of their neighbours in $\mathbb{Z} \times \{t+1\}$ and keep doing so for H turns. We are able to show that over that time period, the disruption will be compensated for completely, so that when the disrupted region returns to simulative mode, the evolution potential inequalities of [16] hold. As Container can only create a fixed amount of disruptions per turn, Spreader can win while still keeping the occupied set sparse enough.

The exact execution of this program involves several additional subtleties, such as maintaining a bulk of alerted vertices near the ends of the occupied set.

In order to exploit this technique to the fullest we must increase h , the sparsity of the simulating vertices, as a function of t . To do so we double the sparsity every once in a while, a procedure which creates additional technical complications, but no essential difficulties.

1.4 Notations

We use the convention $\mathbb{N} := \{0, 1, 2, \dots\}$. For any $x \in \mathbb{R}$ we denote $x_+ := \max\{x, 0\}$. Δg_t denotes the backwards difference of g , defined by $\Delta g_t := g_t - g_{t-1}$ or $\Delta g_t := g_t \setminus g_{t-1}$. $d_G(v, u)$ denotes the graph distance between u and v in G .

Throughout, we naturally extend functions from elements to sets by following the convention that if g has numerical output, we set $g(A) = \sum_{x \in A} g(x)$, while if it produces sets, we set $g(A) = \bigcup_{x \in A} g(x)$.

Finally, we follow the standard convention that $\max \emptyset = -\infty$ and $\min \emptyset = +\infty$.

1.5 Outline of the paper

Section 2 is dedicated to the proof of Theorem 1 on \mathbb{Z}_{\nearrow}^2 , including the definition of Spreader's strategy and its analysis. In Section 3 we introduce the game on the directed half-plane graph \mathbb{Z}_{\searrow}^2 and generalise the results of Section 2 to this setting. In Section 4 we compose a winning strategy on the entire $\mathbb{Z} \boxtimes \mathbb{Z}$ graph from the strategy of Section 3, completing the proof of Theorem 1. Section 5, which is independent of the other sections, is dedicated to the proof of Theorem 2. The paper is accompanied by a notation table, provided as an appendix.

2 The Eighth Plane

This section is dedicated to proving Theorem 1 for $G = \mathbb{Z}_{\nearrow}^2$. We begin by describing a winning Spreader strategy on the graph, then reduce the theorem to several key propositions.

Recalling that the set of (Spreader) occupied vertices at time t is denoted by \mathcal{B}_t , the Spreader strategy will always satisfy $\mathcal{B}_t \subset \mathbb{Z} \times \{t\}$. We therefore refer to the row $\mathbb{Z} \times \{t\}$ as the *front* at time t .

Outline of this section. In Section 2.1 we present the winning Spreader strategy, postponing the precise definition of several key functions to later sections and settling for a description of their required properties. We also state an upper bound on the number of new vertices which the spreader strategy occupies each turn. In Section 2.2 we introduce the key functions used for the analysis of the process and reduce Theorem 1 to several propositions. The final details of the strategy needed to establish its existence are provided in Section 2.3. Once these are given, Section 2.4 is dedicated to prove the propositions stated in Section 2.1. Most of the details of the analysis are given in Section 2.5, leaving out two key notions: the *potential function* and the *debt*. Sections 2.6 to 2.9 are dedicated to develop these notions and establish the remaining propositions. Finally, in Section 2.10 we prove a technical lemma stated in Section 2.5. The section is accompanied by Figure 1, illustrating different aspects of Spreader's strategy.

2.1 The winning Spreader strategy

Knowing that the firefighter player of strength $q = 1$ loses the firefighter problem game on \mathbb{Z}_{\nearrow}^2 (by [19]), the Spreader strategy strives to imitate the fire evolution using a smaller occupied set, so that each occupied vertex represents a fire segment of size $h(t)$ in $\mathbb{Z} \times \{t + h(t)\}$, where $h : \mathbb{N} \rightarrow \mathbb{N}$ is a monotonically non-decreasing function. To introduce the strategy formally we require several definitions.

Segments. We partition \mathbb{Z} into intervals of size h_t of the form $\{0, \dots, h_t - 1\} + h_t \mathbb{Z}$. The set of segments at time t is denoted by \mathcal{S}_t . Given an interval $I \subset \mathbb{Z}$ we denote $\mathcal{S}_t(I) := \{S \in \mathcal{S}_t : S \cap I \neq \emptyset\}$.

Segments naturally inherit the order of \mathbb{Z} , so that $S_1, S_2 \in \mathcal{S}_t$ satisfy $S_1 < S_2$ if for all $s_1 \in S_1, s_2 \in S_2$ we have $s_1 < s_2$. We denote contiguous closed (similarly, open and half-open) intervals of segments by $[S_1, S_2] := \{S \in \mathcal{S}_t : S_1 \leq S \leq S_2\}$, when $S_1, S_2 \in \mathcal{S}_t$. Given $x \in \mathbb{Z}$, denote by $S_t(x)$ the unique segment in \mathcal{S}_t containing x , and extend this to \mathbb{Z}^2 by setting $S_t((x, y)) := S_t(x)$.

Doubling segments. For the construction and analysis of the sub-linear Spreader strategy used to establish Theorem 1, we will allow the size of the segments h to depend on t . To make the analysis simpler, we double the value of h once in a while, rather than change it gradually, so that $h : \mathbb{N} \rightarrow \mathbb{N}$ satisfies $h_t/h_{t-1} \in \{1, 2\}$. Given such h we denote $\mathbb{N}_i = \mathbb{N}_i(h) := \{t \in \mathbb{N} : h_t/h_{t-1} = i\}$ for $i = 1, 2$, such that $\{0\} \cup \mathbb{N}_1 \cup \mathbb{N}_2 = \mathbb{N}$. We refer to \mathbb{N}_2 as *doubling times*.

Over the course of time smaller dyadic segments in \mathcal{S}_s joint to form the segments in \mathcal{S}_t for $t > s$. Thinking of these bigger segments as descendants of their smaller counterparts, given $S \in \mathcal{S}_t$ we write $A_s(S) := \{S' \in \mathcal{S}_s : S' \subset S\}$.

For functions f' on \mathcal{S}_t defined below ($\phi_t(S), d_t(S), f_t(S), F_t(S)$ etc.), for $s < t$ we define, $f'_s(S) := \sum_{S' \in A_s(S)} f'_t(S')$, if f' yields numerical value, and $f'_s(S) := \cup_{S' \in A_s(S)} f'_t(S')$ if it produces sets.

Remark 3. *As per Remark 1, the reader is advised, at first reading, to only consider the case where h is constant, so that $A_t(S) = \{S\}$ for all t and \mathcal{S}_t is independent of t . In this case doubling never occurs, so that $\mathbb{N}_1 = \mathbb{N} \setminus \{0\}$ and $\mathbb{N}_2 = \emptyset$. This removes complications from many of the proofs and definitions, while retaining their essence.*

2.1.1 The strategy

For the remainder of Section 2, fix a constant $q > 0$ and an $h : \mathbb{N} \rightarrow \mathbb{N}$ as above, and set $H_t := 4q^2 h_t^2$. Throughout we make the following assumption concerning doubling times of h .

$$\text{If } t \in \mathbb{N}_2, \text{ then for all } s \in [t - 2H_t, t + 2H_t] \text{ we have } s \notin \mathbb{N}_2. \quad (1)$$

This will be satisfied by our final choice of h in the proof of Theorem 1.

Our strategy divides \mathcal{S}_t into two categories. The first category is *simulative segments*, indicated by $\chi_t(S) = 0$, in which a single occupied vertex simulates the fire on a segment of $\mathbb{Z} \times \{t + h_t\}$. The second category is *spreading segments*, indicated by $\chi_t(S) = 1$, the occupied vertices of which spread to all of their neighbours. The occupied vertex in a simulative segment S is called the *pivot*, and is denoted by $p_t(S)$. For completeness, for spreading segments we set $p_t(S) = \infty$. For $S \in \mathcal{S}_t$, the evolution of B_t is hence as follows.

$$B_t(S) := (S \times \{t\}) \cap \begin{cases} \{(p_t(S), t)\} & \text{if } \chi_t(S) = 0, \\ \{(B_{t-1} + \{(0, 1), (1, 1)\}) \setminus \mathcal{F}_t\} & \text{if } \chi_t(S) = 1, \end{cases} \quad (2)$$

where exact definitions of the evolution of $p_t(S)$ and $\chi_t(S)$ are delayed to Section 2.3. Finally, denote $b_t(S) := |B_t(S)|$, and, with the purpose of simplifying inductive definitions, we extend the definition of B_t and b_t also to $S \in \mathcal{S}_{t-1}$ by $B_{t-1}(S) := \cup_{S' \in A_{t-1}(S)} B_{t-1}(S')$.

Throughout the section we treat our strategy as fixed, and all subsequent propositions assume that the game is played according to it.

2.1.2 Paths

Next we define several notions concerning upward paths emanating from occupied vertices. A (t, ℓ) -*path* is an ℓ -tuple $((x_1, t), \dots, (x_\ell, t + \ell - 1)) \subset G_t$ satisfying $(x_1, t) \in \mathcal{B}_t$ and $x_{i+1} - x_i \in \{0, 1\}$ for every $i = 1, \dots, \ell - 1$. With applications in Sections 3 and 4 in mind, we also define the analogous notion of a *two-sided* (t, ℓ) -*path*, where we allow $x_{i+1} - x_i \in \{-1, 0, 1\}$. To make the distinction clearer we sometimes refer to a (t, ℓ) -path as *one-sided* (t, ℓ) -*path*. We say that such paths *emanate from the column* x_1 .

Observation 2.1. *Let P and Q be i -sided (t, ℓ) -paths, where $i \in \{1, 2\}$, emanating from columns x and y respectively. If $|x - y| \geq i \cdot \ell$ then P and Q are disjoint.*

Avoiding dead-ends. We denote by $B_t^\ell \subset \mathcal{B}_t$ the set of vertices from which a (t, ℓ) -path starts. Observe that, for any $\ell > 0$, restricting Spreader's spread at time t to the vertices of B_t^ℓ does not alter its winning or losing against Container. We shall therefore describe a somewhat inefficient strategy which does not obey the constraint of occupying at most $g(t)$ vertices, and later bound $|B_t^{3H_t}|$ by $g(t)$, so that by "avoiding dead-ends", namely restricting Spreader's spread to this set, this constraint is met. To this end we need the following proposition (established in Section 2.4).

Proposition 2.2. $|\{x \in B_t^{3H_t} : \chi_t(S_t(x)) = 1\}| = O(h_t^6)$

From this we draw an upper bound on $|B_t^{3H_t}|$.

Proposition 2.3. $|B_t^{3Ht}| \leq O(h_t^6) + \frac{t}{h_t}$.

Proof. Partition the vertices of B_t^{3Ht} according to χ_t .

$$|B_t^{3Ht}| = \left| \left\{ x \in B_t^{3Ht} : \chi_t(S_t(x)) = 1 \right\} \right| + \left| \left\{ x \in B_t^{3Ht} : \chi_t(S_t(x)) = 0 \right\} \right|.$$

The first term is of size $O(h_t^6)$ by Proposition 2.2. Since a simulative segment has at most one occupied vertex, and $B_t \subset [0, t]$, the second term is bounded by $\frac{t+1}{h_t}$. Together, these yield the required bound. \square

2.2 Strategy analysis and proof of Theorem 1

For the analysis of our strategy we require two auxiliary functions, F_t and Φ_t . Here we provide brief introduction to these, and reduce Theorem 1 to several propositions concerning their behaviour. We postpone the full definitions of these functions to Section 2.5.

Given a segment $S \in \mathcal{S}_t$, we associate deleted vertices with S at different times. We shall define an increasing sequence of sets $F_t(S) \subset \mathcal{F}_t$, which represent the set of deleted vertices associated with S up to time t , writing $f_t(S) := |F_t(S)|$. It will be useful to define the game with an arbitrary initial set of deleted vertices, denoted by \mathcal{F}_0 .

The fact that we shall never count deleted vertices towards more than one segment $S \in \mathcal{S}_t$ yields the following claim (established in Section 2.5 after (11)).

Claim 2.4. $f_t(\mathcal{S}_t) - |\mathcal{F}_0| \leq qt$ for every $t \in \mathbb{N}$.

We define an additional function $\Phi_t(S)$, initialized as $\Phi_0(S) = b_0(S) + f_0$ which we refer to as the *potential* of S , representing a modified count of simulated burning vertices plus the number of deleted vertices in the segment. We show that this quantity grows globally by 1 whenever the occupied set is non-empty, serving to imitate the analysis of [16]. This is captured by the following proposition (established in Section 2.9).

Proposition 2.5. *Let $s \leq t$. If $b_{s-1}(\mathcal{S}_{s-1}) > 0$ then $\Delta\Phi_s(\mathcal{S}_t) \geq 1$.*

We define $\Phi_t(S)$, so that it must be equal to f_t whenever the occupied set dies out. This is captured in the following proposition (established in Section 2.8).

Proposition 2.6. *Let $t \in \mathbb{N}$. If $\Phi_t(\mathcal{S}_t) - f_t(\mathcal{S}_t) > 0$ then $b_t(\mathcal{S}_t) > 0$.*

With all of these at hand, we are ready to prove Theorem 1.

Proof of Theorem 1 for \mathbb{Z}_{ν}^2 . Fogarty [19] established the fact that $q_{\mathbb{Z}_{\nu}^2} = (1, \infty)$. Thus, to prove the theorem, it remains to show that $1 \notin q(\mathbb{Z}_{\nu}^2, Ct^{6/7})$ for some sufficiently large C , and some initial fire \mathcal{B}_0 , that is, that there exists a Spreader strategy which is winning against $q = 1$, and satisfies $|\mathcal{B}_t| \leq Ct^{6/7}$. Since we can alter our strategy to avoid dead-ends (see discussion before Proposition 2.2), it would suffice to show that in our strategy, $|B_t^{3Ht}| \leq Ct^{6/7}$.

Firstly, set $h_t := 2^{\lceil \log_2(Ct^{1/7}) \rceil}$ for a sufficiently large C so that h satisfies (1). Note that $h_t/h_{t-1} \in \{1, 2\}$ for all t . By Proposition 2.3 we then obtain the required bound

$$|B_t^{3Ht}| \leq O(h_t^6) + \frac{t}{h_t} = O(t^{6/7}).$$

Secondly, let us show that the strategy $\mathcal{B} = (\mathcal{B}_t)_{t \in \mathbb{N}}$ is winning against $q = 1$ for $\mathcal{B}_0 = \{(0, 0)\}$. To so do we prove by induction that $b_t > 0$ for every $t \in \mathbb{N}$ and against any Container strategy $\{\mathcal{F}_t\}_{t \in \mathbb{N}}$. For $t = 0$ this is clear. Next, assume that this holds up to time $t - 1$. By Proposition 2.5 we have $\Delta\Phi_s(\mathcal{S}_t) \geq 1$ for every $1 \leq s \leq t$. Summing this up over $s \leq t$ we obtain $\Phi_t(\mathcal{S}_t) \geq \Phi_0(\mathcal{S}_t) + t$, so that

$$\Phi_t(\mathcal{S}_t) - f_t(\mathcal{S}_t) \geq \Phi_0(\mathcal{S}_t) + t - f_t(\mathcal{S}_t) \geq 1 + t - f_t(\mathcal{S}_t).$$

Since $q = 1$ we know that $f_t(\mathcal{S}_t) \leq t$ by Claim 2.4 and the fact that $\mathcal{F}_0 = \emptyset$. Hence $\Phi_t(\mathcal{S}_t) - f_t(\mathcal{S}_t) \geq 1$. By Proposition 2.6 this implies that $b_t(\mathcal{S}_t) > 0$, as required. \square

Remark 4. *By setting $h \equiv \lceil \frac{1}{\alpha} \rceil$, this proof establishes a linear analog of Theorem 1 for \mathbb{Z}_{ν}^2 , as per Remark 1.*

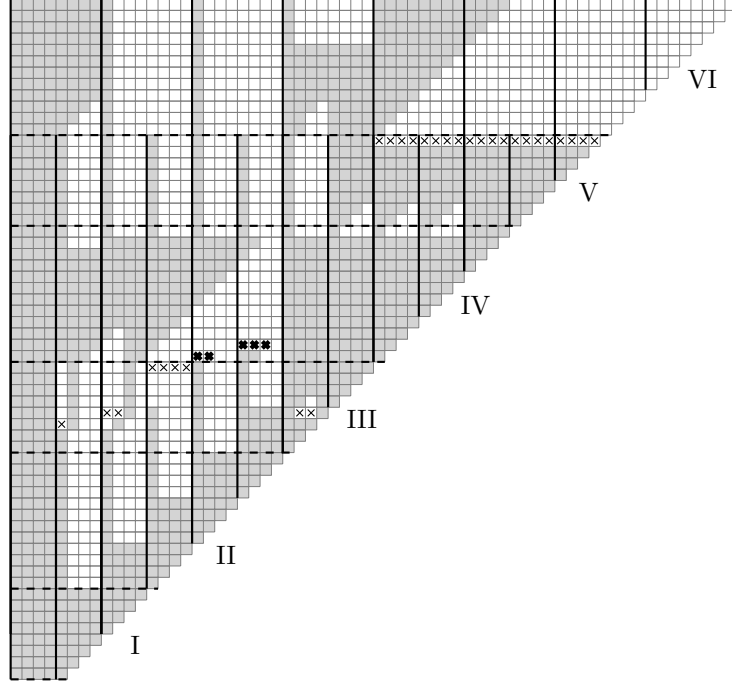


Figure 1: Illustration of different phenomena of the main Spreader strategy. The illustrated phenomena are presented in different phases, marked by horizontal dashed lines, while the partition into segments is marked by vertical lines. In **Phase I**, all segments, being close to the boundary, are spreading. In **Phase II**, the inner segments become simulative, and their occupied vertices consolidate to single pivots. In **Phase III**, the pivots move in response to deleted vertices, those possessing an unblocked path move along it, while those who do not cease playing. All deleted vertices of this phase were removed at least h vertices away from the front, hence simulative segments in their vicinity remain simulative. In **Phase IV**, two disruptions occur, one after another. This causes the segments in their vicinity to become spreading for a while, until they consolidate once again at the end of the step. Note that the later disruption completely blocks a segment at a time in which it is transitional. In **Phase V**, the right boundary is blocked by non-disruptive deleted vertices. This causes the segments at the boundary to become spreading, such that at the end of the step there is still a bulk of occupied vertices at the boundary. **Phase VI** opens with a doubling. In the boundary this causes some segments to become spreading, while the inner simulative segments have half of their pivots extinguished.

2.3 Details of the Spreader strategy: evolution of $\chi(S)$ and $p(S)$

In this section we describe the conditions under which a segment S transitions between simulative and spreading behaviour and the location of the pivot of S in simulative time-steps. This description is accompanied by Figure 1 which can be of use throughout in order to get a more intuitive grasp of the strategy's mechanics.

Counting disjoint (t, ℓ) -paths. We start by developing bounds on the number of deletions needed to block Spreader in a certain region given the current state of the board. This we do by counting disjoint (t, ℓ) -paths, as every deleted vertex can eliminate at most one such path.

Given $R \subset \mathbb{Z}$, denote the size of the largest collection of pairwise disjoint (t, ℓ) -paths emanating from R by $\widehat{b}_t^\ell(R)$. We make two straightforward observations. The first concerns two monotonicity properties of \widehat{b}_t^ℓ .

Observation 2.7. *Let $t, \ell, \ell' \in \mathbb{N}$, and $R, R' \subset \mathbb{Z}$. The following hold.*

- (a) *If $R \subseteq R'$ then $\widehat{b}_t^\ell(R) \leq \widehat{b}_t^\ell(R') \leq \widehat{b}_t^\ell(R) + |R' \setminus R|$.*
- (b) *If $\ell \leq \ell'$ then $\widehat{b}_t^\ell(R) \geq \widehat{b}_t^{\ell'}(R)$.*

The second is concerned with monotonicity with respect to time in regions consisting of spreading segments. It is obtained straightforwardly from (2) together with the fact that at most q vertices are deleted each turn.

Observation 2.8. *Let $t, t', \ell \in \mathbb{N}$, and $R \subset \mathbb{Z}$, and assume that $\chi_\tau(S') = 1$ for all $t' \leq \tau \leq t$ and all $S' \in \mathcal{S}_\tau(R)$. Then $\widehat{b}_t^{\ell-(t-t')}(R) \geq \widehat{b}_{t'}^\ell(R) - q(t-t')$.*

2.3.1 Evolution of $\chi_t(S)$

A *disruption* is the event that a vertex was deleted at vertical distance of less than h_t from the front. The set of segments disrupted at time t is denote by

$$\Omega_t := \left\{ S \in \mathcal{S}_t : (\mathcal{F}_t \setminus \mathcal{F}_{t-1}) \cap (S \times [t, t + h_t - 1]) \neq \emptyset \right\}.$$

Alert interval. The *alert interval* $I_t(S)$ of a segment S , acts as the interval in which, following a disruption at S , segments will start spreading for a prescribed period of time.

Given a segment $S \in \mathcal{S}_t$, $s \leq t$, and $\ell > 0$, we first define the *alert interval of S at time s and height ℓ* as the minimal interval around S such that there are ℓ many (t, ℓ) -paths to the left of S , and 2ℓ many $(t, 2\ell)$ -paths to its right. Namely,

$$I_s^\ell(S) := \left[\max\{x \in h_s \mathbb{Z} : \widehat{b}_s^\ell([x, \min S]) \geq \ell\}, \min\{x \in h_s \mathbb{Z} : \widehat{b}_s^{2\ell}((\max S, x)) \geq 2\ell\} \right]. \quad (3)$$

Observe that $I_s^\ell(S)$ is not always bounded. Using this we define the *alert interval of S at time t* as $I_t(S) := I_t^{\widetilde{H}_t}(S)$ for \widetilde{H}_t defined below.

Consolidation timer. We inductively define an additional auxiliary function, the *consolidation-timer* of a segment S , which we denote by $\tau_t(S)$. This function measures the number of time-steps remaining until a spreading segment returns to simulative status, unless further disruptions ensue. The evolution of $\chi_t(S)$ for $S \in \mathcal{S}_t$, is thus defined via

$$\chi_t(S) := \mathbb{1}\{\tau_t(S) > 0\}. \quad (4)$$

We extend $\chi(S)$ to times $s < t$ by setting $\chi_s(S) := \max\{\chi_s(S') : S' \in A_s(S)\}$.

Evolution of $\tau_t(S)$. We define $\tau_t(S)$ so that whenever a segment $S \in \mathcal{S}_t$ lies in the proximity of a segment disrupted at time $t \in \mathbb{N}$ its consolidation timer is reset to \widetilde{H}_t , if it lies near the edge of the occupied set it is reset to \widetilde{h}_t , while at other times, it ticks away to zero. Formally,

$$\begin{aligned} \tau_{t-1}(S) &:= \max\{\tau_{t-1}(S') : S' \in A_{t-1}(S)\}, \\ \tau_t(S) &:= \begin{cases} \widetilde{H}_t & \text{if exists } S' \in \Omega_t \text{ s.t. } S \subset I_t(S'), \\ \widetilde{h}_t & \text{otherwise, if } |I_t(S)| = \infty, \\ (\tau_{t-1}(S) - 1)_+ & \text{otherwise.} \end{cases} \end{aligned} \quad (5)$$

For $S \in \mathcal{S}_0$ we set $\tau_0(S) = 1$.

The use of \widetilde{H}_t and \widetilde{h}_t is needed to handle doubling times. We define

$$\widetilde{H}_t := \begin{cases} H_t & \text{if } h_t = h_{t+H_t}, \\ H_t + h_t & \text{otherwise.} \end{cases} \quad \widetilde{h}_t := \begin{cases} h_t & \text{if } h_t = h_{t+h_t}, \\ 2h_t & \text{otherwise.} \end{cases} \quad (6)$$

Hence, for constant h we have $\widetilde{H}_t \equiv H_t$ and $\widetilde{h}_t \equiv h_t$.

\widetilde{H}_t and \widetilde{h}_t are designed to satisfy the following claim.

Claim 2.9. *Let $t \in \mathbb{N}_2$ and $S \in \mathcal{S}_t$. Then $\tau_{t-1}(S) = 0$ or $\tau_{t-1}(S) > h_{t-1}$.*

Proof. Assume that $\tau_{t-1}(S) > 0$, and define t_0 as the last time before t such that $\tau_t(S)$ was defined by one of the first two cases of (5), so that $\tau_{t_0}(S) \in \{\widetilde{h}_{t_0}, \widetilde{H}_{t_0}\}$. We proceed under the assumption that $\tau_{t_0}(S) = \widetilde{H}_{t_0}$, as the other case

follows similarly. In this case $t - t_0 \leq \tilde{H}_{t_0}$. By (1), and the fact that $t \in \mathbb{N}_2$, we conclude that in fact $t - t_0 \leq H_{t_0}$. Therefore, by (6), we have $\tilde{H}_{t_0} = H_{t_0} + h_{t_0}$. All in all,

$$\tau_{t-1}(S) \geq \tau_{t_0}(S) - (t - 1 - t_0) \geq \tilde{H}_{t_0} - (H_{t_0} - 1) = h_{t_0} + 1 > h_{t_0} = h_{t-1}. \quad \square$$

For future use, we establish the following proposition concerning simulative segments.

Proposition 2.10. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ such that $\chi_t(S) = 0$. For all $t \leq t' < t + \tilde{H}_t$ we have*

$$b_{t'}((-\infty, S]) \geq \tilde{H}_t - q(t' - t) \quad \text{and} \quad b_{t'}([S, \infty)) \geq 2\tilde{H}_t - q(t' - t).$$

Proof. We first observe that $\hat{b}_t^{\tilde{H}_t}((-\infty, S)) \geq \tilde{H}_t$ and $\hat{b}_t^{2\tilde{H}_t}([S, \infty)) \geq 2\tilde{H}_t$. by the second case of (5). Next, we extend the definition of χ_t to $S \in \mathcal{S}_s$ and times $s < t$ by setting $\chi_t(S) := \chi_t(S')$, where S' is the unique segment in \mathcal{S}_t containing S .

Let $t' \in [t, t + \tilde{H}_t)$. We prove only that $b_{t'}((-\infty, S]) \geq \tilde{H}_t - q(t' - t)$, as the bound of $b_{t'}([S, \infty))$ follows similarly. Denote $\ell_{t'} := \tilde{H}_t - q(t' - t)$ and $x_{t'} := \inf\{x \in \mathbb{Z} : \hat{b}_{t'}^{\tilde{H}_t - (t' - t)}((-\infty, x)) \geq \ell_{t'}\}$. Let $S' \in \mathcal{S}_t((-\infty, x_{t'}))$. Observe that by the definition of χ , we have $\chi_{t'}(S') = 1$. Since $\hat{b}_t^{\tilde{H}_t}((-\infty, S]) \geq \tilde{H}_t$, we have $S_t(x_t) \leq S$. We wish to establish the fact that $S_t(x_{t'+1}) \leq S_t(x_{t'})$. Assume to the contrary that $S_t(x_{t'+1}) > S_t(x_{t'})$ so that $\chi_{t'+1}(S') = 1$. Thus, applying Observation 2.8 with $t \leftarrow t' + 1$, $\ell \leftarrow \tilde{H}_t - (t' - t)$ and $I \leftarrow (-\infty, S(x_{t'}))$, we obtain

$$\hat{b}_{t'+1}^{\tilde{H}_t - (t'+1-t)}(-\infty, S(x_{t'})) \geq \hat{b}_{t'}^{\tilde{H}_t - (t' - t)}(-\infty, S(x_{t'})) - q \geq \ell_{t'} - q = \ell_{t'+1},$$

in contradiction with the definition of $x_{t'+1}$. Hence $S_t(x_{t'+1}) \leq S_t(x_{t'})$, so that $S_t(x_{t'}) \leq S$.

To obtain the proposition, note that

$$b_{t'}((-\infty, S]) = \hat{b}_{t'}^1((-\infty, S]) \geq \hat{b}_{t'}^{\tilde{H}_t - (t' - t)}((-\infty, S]) \geq \hat{b}_{t'}^{\tilde{H}_t - (t' - t)}((-\infty, S(x_{t'}))) \geq \ell_{t'},$$

where the first inequality is by the definition of \hat{b} , the second is by Observation 2.7(b), the third is by $S(x_{t'}) \leq S$ and Observation 2.7(a), and the last by the definition of $x_{t'}$. \square

2.3.2 Evolution of $p_t(S)$

Recall that each simulative segment in \mathcal{S}_t is equipped with a pivot $p_t(S)$, the only occupied vertex in the segment which represents the fire in the entire segment. During spreading periods (i.e. when $\chi_t(S) = 1$), the pivot plays no role and we set $p_t(S) := \infty$ (as is the initial state of all segments).

We call the transition of a segment from a spreading state to a simulative state (i.e. when $\chi_{t-1}(S) = 1$ and $\chi_t(S) = 0$, and the consolidation timer reaches 0) *consolidation*, inspired by the fact that several occupied vertices ‘consolidate’ to a single pivot. When a consolidation occurs we set the pivot to be the leftmost occupied vertex with an unblocked vertical path, namely

$$p_t(S) := \min\{x \in S : (x, t - 1) \in B_{t-1}, C_t^{h_t}(x) \cap \mathcal{F}_t = \emptyset\}, \quad (7)$$

where $C_t^\ell(x) := \{(x, t), (x, t + 1), \dots, (x, t + \ell - 1)\}$.

Between two simulative states of the segment (i.e. when $\chi_{t-1}(S) = 0$ and $\chi_t(S) = 0$), the pivot moves to a neighbour with a (t, ℓ) -path in $S \times \mathbb{Z}$, preferring to go as leftwards as possible. To define this formally, given a segment S , we first denote by $\bar{B}_t^{\ell, S}$ the set of columns of the second vertex in $(t - 1, \ell + 1)$ -paths contained in S , analogously to the definition of B_t^ℓ (see after Observation 2.1). Then, set

$$p_t(S) := \min \bigcup_{S' \in P_{t-1}(S')} \left\{ (p_{t-1}(S') + \{0, 1\}) \cap \bar{B}_t^{h_t, S'} \right\}, \quad (8)$$

which can be ∞ if the minimum is taken over an empty set, in which case by (2), the segment contains no occupied vertices at time t .

2.4 Sparsity of the strategy: proof of Proposition 2.2

This section consists of the proof of Proposition 2.2. We start by introducing a definition pertaining to the evolution of the process which lead to the occupation of a given vertex. Given $y \in B_{t+\ell}$, denote $\beta_t^\ell(y) \in B_t$ for the leftmost starting vertex of a (t, ℓ) -path ending in y . Observe that

$$d_G(\beta_t^\ell(y), y) = \ell. \quad (9)$$

Let $t \in \mathbb{N}$, denote $\Sigma := \{y \in B_t^{3H_t} : \chi_t(S_t(y)) = 1\}$ and observe that by (4) for every $y \in \Sigma$ we have $\tau_t(S_t(y)) > 0$. In view of (5), write

$$\begin{aligned} \Sigma_1 &:= \left\{ y \in B_t^{3H_t} : \exists t' \in [t - H_t, t], S' \in \Omega_{t'} \text{ s.t. } S_{t'}(y) \subset I_{t'}(S') \right\}, \\ \Sigma_2 &:= \left\{ y \in B_t^{3H_t} : \exists t' \in [t - h_t, t] \text{ s.t. } |I_{t'}(S_{t'}(y))| = \infty \right\}, \end{aligned}$$

so that $\Sigma \subset \Sigma_1 \cup \Sigma_2$.

We first find an upper bound for $|\Sigma_1|$. Let $y \in \Sigma_1$, $t' \in [t - H_t, t]$ and $S' \in \Omega_{t'}$ so that we have $S_{t'}(y) \in \mathcal{S}_{t'}(I_{t'}(S'))$. Denote $\ell := t - t'$, and observe that $\beta_{t'}^\ell(y) \in B_{t'}^{3H_t}$, since $y \in B_t^{3H_t}$ and there exists a (t', ℓ) -path from $\beta_{t'}^\ell(y)$ to y . From (9) we obtain $S_{t'}(\beta_{t'}^\ell(y)) \subset I_{t'}(S') + \{-H_t, \dots, 0\}$, as $\ell \leq H_t$. Denote

$$D_{t'} := B_{t'}^{3H_t} \cap \left(\bigcup_{S' \in \Omega_{t'}} (I_{t'}(S') + \{-H_t, \dots, 0\}) \times \{t'\} \right).$$

Observe that there are at most $q(H_t + 1)$ disruptions in the time interval $[t - H_t, t]$, and for every $S' \in \Omega_{t'}$ at a given time t' , we have $|B_{t'}^{3H_t} \cap (I_{t'}(S') + \{-H_t, \dots, 0\})| \leq 3\tilde{H}_{t'} + H_t$. By (9), every $y' \in A_1$ satisfies $\beta_{t'}^\ell(y') \in D_{t'}$, and $|\{y' \in A_1 : \beta_{t'}^\ell(y') = x\}| \leq H_t + 1$, for every $x \in D_{t'}$. Thus,

$$|\Sigma_1| \leq (H_t + 1) \cdot \left| \bigcup_{t'=t-H_t}^t D_{t'} \right| \leq (H_t + 1) \cdot \sum_{t'=t-H_t}^{t-1} |D_{t'}| \leq q(H_t + 1) \cdot (H_t + 1) \cdot (3\tilde{H}_t + H_t) = O(H_t^3) = O(h_t^6),$$

where we used the fact that $\tilde{H}_t \leq H_t + h_t = O(H_t)$.

Next, we bound $|\Sigma_2|$. Given $t' \in [t - h_t, t]$, write

$$E_{t'} := B_{t'}^{3H_t} \cap \left((\{y' \in \mathbb{Z} : |I_{t'}(S_{t'}(y'))| = \infty\} + \{-h_t, \dots, 0\}) \times \{t'\} \right).$$

Observe that every $y' \in \Sigma_2$ satisfies $\beta_{t'}^{t-t'}(y') \in E_{t'}$ for some $t' \in [t - h_t, t]$, and $|\{y' \in \Sigma_2 : \beta_{t'}^{t-t'}(y') = x\}| \leq h_t + 1$ for every such t' and $x \in E_{t'}$. Thus,

$$|\Sigma_2| \leq (h_t + 1) \cdot \left| \bigcup_{t'=t-h_t}^t E_{t'} \right| \leq (h_t + 1) \cdot \sum_{t'=t-h_t}^t |E_{t'}|.$$

We turn to bound $|E_{t'}|$ for each value of t' individually. Recalling (3), and using the fact that $h_{t'} \leq h_t$, we get

$$E_{t'} = \left\{ (x, t') \in B_{t'}^{3H_t} : \hat{b}_{t'}^{\tilde{H}_{t'}}((-\infty, x)) \leq \tilde{H}_{t'} + h_t \right\} \cup \left\{ (x, t') \in B_{t'}^{3H_t} : \hat{b}_{t'}^{\tilde{H}_{t'}}((x, \infty)) \leq 2\tilde{H}_{t'} + 2h_t \right\}.$$

By Observation 2.1, each $(t', 3H_{t'})$ -path emanating from a vertex in $E_{t'}$ can intersect with at most $\tilde{H}_{t'} - 1$ other such paths. Thus, $\hat{b}_{t'}^{\tilde{H}_{t'}}(O_{t'}) \geq \frac{|E_{t'}|}{\tilde{H}_{t'}}$, so that

$$|E_{t'}| \leq \tilde{H}_{t'} \cdot \hat{b}_{t'}^{\tilde{H}_{t'}}(E_{t'}) \leq \tilde{H}_{t'} \cdot (\tilde{H}_{t'} + h_t + 2\tilde{H}_{t'} + 2h_t) = E(H_t^2).$$

Therefore,

$$|\Sigma_2| \leq (h_t + 1) \cdot \sum_{t'=t-h_t}^{t-1} |E_{t'}| = O(h_t^2 \cdot H_t^2) = O(h_t^6). \quad \square$$

2.5 Counting blocked vertices and simulated fire: evolution of $F_t(S)$ and $\phi_t(S)$

In this section we formally define the function $F_t(S)$ mentioned in Section 2.2 and a new function $\phi_t(S)$ representing the size of the simulated fire in S at time t . Firstly, we define a central auxiliary function, $\ell_t(S)$, which we call the *look-ahead* of a segment.

Look-ahead. Let us begin with a rough description of the look-ahead's role. Roughly speaking, $\ell_t(S)$ represents the vertical distance between the front and the simulated fire in the segment S . The box $S \times [t, t + \ell_t(S)]$ serves as the region where we count deleted vertices towards $F_t(S)$ and in a simulative segment, $\ell_t(S) = h_t$.

Once a segment's evolution changes to non-simulative status, it takes time for the front to catch up with the simulated fire. Therefore the look-ahead distance is reduced gradually, one unit at a time, rather than abruptly. The look-ahead $\ell_t(S)$ of a segment $S \in \mathcal{S}_t$ is defined to be h_t for simulative segments. We thus formally define, for times $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$,

$$\begin{aligned} \ell_{t-1}(S) &:= \max\{\ell_{t-1}(S') : S' \in P_{t-1}(S)\}, \\ \ell_t(S) &:= \begin{cases} h_t & \text{if } \chi_t(S) = 0, \\ (\ell_{t-1}(S) - 1)_+ & \text{if } \chi_t(S) = 1, \end{cases} \\ L_t(S) &:= [t, t + \ell_t(S)]. \end{aligned} \tag{10}$$

The look-ahead is initialised as $\ell_0 \equiv 0$ for all segments. We refer to $L_t(S)$ as the *look-ahead region* of S . We generalise the definitions to $S \in \mathcal{S}_t$ and $s < t$ by setting

$$\ell_s(S) := \max_{S' \in A_s(S')} \ell_s(S').$$

Spreading segments: simple & transitional. We call a segment S satisfying $\ell_t(S) = 0$ *simple*, and a segment satisfying $0 < \ell_t(S) < h_t$ *transitional*. See Figure 2 for a depiction of one Spreader step in a single segment in each of the simulative, transitional, and simple states.

Deleted vertices. For each segment $S \in \mathcal{S}_0$ set $F_0(S) = \emptyset$. We inductively define $F_t(S)$ for times $t \in \mathbb{N}$ and segments $S \in \mathcal{S}_{t-1} \cup \mathcal{S}_t$ as follows. In a non-simple segment we add to $F(S)$ any new deleted vertex inside its look-ahead region, and in a simple segment we add only the deleted vertices directly blocking the spread of Spreader. Namely,

$$\begin{aligned} F_{t-1}(S) &:= \bigcup_{S' \in P_{t-1}(S)} F_{t-1}(S'), \\ \Delta F_t(S) &:= (\mathcal{F}_t \setminus F_{t-1}) \cap \begin{cases} S \times L_t(S) & \text{if } \ell_t(S) > 0 \\ (S \times L_t(S)) \cap (B_{t-1} + \{(0, 1), (1, 1)\}) & \text{if } \ell_t(S) = 0. \end{cases} \end{aligned} \tag{11}$$

Also, denote $f_t(S) := |F_t(S)|$.

Observe that Claim 2.4 follows directly from the facts that $F_t(S) \subset \mathcal{F}_t$ for any $S \in \mathcal{S}_t$, and that $\{F_t(S)\}_{S \in \mathcal{S}_t}$ are pair-wise disjoint, since $F_t(S) \subset S \times \mathbb{Z}$.

In non-disruptive segments, F_t exhausts all the deleted vertices that are ever going to be counted in $S \times L_t(S)$. Formally,

Observation 2.11. *If $S \in \mathcal{S}_t \setminus \Omega_t$ then $\Delta F_t(S)$ is disjoint from $S \times L_{t-1}(S)$.*

Proof. As $S \in \mathcal{S}_t \setminus \Omega_t$, and $\ell_{t-1}(S) \leq h$, the set $\Delta \mathcal{F}_t$ is disjoint from $S \times L_{t-1}(S)$. In addition, by (11), we have $F_t \supset \mathcal{F}_{t-1} \cap (S \times L_{t-1}(S))$, from which the observation easily follows. \square

We also establish the following observation concerning simple segments.

Observation 2.12. *Let $t \in \mathbb{N}, S \in \mathcal{S}_t$. If $\ell_{t-1}(S) = 0$ then $F_{t-1}(S)$ is disjoint from $S \times L_t(S)$.*

Proof. Note that $F_{t-1}(S) \subset S \times (-\infty, t-1 + \ell_{t-1}(S)) = S \times (-\infty, t-1)$. As $\min L_t(S) = \{t\}$, the observation follows. \square

Range and simulated fire. Define the *range* of a segment $S \in \mathcal{S}_t$, denoted $r_t(S)$, to be the number of elements in $S \times \{t + \ell_t\}$ which are the endpoints of a $(t, \ell_t(S))$ -path emanating from $B_t(S)$. We generalise the definition for time $t-1$ by setting $r_{t-1}(S) := \sum_{S' \in P_{t-1}(S)} r_{t-1}(S')$.

Next, for any $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ we define $\phi_t(S)$, representing the simulated fire in S . In a simulative segment S , it is reduced by its value by 1 for every newly counted removed vertex and is otherwise fixed, while in spreading segments it measures the size of the range of the S . Formally, we define $\phi_t(S)$ inductively by

$$\begin{aligned} \phi_{t-1}(S) &:= \sum_{S' \in P_{t-1}(S)} \phi_{t-1}(S'), \\ \phi_t(S) &:= \begin{cases} (\phi_{t-1}(S) - \Delta f_t(S))_+ & \text{if } \chi_t(S) = 0, \\ r_t(S) & \text{if } \chi_t(S) = 1. \end{cases} \end{aligned} \quad (12)$$

Observe that this fully defines ϕ , as $\chi_t(S) = 1$ for all $S \in \mathcal{S}_0$.

Note that $\phi_t(S) \leq h_t$ for all t and that in case that $\ell_t(S) = 0$ we have

$$\phi_t(S) = b_t(S). \quad (13)$$

We conclude this section with the following lemma.

Lemma 2.13. *For all $S \in \mathcal{S}_t$ we have $\phi_t(S) \leq r_t(S)$.*

Since the proof is somewhat technical, we postpone it to Section 2.10.

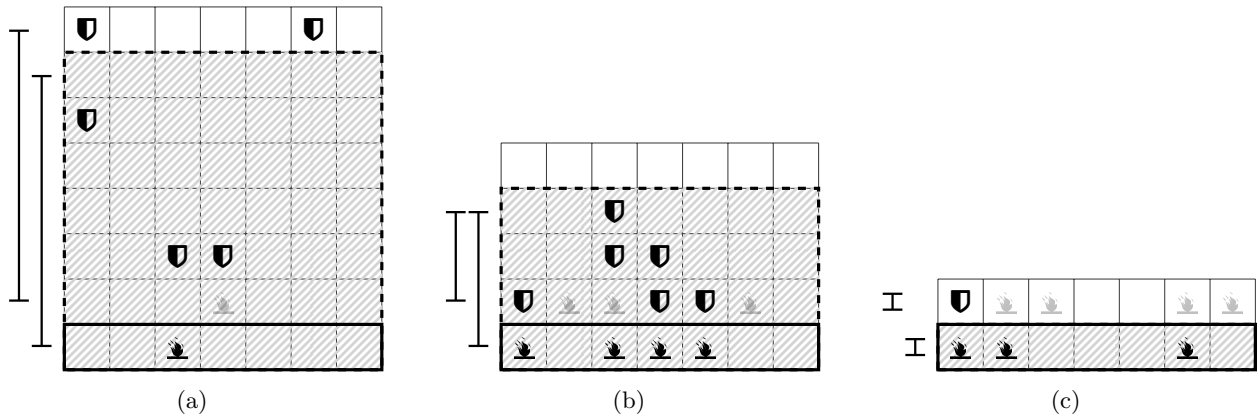


Figure 2: In each sub-figure, a segment and its look-ahead region are depicted in two consecutive times $t-1$ and t . The segment is simulative in (2a), transitional in (2b), and simple in (2c). The look-ahead distances are marked by a vertical lines, and $S \times L_{t-1}(S)$ by a filling pattern.

2.6 First notion of a potential

Following [16], for any segment $S \in \mathcal{S}_t$ we define its *pre-potential* by $\Phi'_t(S) := \phi_t(S) + f_t(S)$. This function played a key role in previous works on the firefighter problem, and its evolution was the main instrument in establishing victory for the fire. In particular, on each finite segment it was non-decreasing. In our setting, this is indeed the case for simple segments, as the following variation on [19, Theorem 1] indicates.

Proposition 2.14. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ be such that $\ell_{t-1}(S) = \ell_t(S) = 0$ then*

(a) $\Delta\Phi'_t(S) \geq 0$,

(b) *If there exists $x \in B_{t-1}$ such that $x + (1, 0) \notin B_{t-1}(S)$, then $\Delta\Phi'_t(S) \geq 1$.*

Proof. Let a be an indicator to the fact that the condition of Item (b) holds. By (2), any neighbour in S of a B_{t-1} is either occupied or deleted at time t . There are always at least $b_{t-1}(S)$ such neighbours which are vertically above a vertex in B_{t-1} , and in case that $a = 1$, at least one additional such neighbour exists. By Observation 2.12 we know that every such blocked vertex is counted towards $\Delta F_t(S)$. We thus have $b_{t-1}(S) \leq b_t(S) + \Delta f_t(S)$. As $\phi_{t'}(S) = b_{t'}(S)$ for $t' \in \{t-1, t\}$ by (13), we conclude that

$$\Delta\Phi'_t(S) = \Delta\phi_t(S) + \Delta f_t(S) = \Delta b_t(S) + \Delta f_t(S) \geq a,$$

as required. □

The same property holds for transitional or simulative segments as long as no disruption occurs, as stated in the following proposition.

Proposition 2.15. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t \setminus \Omega_t$. Then*

(a) $\Delta\Phi'_t(S) \geq 0$.

(b) *if $\chi_t(S) = 1$ then $\Delta r_t(S) + \Delta f_t(S) \geq 0$.*

Proof. We first observe that Item (b) implies Item (a). We consider two cases according to the value of $\chi_t(S)$. In case that $\chi_t(S) = 0$, we have, by (12), the definition of ϕ ,

$$\Delta\Phi'_t(S) = \Delta\phi_t(S) + \Delta f_t(S) = \phi_t(S) - (\phi_{t-1}(S) - \Delta f_t(S)) \geq \phi_t(S) - (\phi_{t-1}(S) - \Delta f_t(S))_+ = 0.$$

Otherwise, when $\chi_t(S) = 1$, we have $\phi_t(S) = r_t(S)$, while, by Lemma 2.13, $\phi_{t-1}(S) \leq r_{t-1}(S)$. By Item (a) we conclude again that

$$\Delta\Phi'_t(S) = \Delta\phi_t(S) + \Delta f_t(S) \geq \Delta r_t(S) + \Delta f_t(S) \geq 0.$$

Next, we prove Item (b). Every vertex counted towards $r_{t-1}(S)$ is the endpoint of a path emanating from a vertex in $B_{t-1}(S)$. We denote the collection of these by \mathcal{P} . Given a path $(p_1, \dots, p_{\ell'}) \in \mathcal{P}$ denote $\ell' = \ell_{t-1}(S_{t-1}(p_1))$, its segment's look-ahead at time $t-1$ and define a new path $p'_1, \dots, p'_{\ell'_t(S)}$ where

$$p'_i = \begin{cases} p_{i+1} & \text{if } i < \ell', \\ p_{\ell'} & \text{otherwise.} \end{cases}$$

Denote the collection of all such paths by \mathcal{P}' , and the collection of those contained in G_t by \mathcal{P}'' . Since $\chi_t(S) = 1$, we have, by (2),

$$B_t(S) = (S \times \{t\}) \cap (B_{t-1} + \{(0, 1), (1, 1)\}) \setminus \mathcal{F}_t.$$

Hence $r_t(S) \geq |\mathcal{P}''|$.

Observe that, as $S \notin \Omega_t$, for every such path, $\Delta F_t(S)$ is disjoint from $p'_1, \dots, p'_{\ell'-1}$, by Observation 2.11. As from $p'_{\ell'-1}$ and on, each such path consists of a distinct column, we deduce that every element of $\Delta F_t(S)$ can take part in at most a single path in \mathcal{P}' .

Putting all of this together with the fact that $|\mathcal{P}| = |\mathcal{P}'|$, we conclude that

$$r_t(S) \geq |\mathcal{P}''| \geq |\mathcal{P}'| - |\Delta F_t(S)| \geq |\mathcal{P}| - |\Delta F_t(S)| = r_{t-1}(S) - \Delta f_t(S);$$

the proposition follows. □

Disruptions, however, may cause a drastic reduction in $\phi_t(S)$, with only a small increase to $f_t(S)$, resulting in a reduction in $\Phi'_t(S)$. Subsequent sections will introduce the debt $d_t(S)$ to handle these in an amortised fashion.

2.7 Debt

Here we define $d_t(S)$, the *debt*, used to keep track of reductions in Φ' and see that these are compensated for. Hence we set $d_0 \equiv 0$, and let Δd_t increase at every time-step by $(-\Delta\Phi'_t(S))_+$. To reduce the debt, we identify $\lambda_t(S)$, the nearest segment to the left of S with positive $\Delta\Phi'_t(S)$. The increase in Φ'_t of $\lambda_t(S)$ will compensate for the decrease in Φ'_t of S . For use in Section 3, it will be convenient not to allow this compensation to come from the leftmost occupied segment. Formally,

$$\lambda_t(S) := \max\{Q \in \mathcal{S}_t : Q \leq S, \Delta\Phi'_t(Q) > 0, \exists Q' < Q : b_t(Q') > 0\}. \quad (14)$$

This definition allows $\lambda_t(S)$ to take the value $-\infty$ in case that the maximum is taken over an empty set.

We now define d_t formally via

$$\begin{aligned} d_{t-1}(S) &:= \sum_{S' \in P_{t-1}(S)} d_{t'}(S), \\ \tilde{d}_t(S) &:= d_{t-1}(S) + (-\Delta\Phi'_t(S))_+, \\ d_t(S) &:= \begin{cases} \left(\tilde{d}_t(S) - \Delta\Phi'_t(\lambda_t(S))\right)_+ & \text{if } \lambda_t(S) > -\infty, \tilde{d}_t(Q) = 0 \text{ for every } \lambda_t(S) \leq Q < S, \text{ and } \tilde{d}_t(S) > 0, \\ \tilde{d}_t(S) & \text{otherwise.} \end{cases} \end{aligned} \quad (15)$$

Observe that $d_t(S) \geq 0$ for all t . Let us make formal the fact that the increase of the debt is governed by $(-\Delta\Phi'_t(S))_+$, by making the following straightforward observation.

Observation 2.16. *For all $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ we have $\Delta d_t(S) \leq (-\Delta\Phi'_t(S))_+$.*

Next, we provide the following criteria for identifying segments in which the debt does not increase.

Corollary 2.17. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$. A sufficient condition for $\Delta d_t(S) \leq 0$ is any of the following:*

- (a) $\Delta\Phi'_t(S) \geq 0$,
- (b) S is simple in times $\{t-1, t\}$, i.e., $\ell_{t-1}(S) = \ell_t(S) = 0$.
- (c) $S \notin \Omega_t$.

Proof. (a) follows immediately from Observation 2.16. Items (b) and (c) follow by combining (a) with Propositions 2.14 and 2.15 respectively. \square

Finally, we show that the total simulated fire together with the debt cannot exceed the width of a segment.

Proposition 2.18. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$. Then $\phi_t(S) + d_t(S) \leq h_t$.*

Proof. We start by proving the following property for any $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$.

$$d_t(S) > 0 \implies \Delta d_t(S) + \Delta\phi_t(S) \leq 0. \quad (16)$$

In case that $\Delta\Phi'_t(S) \leq 0$, (16) is immediate from Observation 2.16 and the fact that $\Delta\phi_t(S) \leq \Delta\Phi'_t(S)$. We thus consider the case $\Delta\Phi'_t(S) > 0$ in which $\lambda_t(S) = S$, by (14). Let us verify that in this case $d_t(S) = \tilde{d}_t(S) - \Delta\Phi'_t(\lambda_t(S))$ via (15) and using the fact that $d_t(S) > 0$ by our assumption.

As $\lambda_t(S) = S$, the fact that $\lambda_t(S) \neq -\infty$ is straightforward and the condition that $\tilde{d}_t(Q) = 0$ for every $\lambda_t(S) \leq Q < S$ holds vacuously. Finally, observe that

$$\tilde{d}_t(S) = d_{t-1}(S) + (-\Delta\Phi'_t(S))_+ = d_{t-1}(S) = d_t(S) - \Delta d_t(S) > -\Delta d_t(S) \geq 0,$$

where the last inequality is by Corollary 2.17(a). Therefore,

$$\Delta d_t(S) = \tilde{d}_t(S) - \Delta\Phi'_t(\lambda_t(S)) - d_{t-1}(S) = -\Delta\Phi'_t(S) \leq -\Delta\phi_t(S);$$

from which (16) follows.

To prove the proposition, we use induction on t . If $d_t(S) = 0$, which holds also at time $t = 0$, the proposition holds, as $\phi_t(S) \leq h_t$ by (12). Otherwise, assume that it holds up to time $t - 1$. By using this and applying (16) we conclude that

$$d_t(S) + \phi_t(S) = d_{t-1}(S) + \phi_{t-1}(S) + \Delta d_t(S) + \Delta \phi_t(S) \leq \sum_{S' \in A_{t-1}(S)} (\phi_{t-1}(S') + d_{t-1}(S')) \leq \sum_{S' \in A_{t-1}(S)} h_{t-1} = h_t. \quad \square$$

Using d_t , we now define $\Phi_t(S)$ by

$$\Phi_t(S) := \Phi'_t(S) + d_t(S).$$

2.8 Bounding the debt

The purpose of this section is to show that $d_t(S)$, the debt of a given section, once generated, is bound to diminish and eventually nullify. In particular we will establish Proposition 2.6, showing that $b_t(\mathcal{S}_t) = 0$ implies $\phi_t(\mathcal{S}_t) = 0$ and $d_t(\mathcal{S}_t) = 0$.

The main idea here is that after a debt is created, the front must contain cavities in the occupied set, which are separated by occupied vertices. Each indebted segment S is contained in such a cavity, while to the left of this cavity lies the segment $\lambda_t(S)$. Proposition 2.14(b) tells us that the evolution of the occupied set in $\lambda_t(S)$ results in an increase of Φ'_t – one which is not expected in a fully occupied front. This additional increase in Φ'_t compensates for the debt of S and eventually eliminates it. From this we may deduce that a Container strategy which generates debt, and hence cavities is, in a way, inefficient.

To establish Proposition 2.6 we require the following proposition (established in Section 2.8.1 below), which is the main proposition of this section, stating that an indebted segment will pay its debt within a constant time-frame.

Proposition 2.19. *For all $t > 0$ and $S \in \mathcal{S}_t$ we have $\max\{s \leq t : d_{s-1}(S) = 0\} \geq t - qh_t^2 - h_t$.*

From this we draw the following useful proposition which states that a segment that on one of its sides there are only a few occupied vertices is simple and free of debt.

Proposition 2.20. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ such that $\min(b_t([S, \infty)), b_t((-\infty, S])) \leq 2h_t$. Then $\ell_{t'}(S) = 0$ and $d_{t'}(S) = 0$ for $t' = t$ and $t' = t - 1$.*

Proof. Let $t' \in [t - 1 - qh_t^2 - h_t, t]$. Under the assumptions of the proposition we obtain, by Proposition 2.10, $\chi_{t'}(S) = 1$. This implies that $\ell_{t'}(S) = 0$, by (10). From this we deduce, by using Corollary 2.17(b) that $d_t(S) \leq d_{t'}(S)$. This yields the desired statement by Proposition 2.19. \square

Using this we easily establish Proposition 2.6.

Proof of Proposition 2.6. Assume that $b_t(\mathcal{S}_t) = 0$ and let $S \in \mathcal{S}_t$. By Lemma 2.13 we have $\phi_t(S) = 0$. In addition, $b_t([S, \infty)) = 0 \leq 2h_t$, and hence, by Proposition 2.20, $d_t(S) = 0$. Putting these together we obtain that $\phi_t(\mathcal{S}_t) + d_t(\mathcal{S}_t) = 0$, as required. \square

2.8.1 Upper bound on the duration of the debt

For the proof of Proposition 2.19 we require the following claim, which guarantees a reduction of the debt in a simple interval containing an occupied vertex to the left of an indebted segment.

Claim 2.21. *Let $t \in \mathbb{N}$, $S_0, S_1, S_2 \in \mathcal{S}_t$ satisfying $S_0 < S_1 < S_2$, $b_{t-1}(S_0), b_{t-1}(S_1) > 0$, and $d_{t-1}(S_2) > 0$. Moreover, assume that $\ell_{t-1}(Q) = \ell_t(Q) = 0$ for all $Q \in \mathcal{S}_t([S_1, S_2])$. Then, $\Delta d_t(\mathcal{S}_t([S_1, S_2])) \leq -1$.*

Proof. Write $I := \mathcal{S}_t([S_1, S_2])$. We have $\Delta d_t(Q) \leq 0$ for every $Q \in I$, as $\ell_{t-1}(Q) = \ell_t(Q) = 0$. To prove the proposition, it would thus be enough to show that $\Delta d_t(S) < 0$ for a particular segment $S \in I$.

Let $S := \min\{Q \in I : S_1 < Q, d_{t-1}(Q) > 0\}$, and observe that since S_2 belongs to the set of which minimum is taken, S is a valid segment. In addition, $\Delta \Phi'_t(S) \geq 0$ by Proposition 2.14 (a), so that $\tilde{d}_t(S) = d_{t-1}(S) > 0$. With (14) and (15) in mind, to complete the proposition we must show two things:

- that $\lambda_t(S)$ is a valid segment, i.e. that $\{Q \in \mathcal{S}_t : Q \leq S \wedge \Delta\Phi'_t(Q) > 0\}$ is nonempty,
- that $\tilde{d}_t(\mathcal{S}_t([\lambda_t(S), S])) = 0$.

We start with the first item. Observe that $b_t(S) = \phi_t(S) \leq h_t - d_t(S) < h_t = |S|$, by Proposition 2.18 and the fact $\ell_t(S) = 0$. Hence $B_t(S) \subsetneq S$. As $B_{t-1}(S_1)$ is non-empty, we conclude that there exists a segment $S_1 \leq Q' \leq S$ and some $x \in B_{t-1}$ such that $x + (1, 0) \in Q' \setminus B_{t-1}(Q')$. By Proposition 2.14(b) this implies that $\Delta\Phi'_t(Q') \geq 1$, so that together with the existence of S_0 , we deduce that $\lambda_t(S)$ is a valid segment in I .

To see the second item, using $\ell_{t-1}(Q) = \ell_t(Q) = 0$ together with Proposition 2.14(a), we obtain that for all $Q \in \mathcal{S}_t([\lambda_t(S), S])$ we have $\tilde{d}_t(Q) = d_{t-1}(Q)$. The item then follows by the definition of S . \square

We are finally ready to establish Proposition 2.19, bounding the duration of the debt. In fact, we shall prove the following stronger proposition.

Proposition 2.22. *Let $t \geq 0$ and $S \in \mathcal{S}_t$.*

- (a) $\max\{s \leq t : d_{s-1}(S) = 0\} \geq t - qh_t^2 - h_t$.
- (b) *If $\chi_t(S) = 0$ then $d_t(\mathcal{S}_t(I_t(S) \cap (-\infty, S + h_t])) = 0$.*

Proof. We prove the proposition inductively on t . The base case easily follows, as $d_t(S) = 0$ for all $S \in \mathcal{S}_0$. Assume that the proposition holds for all $t' \leq t - 1$. We first prove the contra-positive of (b). Let $S \in \mathcal{S}_t$, denote $I_S := I_t(S) \cap (-\infty, S + h_t]$ and assume that there exists some $Q \in \mathcal{S}_t(I_S)$ such that $d_t(Q) > 0$. Denote $t_Q := \max\{s \leq t : d_{s-1}(Q) = 0\}$. By applying (a) for time $t - 1$, we obtain

$$t_Q \geq \max\{s \leq t - 1 : d_{s-1}(Q) = 0\} \geq t - 1 - qh_{t-1}^2 - h_{t-1} > t - 2H_t, \quad (17)$$

from which we deduce, by (1), that $h_t \leq 2h_{t_Q}$. By Corollary 2.17(c) we have $Q \in \Omega_{t_Q}$. Denoting $\tilde{J} := I_{t_Q}(Q) \cap [Q, \infty)$, we thus obtain, by (5), that $\chi_{t'}(S') = 1$ for all $S' \in \mathcal{S}_{t_Q}(J)$ and $t' \in [t_Q, t]$. Thus, it suffices to show that $S \subset \tilde{J}$. When I_Q is infinite this is straightforward; otherwise, by (3), we have $\widehat{b}_{t_Q}^{2\tilde{H}_{t_Q}}(J) \geq 2\tilde{H}_{t_Q}$. Hence

$$\widehat{b}_t^{\tilde{H}_t}(\tilde{J}) \geq \widehat{b}_t^{2\tilde{H}_{t_Q} - (t-t_Q)}(\tilde{J}) \geq \widehat{b}_{t_Q}^{2\tilde{H}_{t_Q}}(\tilde{J}) - q \cdot (t - t_Q) \geq 2\tilde{H}_{t_Q} - q \cdot (t - t_Q) > \tilde{H}_t + 2h_t,$$

where the first inequality is by Observation 2.7(b), the second is by Observation 2.8, the third is using the fact that J is finite, and the fourth is obtained by plugging in (17) and $h_t \leq 2h_{t_Q}$. On the other hand, as $Q \in \mathcal{S}_t(I_S)$, we have $\widehat{b}_t^{\tilde{H}_t}([\max Q, \min S]) \leq \tilde{H}_t$. Therefore,

$$\widehat{b}_t^{\tilde{H}_t}(\tilde{J} \cap (-\infty, S]) \leq \widehat{b}_t^{\tilde{H}_t}([Q, S]) \leq \widehat{b}_t^{\tilde{H}_t}([\max Q, \min S]) + 2h_t \leq \tilde{H}_t + 2h_t.$$

from which we conclude that $\tilde{J} \cap (-\infty, S] \subsetneq \tilde{J}$, so that indeed $S \subset \tilde{J}$.

Next, towards proving (a), denote $t_S := \max\{s \leq t : d_{s-1}(S) = 0\}$ and $t_0 := \max\{s \leq t_S : \chi_s(S) = 0\}$. Observe that, as before, $S \in \Omega_{t_S}$. Hence, S is simple at times $t_0 + h_{t_0} + 1, \dots, t_S + \tilde{H}_{t_S}$, where, by Corollary 2.17(b), it can gain no new debt. In particular,

$$t_0 < t_S \leq t_0 + h_{t_0}. \quad (18)$$

Also, as in (17) (with t_S in the role of t_Q), we obtain, using (18), that

$$t - t_0 \leq t - t_S + h_{t_0} \leq qh_{t-1}^2 + h_{t-1} + 1 + h_{t_0} < 2H_t. \quad (19)$$

Using (1), this implies that $h_t \leq 2h_{t_0}$.

Next, writing $S' = \min \mathcal{S}_{t_0}(I_{t_0}(S))$, namely the leftmost segment in $I_{t_0}(S)$, we define

$$\tilde{J} := (I_{t_0}(S) \cap (-\infty, S]) \setminus S'.$$

Let $S_0 := \min P_{t_0}(S)$. Observe that, since $h_t \leq 2h_{t_0}$, we have $\tilde{J} \subseteq I_{t_0}(S_0) \cap (-\infty, S_0 + h_{t_0}]$. As $\chi_{t_0}(S) = 0$, we have, by definition, $\chi_{t_0}(S_0) = 0$, so that, from (b) we obtain $d_{t_0}(\tilde{J}) = 0$. From this we obtain that $d_{t_0+h_{t_0}}(\tilde{J}) \leq qh_{t_0}^2$, by

observing that at most qh_t segments could be disrupted at times $[t_0 + 1, \dots, t_0 + h_{t_0}]$, each disrupted segment has at most h_t debt (by Proposition 2.18), and debt can only be created via a disruption (by Corollary 2.17(c)).

Let $t_0 + h_{t_0} \leq t' \leq t$. By the definition of \tilde{J} and the fact that $h_t \leq 2h_{t_0}$, note that all segments in $S'_t(\tilde{J})$ are contained in $I_{t_0}(S)$, and are thus simple at times $t_0 + h_{t_0}, \dots, t$. We therefore obtain,

$$b_{t'}(\tilde{J} \setminus \{S\}) \geq b_{t'}(\tilde{J}) - h_{t_0} \geq \tilde{H}_{t_0} - h_{t_0} - q \cdot (t' - t_0) > h_t,$$

where the first inequality is by containment, the second is by Observation 2.8, and the third is obtained by plugging in (19). Hence there are at least two segments containing occupied vertices in \tilde{J} . Hence, by Claim 2.21, we conclude that $\Delta d_{t'}(\tilde{J}) \leq -1$ for any $t_0 + h_{t_0} \leq t' \leq t$. Since $d_t(\tilde{J}) \geq 0$ and $d_{t_0+h_{t_0}}(\tilde{J}) \leq qh_t^2$, we obtain, using (18), the required bound:

$$t - t_S \leq t - t_0 \leq qh_t^2 + h_{t_0} \leq qh_t^2 + h_t. \quad \square$$

2.9 Potential growth: proof of Proposition 2.5

This section consists of the proof of Proposition 2.5, showing that the potential grows by 1 whenever there exist occupied vertices on the front, which will rely upon Proposition 2.20. Recalling that $\Phi_s(S) = \sum_{S' \in A_s(S)} \Phi_s(S')$, it suffices to prove the proposition only for the case $s = t$.

Fix $t \in \mathbb{N}$ and denote

$$\bar{\mathcal{S}} := \{S \in \mathcal{S}_t : \Delta\Phi_t(S) < 0\} \quad \text{and} \quad S_t^{\rightarrow} := \max\{S \in \mathcal{S}_t : ((S-1) \times \{t-1\}) \cap B_{t-1} \neq \emptyset\}.$$

Let $S \in \bar{\mathcal{S}}$ and observe that by (15), we must have $\lambda_t(S) \neq -\infty$ and $\tilde{d}_t(Q) = 0$ for every $\lambda(S) \leq Q < S$, and $\tilde{d}_t(S) > 0$, as otherwise we would have

$$\Delta\Phi_t(S) = \Delta\Phi'_t(S) + \Delta d_t(S) = \Delta\Phi'_t(S) + (-\Delta\Phi'_t(S))_+ \geq 0.$$

In particular, this implies that

$$\Delta\Phi_t(S) = \Delta\Phi'_t(S) + \Delta d_t(S) = \Delta\Phi'_t(S) + (-\Delta\Phi'_t(S))_+ - \Delta\Phi'_t(\lambda_t(S)) = (\Delta\Phi'_t(S))_+ - \Delta\Phi'_t(\lambda_t(S)).$$

This implies that $\lambda_t(S) \neq S$, as otherwise we would immediately get $\Delta\Phi_t(S) \geq 0$. Hence, by (15), we have $\tilde{d}_t(\lambda_t(S)) = 0$. Using this we obtain,

$$\Delta\Phi_t(S) + \Delta\Phi_t(\lambda_t(S)) = (\Delta\Phi'_t(S))_+ - \Delta\Phi'_t(\lambda_t(S)) + \Delta\Phi'_t(\lambda_t(S)) + \Delta d_t(\lambda_t(S)) = (\Delta\Phi'_t(S))_+ + \Delta d_t(\lambda_t(S)) \geq 0.$$

Next, we show that the map $Q \mapsto \lambda_t(Q)$ is one-to-one in $\bar{\mathcal{S}}$. Since $\lambda_t(Q) \leq Q$ for all $Q \in \bar{\mathcal{S}}$, it would suffice to show that for all segments $Q \in \mathcal{S}_t([\lambda_t(S), S])$ we have $Q \notin \bar{\mathcal{S}}$, i.e. that $\Delta\Phi_t(Q) \geq 0$. Given such $Q \in \mathcal{S}_t([\lambda_t(S), S])$, using the fact that $\tilde{d}_t(Q) = 0$, we obtain $d_{t-1}(Q) = 0$ so that $\Delta d_t(Q) \geq 0$, and $\Delta\Phi'_t(Q) \geq 0$, and hence, indeed, $\Delta\Phi_t(Q) \geq 0$.

By the fact that all segments to the right of S_t^{\rightarrow} are non-occupied and are thus free of debt by Proposition 2.20, we observe that $S_t^{\rightarrow} \neq \lambda_t(S)$ for any $S \in \mathcal{S}_t$. Since for every $Q \in \bar{\mathcal{S}}$ we have found a distinct counterpart $\lambda_t(Q)$ such that $\Delta\Phi_t(S) + \Delta\Phi_t(\lambda_t(S)) \geq 0$, we deduce that $\Delta\Phi_t(\mathcal{S}_t \setminus \{S_t^{\rightarrow}\}) \geq 0$. To complete the proof, it would suffice to show that $\Delta\Phi_t(S_t^{\rightarrow}) \geq 1$. To see that this, observe that, by Proposition 2.20, $d_{t-1}(S_t^{\rightarrow}) = 0$ so that $\Delta\Phi_t(S_t^{\rightarrow}) \geq \Delta\Phi'_t(S_t^{\rightarrow})$ and use Proposition 2.14(b). \square

2.10 Controlling the range: proof of Lemma 2.13

This section is dedicated to the proof of Lemma 2.13, stating that $\phi_t(S) \leq r_t(S)$. In Part 2.10.1 we prove several auxiliary claims, while in Part 2.10.2 we use these to establish the lemma itself.

2.10.1 Auxiliary claims

Claim 2.23. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$, such that $\chi_t(S) = 1$. Then $\ell_t(S) < \tau_t(S)$.*

Proof. Let $t_0 := \max\{t' \leq t : \chi_{t'-1}(S) = 0\}$, so that $\tau_{t_0-1}(S) = 0$ and $\tau_{t_0}(S) > 0$. By (5) we have $\tau_{t_0}(S) \geq \tilde{h}_{t_0} \geq h_{t_0}$. Hence by (10), $\ell_{t_0}(S) = (\ell_{t_0-1}(S) - 1)_+ = h_{t_0-1} - 1 < \tau_{t_0}(S)$. At any later time $t_0 < t' \leq t$ we have either $\ell_{t'}(S) = 0$ so that, by (4), $\ell_{t'}(S) < \tau_{t'}(S)$, or $\Delta\tau_{t'}(S) = \Delta\ell_{t'}(S) = -1$. The claim follows. \square

Claim 2.24. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ such that $\chi_{t-1}(S) = 1$ and $\chi_t(S) = 0$. Then $\ell_{t-1}(S) = 0$ and $t \in \mathbb{N}_1$.*

Proof. The fact that $\chi_{t-1}(S) = 1$ and $\chi_t(S) = 0$ implies that $\tau_{t-1}(S) = 1$. By Claim 2.23 this implies that $\ell_{t-1}(S) = 0$, and by Claim 2.9 – that $t \in \mathbb{N}_1$. \square

Claim 2.25. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ such that $\chi_t(S) = 0$. Then $\phi_t(S) \leq h_t - |\mathcal{F}_t \cap (S \times L_t(S))|$.*

Proof. We prove by using induction on t . If $\chi_{t-1}(S) = 1$, then by Claim 2.24 we know that $\ell_{t-1}(S) = 0$ and $t \in \mathbb{N}_1$. Hence, by Observation 2.12 we know that $\Delta f_t(S) = |\mathcal{F}_t \cap (S \times L_t(S))|$, so that,

$$\phi_t(S) = \phi_{t-1}(S) - \Delta f_t(S) \leq h_{t-1} - \Delta f_t(S) = h_t - |\mathcal{F}_t \cap (S \times L_t(S))|.$$

Otherwise, if $\chi_{t-1}(S) = 0$, we have $\Delta f_t(S) = |\mathcal{F}_t \cap (S \times \{t + \ell_t(S)\})|$. By the induction hypothesis, we thus obtain

$$\begin{aligned} \phi_t(S) &= \phi_{t-1}(S) - \Delta f_t(S) \leq h_{t-1}(S) - |\mathcal{F}_{t-1} \cap (S \times L_{t-1}(S))| - |\mathcal{F}_t \cap (S \times \{t + \ell_t(S)\})| \\ &\leq h_t - |\mathcal{F}_t \cap (S \times L_t(S))|. \end{aligned} \quad \square$$

2.10.2 Proof of Lemma 2.13

We introduce the following notations used only for the proof of the lemma. Given $S \in \mathcal{S}_t$ such that $\phi_t(S) > 0$ and $p_t(S) \neq \infty$, we write

$$w_t(S) := |[p_t(S), \max S]| \quad \text{and} \quad \tilde{f}_t(S) := |\mathcal{F}_t \cap ([p_t(S), \max S] \times L_t(S))|.$$

Next, we prove the following connection between $w_t(S)$, $r_t(S)$, and $\tilde{f}_t(S)$ for a simulative segment S containing an occupied vertex.

Claim 2.26. *Let $t \in \mathbb{N}$ and $S \in \mathcal{S}_t$ such that $p_t(S) \neq \infty$. Then*

$$w_t(S) \leq r_t(S) + \tilde{f}_t(S).$$

Proof. Let S be as in the claim. By the definitions of $p_t(S)$ and $r_t(S)$ we have $\chi_t(S) = 0$ and $r_t(S) > 0$. As $\chi_t(S) = 0$, we know that $B_t(S) = \{p_t(S)\}$, and as $r_t(S) > 0$, there must exist a (t, h_t) -path in G_t which we denote by

$$P = ((p_t(S), t) = (p_1, t), (p_2, t+1), \dots, (p_{h_t}, t+h_t-1)).$$

We construct a family of $w_t(S)$ many paths in \mathbb{Z}_{ν}^2 , each of length h_t , as follows (see the accompanying Figure 3).

Define a vector $(a_1, \dots, a_{h_t-1}) \in \{0, 1\}^{h_t-1}$ via $a_j := p_{j+1} - p_j$, and $(h_t - 1)$ -tuples $a^1, \dots, a^{w_t(S)}$ by

$$a_j^k := \begin{cases} a_j & \text{if } j < \gamma_k, \\ 1 - a_k & \text{if } j \geq \gamma_k, \end{cases} \quad \text{where } \gamma_k := \min\{m : P_m \text{ is not connected in } \mathbb{Z}_{\nu}^2 \text{ to } (p_1 + k - 1, t + h_t - 1)\}.$$

Consider the paths $((p_t(S), t), (p_t(S) + a_1^k, t+1), \dots, (p_t(S) + a_{h_t-1}^k, t+h_t-1))$ for $k = 1, \dots, w_t(S)$. As these paths intersect on $P \subset G_t$, we conclude that every deleted vertex in $S \times [h_t]$ can block at most one of these paths, so that each path must be either in G_t , in which case its endpoint is counted towards $r_t(S)$, or contain a distinct deleted vertex counted towards $\tilde{f}_t(S)$. We conclude that $w_t(S) \leq \tilde{f}_t(S) + r_t(S)$, as required. \square

We extend the definitions of w and \tilde{f} for $S \in \mathcal{S}_t$ and time $t - 1$ by

$$w_{t-1}(S) := \sum_{S' \in P_{t-1}(S)} w_{t-1}(S') \quad \text{and} \quad \tilde{f}_{t-1}(S) := \sum_{S' \in P_{t-1}(S)} \tilde{f}_{t-1}(S').$$

We now turn to prove the lemma.

Proof of Lemma 2.13. Let $S \in \mathcal{S}_t$. Observe that the lemma is straightforward when $\chi_t(S) = 1$ (as in that case $\phi_t(S) = r_t(S)$ by (12)), and when $\phi_t(S) = 0$ (as $r_t(S) \geq 0$ by definition). Hence we assume $\chi_t(S) = 0$ and $\phi_t(S) > 0$.

To establish the lemma, we inductively prove the following:

- (a) $p_t(S) \neq \infty$ and $r_t(S) > 0$,
- (b) $\phi_t(S) \leq w_t(S) - \tilde{f}_t(S)$.

This would conclude the proof, as (a) and (b), together with Claim 2.26 imply that $\phi_t(S) \leq r_t(S)$.

We study two cases according to the value of $\chi_{t-1}(S)$, the first of which is proved directly and serves also as the basis of the induction.

Case 1 – consolidation: $\chi_{t-1}(S) = 1$. By Corollary 2.24 we know that $t \in \mathbb{N}_1$ and $\ell_{t-1}(S) = 0$, so that $\phi_{t-1}(S) = b_{t-1}(S)$ by (12). Again by (12) and the fact that $\phi_t(S) > 0$, this implies that $\Delta f_t(S) < b_{t-1}(S)$, while by Observation 2.12, we have $\Delta f_t(S) = |\mathcal{F}_t \cap (S \times L_t(S))|$.

Recall that $p_t(S) := \min\{x \in S : (x, t-1) \in B_{t-1}, C_t^{h_t}(x) \cap \mathcal{F}_t = \emptyset\}$ where $C_t^{h_t}(x)$ is the vertical path $\{(x, t), (x, t+1), \dots, (x, t+h_t-1)\}$ (see (8)). Hence $p_t \neq \infty$, as $\Delta f_t(S) < b_{t-1}(S)$. This implies (a).

To prove (b), writing $\phi_t(S) = \phi_{t-1}(S) - \Delta f_t(S)$, we must show that

$$\phi_{t-1}(S) = b_{t-1}(S) \leq w_t(S) + \Delta f_t(S) - \tilde{f}_t(S) = w_t(S) + |\mathcal{F}_t \cap ([\min S, p_t(S)] \times L_t(S))|.$$

To see this, observe that $b_{t-1}([p_t(S), \max S]) \leq |[p_t(S), \max S]| = w_t(S)$, while $C_t^{h_t}(x) \cap \mathcal{F}_t \neq \emptyset$ for all $(x, t-1) \in B_{t-1}([\min S, p_t(S)])$, by (7). Hence

$$b_{t-1}([\min S, p_t(S)]) \leq |\mathcal{F}_t \cap ([\min S, p_t(S)] \times L_t(S))|.$$

Since, $b_{t-1}(S) = b_{t-1}([\min S, p_t(S)]) + b_{t-1}([p_t(S), \max S])$, (b) follows.

Case 2 – simulated step: We begin by establishing (a). By the definition of $r_{t-1}(S)$ there are this many points $s \in S \times \{t-1 + \ell_{t-1}\}$ which are the end points of a $(t-1, \ell_{t-1}(S))$ -paths emanating from $B_{t-1}(S)$; denote their columns by R . By the induction hypothesis we have $|R| \geq r_{t-1}(S) \geq \phi_{t-1}(S)$. For every $s \in R$, consider the column segment $(s, t-1 + \ell_{t-1}), (s, t + \ell_{t-1}), \dots, (s, t + \ell_t)$, and observe that these are disjoint from each other and from $F_{t-1}(S)$. Any vertex in $\Delta F_t(S)$ can intersect with at most one such column segment, so that at least

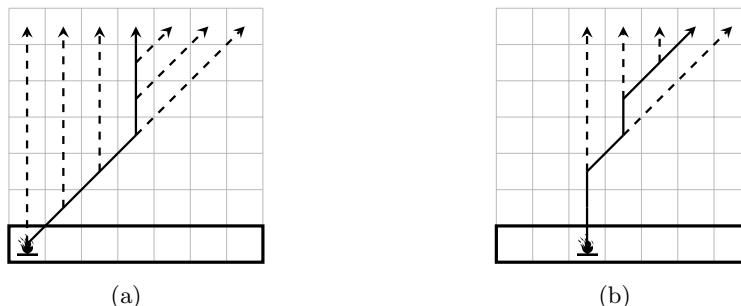


Figure 3: Two possible cases for the location of $p_t(S)$ within S , and the corresponding paths. In each sub-figure, the central path P is indicated by an unbroken line, while the other paths in are indicated by dashed lines.

$\phi_{t-1}(S) - \Delta f_t(S) = \phi_t(S) > 0$ of these column segments appear in G_t . Since $\chi_t(S) = 0$, no vertices in $S \times L_{t-1}(S)$ can be deleted at time t . We conclude that there is at least one path in G_t emanating from $B_{t-1}(S)$ and terminating in $S \times \{t + \ell_t\}$. By (8) we thus obtain that $p_t(S) \neq \infty$ and $r_t(S) > 0$, proving (a).

To prove (b), let us first consider the case $t \in \mathbb{N}_1$. As $\chi_t(S) = 0$, we have in this case, $\Delta f_t(S) = |\mathcal{F}_t \cap (S \times \{t + h_t\})|$. We thus obtain

$$\tilde{f}_t(S) \leq \tilde{f}_{t-1}(S) + \Delta f_t(S) - \Delta p_t(S), \quad (20)$$

using the fact that $\Delta p_t(S) = 1$ only if $\mathcal{F}_{t-1} \cap C_{t-1}^{h_{t-1}}(p_{t-1}(S)) \neq \emptyset$, in which case at least one such vertex is counted towards $\tilde{f}_{t-1}(S)$ but not towards $\tilde{f}_t(S)$. We obtain

$$\phi_t(S) = \phi_{t-1}(S) - \Delta f_t(S) \leq w_{t-1}(S) - \tilde{f}_{t-1}(S) - \Delta f_t(S) \leq w_{t-1}(S) + \Delta p_t - f_t(S) \leq w_t(S) - \tilde{f}_t(S),$$

Where the equality follows from (12), the next inequality – from the induction hypothesis, the following one – from (20) and the last one – from the fact that $\Delta w_t(S) = -\Delta p_t(S)$. Observe that this concludes the proof of the lemma for the linear Spreader strategy (as per Remark 1).

Next, we consider the case of $t \in \mathbb{N}_2$. Observe that in this case we have $\Delta f_t(S) = |\mathcal{F}_t \cap (S \times \{t + h_{t-1}, \dots, t + h_t\})|$. Denote $\{S_1, S_2\} := P_{t-1}(S)$, where $S_1 < S_2$. We consider two sub-cases according to value of $p_t(S)$. Firstly, consider the case $p_t(S) \in p_{t-1}(S_1) + \{0, 1\}$. In this case,

$$\tilde{f}_t(S) - \tilde{f}_{t-1}(S_1) \leq |\mathcal{F}_{t-1} \cap (S_2 \times L_{t-1}(S_2))| + \Delta f_t(S) - (p_t(S) - p_{t-1}(S_1)), \quad (21)$$

using once again the fact that $p_t(S) - p_{t-1}(S_1) = 1$ only if $\mathcal{F}_{t-1} \cap C_{t-1}^{h_{t-1}}(p_{t-1}(S_1)) \neq \emptyset$, and the fact that all firefighters above S_2 are counted towards $\tilde{f}_t(S)$. Applying the induction hypothesis to S_1 and Claim 2.25 to S_2 , we obtain

$$\begin{aligned} \phi_t(S) &= \phi_{t-1}(S_1) + \phi_{t-1}(S_2) - \Delta f_t(S) \\ &\leq w_{t-1}(S_1) - \tilde{f}_{t-1}(S_1) + h_{t-1} - |\mathcal{F}_{t-1} \cap (S_2 \times L_{t-1}(S_2))| - \Delta f_t(S) \\ &\leq w_{t-1}(S_1) - \tilde{f}_t(S_1) + h_{t-1} - (p_t(S) - p_{t-1}(S_1)), \end{aligned}$$

where the last inequality follows from (21). Observing that $w_t(S) = w_{t-1}(S_1) + h_{t-1} - (p_t(S) - p_{t-1}(S_1))$ then concludes the proof of (b).

Finally, consider the case $p_t(S) \in p_{t-1}(S_2) + \{0, 1\}$. By repeating the argument of the proof of part (a) in the contra-positive, we obtain that, since $p_t(S) \notin S_1$, we necessarily have $r_{t-1}(S_1) \leq \Delta f_t(S_1)$, so that, by the induction hypothesis,

$$\phi_{t-1}(S_1) \leq \Delta f_t(S_1). \quad (22)$$

In addition,

$$\tilde{f}_t(S) - \tilde{f}_{t-1}(S_2) \leq \Delta f_t(S_2) + w_t(S) - w_{t-1}(S_2), \quad (23)$$

using the fact that $w_t(S) - w_{t-1}(S_2) = -1$ only if $\mathcal{F}_{t-1} \cap C_{t-1}^{h_{t-1}}(p_{t-1}(S_2)) \neq \emptyset$.

We conclude that

$$\begin{aligned} \phi_t(S) &= \phi_{t-1}(S_1) + \phi_{t-1}(S_2) - \Delta f_t(S) \\ &\leq \Delta f_t(S_1) + \phi_{t-1}(S_2) - \Delta f_t(S) \\ &\leq w_{t-1}(S_2) + \Delta f_t(S_1) - \tilde{f}_{t-1}(S_2) - \Delta f_t(S) \\ &= w_{t-1}(S_2) - \tilde{f}_{t-1}(S_2) - \Delta f_t(S_2) \\ &\leq w_t(S) - \tilde{f}_t(S), \end{aligned}$$

where the first equality is from (12), the next inequality follows from (22), the one that follows – from part (b) applied to S_2 at time $t - 1$, the second equality is straightforward and the final inequality is an application of (23). \square

3 The Directed Half Plane

This section is dedicated to the study of the containment game on the *directed half plane* $\mathbb{Z}_{\searrow}^2 = (V, E)$, the sub-graph of $\mathbb{Z} \boxtimes \mathbb{Z}$ given by

$$V = \{(x, y) \in \mathbb{Z}^2 : y \geq 0\}, \quad E = \{(x, y), (x + i, y + 1)\} : |x| \leq y, i \in \{-1, 0, 1\}\}.$$

Section 3.1 depicts the game strategy on this graph. In Section 3.2 we provide three key propositions concerning this strategy which will play a prominent role in the proof of Theorem 1 in Section 4 below. Sections 3.3, 3.4 and 3.5 are dedicated to the proofs of these proposition.

3.1 The directed half plane strategy

Here we present the Spreader strategy on \mathbb{Z}_{\searrow}^2 , and declare several of its key properties, later established in subsequent sections. This is not an independent section, as it strongly relies on Section 2 for definitions and results. In particular, our strategy maintains the convention that at time t all of the occupied vertices are on the row $\mathbb{Z} \times \{t\}$. This allows us to keep using the one-dimensional notations \mathcal{S}_t and p_t . For our application we must analyze the game also with arbitrary starting conditions, given by $\mathcal{F}_0, F_0 \subset \mathcal{F}_0$, and $B_0 \subset \mathbb{Z} \times \{0\}$.

Min and Max notation. Throughout the section it will also be useful to denote the leftmost and rightmost vertices in a set D by $\min D$ and $\max D$, where formally minimum and maximum are taken with respect to the lexicographical order.

The \mathbb{Z}_{\searrow}^2 strategy. We begin by describing a preliminary \mathbb{Z}_{\searrow}^2 -strategy of Spreader, subject to starting conditions \mathcal{F}_0, F_0, B_0 . We will later amend this strategy to avoid dead-ends in Section 3.4. This preliminary strategy is rather similar to the one used in \mathbb{Z}_{\searrow}^2 (see (2)), modified to account for the expansion of the occupied set also northwestwards. Somewhat surprisingly, it suffices for our purposes to allow only the single leftmost occupied vertex to expand also to the northwest, while all other vertices expand only to the north and the northeast, as before. Indeed, for every $S \in \mathcal{S}_t$, we set

$$B_t(S) := (S \times \{t\}) \cap \begin{cases} \{(p_t(S), t)\} & \text{if } \chi_t(S) = 0, \\ \{(B_{t-1} + \{(0, 1), (1, 1)\}) \cup \{\min B_{t-1} + (-1, 1)\}\} \setminus \mathcal{F}_t & \text{if } \chi_t(S) = 1, \end{cases}$$

where the evolution of p_t and χ_t remains unchanged from Section 2.3.

3.2 The directed half plane analysis

Much of the analysis conducted in Section 2 remains valid after slight modifications. We begin by modifying the definition of F_t , originally defined in (11), to account for westwards expansion. We define

$$\Delta F_t(S) := (\mathcal{F}_t \setminus F_{t-1}) \cap \begin{cases} S \times L_t(S) & \text{if } \ell_t(S) > 0 \\ (S \times L_t(S)) \cap (B_{t-1} + \{(-1, 1), (0, 1), (1, 1)\}) & \text{if } \ell_t(S) = 0, \end{cases} \quad (24)$$

while maintaining the definitions of $\phi_t, \Phi_t, \ell_t, L_t, b_t$, and $\lambda_t(S)$.

Next, we verify that Proposition 2.2 remains valid on \mathbb{Z}_{\searrow}^2 . We maintain the notation of A_1, A_2 , and E_t from the original proof given in Section 2.4. That proof is still valid, except we we must consider an additional case where $\beta_t^\ell(y)$ is no longer defined for some y and ℓ . This is possible if at some time $t' \in [t, t + \ell]$, the ancestor of y was a northwestwards expansion of a vertex. Nevertheless, if $y \in A_1$, then by applying the contra-positive of Proposition 2.10 we have $\beta_t^{Ht}(y) \in B_t^{3Ht}$ while if $y \in A_2$, the statement that $y' \in A_2$ satisfies $\beta_{t'}^{t-t'}(y') \in E_{t'}$ for some $t' \in [t - h_t, t]$ remains valid.

Propositions 2.5, 2.6, 2.18, 2.19 and 2.20 and the propositions and observations on which they rely remain valid for the directed half plane with their original proofs. This is due to the fact that they only difference between the

process on $\mathbb{Z}_{\nearrow\searrow}^2$ and on $\mathbb{Z}_{\nwarrow\swarrow}^2$, is that on every turn there is potentially a single additional spreading vertex occupied or deleted in $\mathbb{Z}_{\nwarrow\swarrow}^2$ (the northwestern neighbour of the min B_t).

Proposition 2.5, however, is too weak for our needs and we require the following analogue, established in Section 3.3 below.

Proposition 3.1. *Let $t \in \mathbb{N}$. If $b_{t-1}(S_{t-1}) > 0$ then $\Delta\Phi_t(S_t) \geq 2$.*

Avoiding dead-ends. Denote B for the $\mathbb{Z}_{\nwarrow\swarrow}^2$ Spreader strategy described above, used against Container of strength q , and subject to the starting conditions B_0, \mathcal{F}_0, F_0 for some $h : \mathbb{N} \rightarrow \mathbb{N}$. In Section 2.1.2 we showed that on $\mathbb{Z}_{\nwarrow\swarrow}^2$ we can reduced the number of vertices in $|B_t|$ in the final strategy by actually spreading only to vertices in $|B_t^{3H_t}|$. The analogue in $\mathbb{Z}_{\nwarrow\swarrow}^2$ is more involved, as we must spread also into vertices that have the potential of becoming min B_s for some $s \in [t, t + 3H_t]$. This is captured by the following analogue of Proposition 2.3 which is proved in Section 3.4.

Proposition 3.2. *There exists a Spreader strategy $B' = (B'_t)_{t \in \mathbb{N}}$ satisfying the following for every $t \in \mathbb{N}$.*

- (a) $|B'_t| \leq O(h_t^6) + \frac{2t}{h_t} + |B_0|$.
- (b) $B'_t \neq \emptyset \iff B_t \neq \emptyset$.

The play area. In Section 4 we shall play in parallel several copies of the containment game on $\mathbb{Z}_{\nwarrow\swarrow}^2$. Towards analysing these we require a new notion of the *play area* of a game \mathcal{G} at time t , which we denote by

$$\mathcal{A}_t^{\mathcal{G}} := \bigcup_{t'=0}^t \mathcal{B}_{t'} \cup F_{t'}.$$

Namely, the occupied set together with all deleted vertices counted by Spreader up to time t . Here we often omit the superscript \mathcal{G} as only a single game is considered, so that the full notation will be used only in Section 4. For the analysis of the play area, we introduce the following notion. Letting $D \subset \mathbb{Z}^2$, we define the *infinite trapezoid generated by D* , denoted by $\mathcal{T}(D)$, to be the convex hull in \mathbb{Z}^2 of $D + \{k(1, 1), k(-1, 1) : k \in \mathbb{N}\}$. The following proposition, established in Section 3.5, bounds $\mathcal{A}_t^{\mathcal{G}}$.

Proposition 3.3. *The following hold for every t .*

- (a) If $B_t = \emptyset$ then $\mathcal{A}_t \subset \mathbb{Z} \times (-\infty, t]$,
- (b) For every $t' > t$ we have $\mathcal{A}_{t'} \setminus \mathcal{A}_t \subseteq \mathcal{T}(B_t)$,
- (c) Denote $\mathcal{B}' = (\bigcup_{\tau=0}^{t-1} B_\tau)$ and write x_{\min} and x_{\max} for the first coordinates of min \mathcal{B}' and max \mathcal{B}' respectively. We have $\mathcal{A}_t \subseteq [x_{\min} - 1, x_{\max} + 1] \times \mathbb{Z}$.

3.3 Potential growth – proof of Proposition 3.1

In this section we prove Proposition 3.1, relying upon Proposition 2.5.

Proof of Proposition 3.1. Let $t \in \mathbb{N}$. Denote $S_{\min} = S_t(\min B_t)$ and observe that by Proposition 2.20 we have

$$\ell_t(S_{\min}), \ell_{t-1}(S_{\min}), d_t(S_{\min}), d_{t-1}(S_{\min}) = 0.$$

Denoting $x := \min B_{t-1} + (-1, 1)$, we have, by definition, $x \in B_t(S_{\min}) \cup F_t(S_{\min}) \setminus B_{t-1}$. Hence, by a variation on Proposition 2.14(b) expanding leftwards instead of rightwards, and using (24), we obtain $\Delta\Phi'_t(S_{\min}) \geq 1$ which thus implies $\Delta\Phi_t(S_{\min}) \geq 1$.

In addition, by (14) the definition of λ_t , we know that S_{\min} is not $\lambda_t(S)$ for any S , so that, by Proposition 2.5, $\Delta\Phi_t(S_t \setminus \{S_{\min}\}) \geq 1$. Therefore, $\Delta\Phi_t(S_t) = \Delta\Phi_t(S_t \setminus \{S_{\min}\}) + \Delta\Phi_t(S_{\min}) \geq 2$, as required. \square

3.4 Avoiding dead-ends – proof of Proposition 3.2

This section is dedicated to the proof of Proposition 3.2. When constructing the modified strategy B' , we aim to ensure that B'_t always contains $\min B_t$, the leftmost occupied vertex, by including any vertex that could take this role within a bounded number of turns. We proceed to define this formally.

Recall the definition of $\beta_t^\ell(y)$, the leftmost starting vertex of a (t, ℓ) -path ending in a given $y \in B_{t+\ell}$, defined in Section 2.4. Define D_t^ℓ to be the set of vertices $x \in B_t$ for which there exists a q -Container strategy under which $x = \beta_t^\ell(\min B_{t+\ell})$ when the strategy B is played against it.

Define $B' = (B'_t)_{t \in \mathbb{N}}$ inductively as follows.

$$\begin{aligned} B'_0 &= B_0, \\ B'_t &= B_t^{3H_t} \cup \left((B'_{t-1} + \{(-1, 1), (0, 1), (1, 1)\}) \cap \bigcup_{\ell=0}^{3H_t} D_t^\ell \right). \end{aligned} \quad (25)$$

Proof of Proposition 3.2. Let $t \in \mathbb{N}$. Firstly we show that $\min B_t \in B'_t$ which implies $B'_t \neq \emptyset \iff B_t \neq \emptyset$ (recalling that $D_t^{3H_t} \subseteq B_t$). This we show using induction. Since $B'_0 = B_0$, we clearly have $\min B_0 \in B'_0$. Assume this holds up to time $t-1$, and let $y := \min B_t$. Let $x := \beta_{t-3H_t}^{3H_t}(y)$ and denote by

$$P = (x = p_{t-3H_t}, \dots, p_t = y)$$

the associated minimal two-sided $(t-3H_t, 3H_t)$ -path of occupied vertices associated with x . Note that $p_{t-i} = \beta_{t-i}^i(y)$, so that the vertices of P satisfy $p_i \in D_i^{t-i}$. To show that $y \in B'_t$ it would therefore suffice to show the existence of i such that $p_i \in B'_i$, as this would follow for every subsequent vertex, by (25). If there exists some $i \in [t-3H_t+1, t]$ such that $p_i = p_{i-1} - 1$, this follows directly from the induction hypothesis, as in this case only the leftmost occupied vertex can expand to the left and we have $p_{i-1} = \min B_{i-1}$. Otherwise, P is a one-sided $(t-3H_t, 3H_t)$ -path, so that

$$p_{t-3H_t} = x \in B_{t-3H_t}^{3H_t} \subseteq B_{t-3H_t}^{3H_t-3H_t} \subseteq B'_{t-3H_t},$$

by (25) and the monotonicity of H .

Next we show that B' is a valid strategy, i.e., that $B'_t \subset (B'_{t-1} + \{(-1, 1), (0, 1), (1, 1)\}) \setminus \mathcal{F}_t$. Let $y \in B'_t$. Since $B'_t \subset B_t$, we clearly have $y \notin \mathcal{F}_t$. Hence we must show only that $y \in B'_{t-1} + \{(-1, 1), (0, 1), (1, 1)\}$. By (25), the only non-trivial case is when $y \in B_t^{3H_t}$. In this case, as B is a valid strategy, there must exist $x \in B_{t-1}$ such that $y \in x + \{(-1, 1), (0, 1), (1, 1)\}$. If $y \in x + \{(0, 1), (1, 1)\}$ then $x \in B_{t-1}^{3H_t-1}$ so that indeed $x \in B'_{t-1}$. Otherwise, if $y = x + (-1, 1)$, then $x = \min B_{t-1} \in B'_{t-1}$.

Finally, we prove the bound on $|B'_t|$. Let $\ell \in [0, 3H_t]$. Given any $x \in D_t^\ell$ we denote the minimal two-sided (t, ℓ) -path starting from x by P_x , and its endpoint by y_x . Such a path must exist, since there exists a Container strategy under which $x = \beta_t^\ell(\min B_{t+\ell})$. Next, denote the points of D_t^ℓ by $\{x_1, \dots, x_N\}$, sorted from left to right. Since $x_N \in D_t^\ell$, Container must have a strategy which blocks P_{x_i} , for every $i \in 1, \dots, N-1$ such that $y_{x_i} < y_N$ (lexicographically). As $d_G(y_{x_i}, x_i) \leq \ell$ for all i , we must have $y_{x_i} < y_N$ for every $i < N-2\ell$. In addition, by Observation 2.1, at most $2\ell-1$ distinct two-sided (t, ℓ) -paths can intersect at a point. Thus, observing that $P_{x_i} \cap \mathcal{F}_{t+\ell} = P_{x_i} \cap (\mathcal{F}_{t+\ell} \setminus \mathcal{F}_t)$, we have

$$|\{i : P_{x_i} \cap \mathcal{F}_{t+\ell} \neq \emptyset\}| \leq 2\ell \cdot q\ell = 2\ell^2 q.$$

We deduce that $N-2\ell \leq 2\ell^2 q$, so that $|D_t^\ell| = N \leq 2\ell^2 q + 2\ell = O(\ell^2)$. Therefore,

$$\left| \bigcup_{\ell=0}^{3H_t} D_t^\ell \right| \leq (3H_t + 2H_t)(2H_t^2 q + 1) = O(H_t^3) = O(h_t^6).$$

To bound $|B_t \cap R_t^{3H_t}|$, note that at time $t=0$ we clearly have $|B_0^{3H_t}| \leq |B_0|$. To prove the case $t \geq 1$, we follow

the footsteps of the proof of Proposition 2.3. Recall that

$$|B_t^{3H_t}| = \left| \{x \in B_t^{3H_t} : \chi_t(S_t(x)) = 1\} \right| + \left| \{x \in B_t^{3H_t} : \chi_t(S_t(x)) = 0\} \right|.$$

Proposition 2.2 yields an $O(h_t^6)$ bound on the first term. Since a simulative segment contains at most one occupied vertex, and B_t is contained in the discrete interval whose end points are $\min B_0 - (0, t)$ and $\max B_0 + (0, t)$, the second term is bounded by $\frac{2t+|B_0|}{h_t}$. The proposition follows. \square

3.5 Bounding the play area – proof of Proposition 3.3

Let $t \in \mathbb{N}$. To prove (a), assume that $B_t = \emptyset$. We claim that $\ell_{t'}(S) = 0$ for every $t' \in [t - h_t, t] \cap \mathbb{N}$ and $S \in \mathcal{S}_t$. Indeed, as $b_t = 0$, this is a direct consequence of Proposition 2.10. By definition, $F_s(S) \subset S \times (-\infty, s + \ell_s(S)]$. Since $\ell_s(S) \leq h_s$ for all s , item (a) follows.

To prove (b), note that for every $n \in \mathbb{N}$ for which $\mathcal{B}_{t+n} \neq \emptyset$, the interval connecting $\min B_t + (-n, n)$ and $\max B_t + (n, n)$ must contain \mathcal{B}_{t+n} . Next, let $n \geq 1$, $S \in \mathcal{S}_{t+n}$ and $v \in \Delta F_{t+n}(S)$. Clearly $v \in S \times \{t+n, \dots, t+n+h_{t+n}\}$. Hence if S is neither the leftmost occupied segment nor the rightmost, we obtain $v \in \mathcal{T}(B_t)$. Otherwise, S must be simple (by Proposition 2.20), so that $v \in B_{t+n-1} + \{(-1, 1), (0, 1), (1, 1)\}$ by (11), and hence $v \in \mathcal{T}(B_t)$.

To prove (c), denote x_{\min}^s, x_{\max}^s for the first coordinate of $\min B_s$ and $\max B_s$, respectively. Note that a deleted vertex counted towards F at time s outside the rectangle $[x_{\min}^{s-1}, x_{\max}^{s-1}] \times \mathbb{Z}$ must have been counted by either the leftmost occupied segment or the rightmost, which are always simple (again, by Proposition 2.20). Thus, by (11), it is of distance at most 1 from the rectangle. As this holds for any $s \leq t$, item (c) follows. \square

4 The Plane

4.1 Winning in the plane

In this section we construct a winning Spreader strategy in $\mathbb{Z} \boxtimes \mathbb{Z}$ from the $\mathbb{Z}_{\downarrow}^2$ strategy described in Section 3.1. Given the $\mathbb{Z}_{\downarrow}^2$ results, the proof of Theorem 1 will closely follow the firefighter analogue of [16, Theorem 1], showing that at least three fronts contain occupied vertices at all times.

4.1.1 The strategy

The strategy of Spreader in the plane consists of two components. The first is a simultaneous implementation of a rotated and translated directed half plane strategy in up to four disjoint play areas, each corresponding to an occupied front. The second component is the re-ignition of an extinguished front by an adjacent occupied front. We proceed to write this formally.

Writing $\mathcal{J} := \{0, 1, 2, 3\}$, we denote the four cardinal directions $\{\theta^i\}_{i \in \mathcal{J}}$ by

$$\theta^0 := (0, 1), \quad \theta^1 := (1, 0), \quad \theta^2 := (0, -1), \quad \theta^3 := (-1, 0),$$

in a clockwise fashion. Superscripts in \mathcal{J} are always taken modulo 4; in particular we write $|i - j| = 1$ if i and j are consecutive modulo 4. We also denote the four secondary directions $\{\theta^{i,i+1}\}_{i \in \mathcal{J}}$ by

$$\theta^{0,1} := (1, 1), \quad \theta^{1,2} := (1, -1), \quad \theta^{2,3} := (-1, -1), \quad \theta^{3,0} := (-1, 1).$$

The infinite line of distance d from the origin, perpendicular to the i -th direction, is denoted by

$$L^i(d) := d\theta^i + \mathbb{Z}\theta^{i+1}.$$

We also introduce a notion of directed *occupation radius* (denoted by ρ_t^i) and *front line* (denoted by L_t^i), keeping

track of the distance of the front from the origin in every cardinal direction. These are defined as follows.

$$\begin{aligned}\rho_t^i &:= \min \left\{ r : L^i(r) \cap \bigcup_{t'=0}^{t-1} \mathcal{B}_{t'} = \emptyset \right\}, \\ L_t^i &:= L^i(\rho_t^i).\end{aligned}\tag{26}$$

Throughout the section, we denote by \mathcal{G}_t^i the half-plane game played in direction i at time t employed by the plane strategy. We write $\mathcal{B}_t^i := \mathcal{B}_t(\mathcal{S}_t^i)$ and use a similar convention for $\phi, d, \Delta f$ and b to denote the corresponding sets and quantities used for the analysis of the game \mathcal{G}_t^i . Observing that on a termination time-step t , $\mathcal{B}_t^i = \emptyset$ and all of the remaining quantities are 0, by Proposition 2.6, we set these as default values for all times s in which no game is played on the i -th front. Once a front game is *initialised*, it plays until $\mathcal{B}_t^i = \emptyset$ at which step it *terminates*. We use the equality $\mathcal{G}_t^i = \mathcal{G}_s^i$ to indicate that the game played at time t didn't terminate before time s . For each front $i \in \mathcal{J}$, the fact that \mathcal{G}_t^i is *active* (rather than terminated) is indicated by $a_t^i := \mathbb{1}\{\mathcal{B}_t^i \neq \emptyset\}$. We also set $\mathcal{B}_t := \bigcup_{i \in \mathcal{J}} \mathcal{B}_t^i$.

The $\mathbb{Z} \boxtimes \mathbb{Z}$ game starts in turn 1, where the half plane games are initialised with $\mathcal{B}_1^i := \{\theta^i, \theta^{i,i+1}\} \setminus \mathcal{F}_1$ (See Figure 4a). In every turn $t > 1$, the strategy plays as follows.

1. If $a_{t-1}^i = 1$, a step is played in the i -th front game, according to the \mathbb{Z}_{\swarrow}^2 strategy on the corresponding shifted and rotated piece of $\mathbb{Z} \boxtimes \mathbb{Z}$.
2. If $a_{t-1}^i = 0$ then,
 - in case that $\mathcal{B}_{t-1} \cap L_{t-1}^i = \emptyset$, nothing happens in this front,
 - otherwise, when $\mathcal{B}_{t-1} \cap L_{t-1}^i \neq \emptyset$, a new \mathbb{Z}_{\swarrow}^2 game \mathcal{G} is initialised, rotated towards direction i and shifted by ρ_t^i , with starting conditions $\mathcal{B}_0 := O \setminus \mathcal{F}_t$, $\mathcal{F}_0 := \mathcal{F}_t$ and $F_0 := O \cap \mathcal{F}_t$, where

$$O := L_t^i \cap \left((\mathcal{B}_{t-1}^{i+1} + \{\theta^i, \theta^{i-1,i}\}) \cup (\mathcal{B}_{t-1}^{i-1} + \{\theta^i, \theta^{i,i+1}\}) \right).\tag{27}$$

Namely, a terminated game can be re-initialised when the front L_{t-1}^i a vertex occupied by Spreader as part of the game played on an adjacent front. In this case $\Delta \rho_t^i = 1$, and the initial play area consists of all of the neighbours of this occupied vertex on $L_{t-1}^i + \theta_i$ which are not in the play area of the adjacent front's game (see Figure 4b). Note that this new game may terminate immediately upon creation, but at a given front, a game cannot be initialised on the same turn that another game terminates.

4.1.2 Analysis of the strategy

We start by making the following claim, concerning rotated infinite trapezoids. Given a set $S \subset \mathbb{Z}^2$ we denote the infinite trapezoid of S rotated to direction i by $\mathcal{T}_i(S)$. This is given by the convex hull of $S + \{k\theta^{i,i+1}, k\theta^{i,i-1} : k \in \mathbb{N}\}$. We omit the subscript i when it is clear from context.

Claim 4.1. *Let $a_1 < a_2 < a_3, b_1 < b_2$, and define $C := [a_1, a_2] \times \{b_2\}, D := \{a_3\} \times [b_1, b_2]$. Then $\mathcal{T}_0(C) \cap \mathcal{T}_1(D) = \emptyset$.*

Proof. Observe that for all $(x, y) \in \mathcal{T}_0(C)$ we have $y - x \geq b_2 - a_2$ while for all $(x, y) \in \mathcal{T}_0(D)$ we have $y - x \leq b_2 - a_3$. \square

The analysis of the plane strategy is based on two key propositions. The first states that the play areas of all \mathbb{Z}_{\swarrow}^2 games employed by Spreader are disjoint.

Proposition 4.2. *Let $t_0, t_1 \in \mathbb{N}$, $i_0, i_1 \in \mathcal{J}$. If $\mathcal{G}_{t_0}^{i_0} \neq \mathcal{G}_{t_1}^{i_1}$ then $\mathcal{A}_{t_0}^{i_0} \cap \mathcal{A}_{t_1}^{i_1} = \emptyset$.*

Proof. Without loss of generality assume that $i_0 = 0$ and $t_0 \leq t_1$. Firstly, we consider the case $i_1 = 0$, under which $t_0 \neq t_1$. Let $t_0 < s \leq t_1$ be the turn in which the game $\mathcal{G}_{t_1}^0$ was initialised, so that $\mathcal{B}_{s-1}^0 = \emptyset$. As $\mathcal{G}_{t_0}^0$ must have terminated before turn s , we obtain, by Proposition 3.3(a), that $\mathcal{A}_{t_0}^0 \subseteq (-\infty, \rho_{s-1}^0]$. In contrast, $\mathcal{A}_{t_1}^0 \subset \mathcal{T}(\mathcal{B}_s^0)$ by Proposition 3.3(b), and $\mathcal{T}(\mathcal{B}_s^0) \subset \mathbb{Z} \times [\rho_s^0, \infty) = [\rho_{s-1}^0 + 1, \infty)$. Thus, the two play areas are disjoint.

We are left with the case $i_0 \neq i_1$. Clearly, areas of non-adjacent fronts cannot intersect, so we assume that $i_1 = 1$. The case $i_1 = -1$ follows by similar arguments. Observe that at turn 1, when the four starting games are initialised,

their initial occupied sets generate disjoint trapezoids, and hence they have disjoint play areas. Next, let $s \leq t_1$ be the turn in which $\mathcal{G}_{t_1}^1$ was initialised, so that $\mathcal{B}_{s-1}^1 = 0$ and $\mathcal{B}_{s-1} \cap L_{s-1}^1 \neq \emptyset$. Recall that, by (27), $\mathcal{B}_s^1 = O \setminus \mathcal{F}_s$, where

$$O := L_s^1 \cap ((\mathcal{B}_{s-1}^2 + \{\theta^1, \theta^{0,1}\}) \cup (\mathcal{B}_{s-1}^0 + \{\theta^1, \theta^{1,2}\})).$$

By Proposition 3.3(b) we have $\mathcal{T}(\mathcal{B}_{t_1}^1) \subset \mathcal{T}(\mathcal{B}_s^1) \subset \mathcal{T}(O)$. To complete the proof it would thus suffice to show that $\mathcal{A}_{t_0}^0 \cap \mathcal{T}(O) = \emptyset$.

Indeed, by Proposition 3.3(c) and (26) we have $\mathcal{A}_{s-1}^0 \subset [-\rho_{s-1}^3, \rho_{s-1}^1] \times \mathbb{Z}$. This latter set is disjoint from $\mathcal{T}(O)$, as $O \subset \{\rho_s^1\} \times \mathbb{Z} = \{\rho_{s-1}^1 + 1\} \times \mathbb{Z}$. Hence, if $t_0 \leq s - 1$, then $\mathcal{A}_{t_0}^0 \subset \mathcal{A}_{s-1}^0$ and we are done. Otherwise, if $t_0 > s - 1$, we have $\mathcal{A}_{t_0}^0 \setminus \mathcal{A}_{s-1}^0 \subset \mathcal{T}(\mathcal{B}_{s-1}^0)$, by Proposition 3.3(b). However, $\mathcal{T}(\mathcal{B}_{s-1}^0)$ is disjoint from $\mathcal{T}(O)$ as these satisfy Claim 4.1. The proposition follows. \square

The second proposition states that when the front advances in direction i , the potential in that direction grows by 2.

Proposition 4.3. $\Delta\Phi_t^i \geq 2\Delta\rho_t^i$.

Proof. If $\Delta\rho_t^i = 0$, then, by (26), we have $\mathcal{B}_{t-1}^i = \emptyset$. By Proposition 2.6, this implies that $\phi_{t-1}^i(\mathcal{S}_t^i) = 0$ and $d_{t-1}^i(\mathcal{S}_t^i) = 0$. Thus, indeed, $\Delta\Phi_t(\mathcal{S}_t^i) \geq 0$.

We therefore assume $\Delta\rho_t^i = 1$. If $a_{t-1}^i = 1$ then \mathcal{G}_t^i is active and takes a step according to its \mathbb{Z}_{\nearrow}^2 strategy, in which case, the proposition follows directly from Proposition 3.1. Otherwise, if $a_{t-1}^i = 0$, we must have $\mathcal{B}_{t-1} \cap L_{t-1}^i \neq \emptyset$. According to the strategy, a new game starts on the trapezoid generated by

$$O = L_t^i \cap ((\mathcal{B}_{t-1}^{i+1} + \{\theta^i, \theta^{i-1,i}\}) \cup (\mathcal{B}_{t-1}^{i-1} + \{\theta^i, \theta^{i,i+1}\})),$$

where $\mathcal{B}_t^i \cup F_t^i = O$. Observe that, in this case $|O| \in \{2, 4\}$. Since $a_{t-1}^i = 0$, we have $d_{t-1}^i = \phi_{t-1}^i = 0$, and, in addition $\Delta f_t^i = |F_t^i|$, by Proposition 4.2. Since all segments in \mathcal{S}_t^i are initialised to be simple, and $a_{t-1}^i = 0$, we have $\Delta\phi_t^i = b_t^i$. Putting all of these together, we have $\Delta\Phi_t^i \geq b_t^i + \Delta f_t^i = |O| \geq 2$. \square

4.2 Proof of Theorem 1 on $\mathbb{Z} \boxtimes \mathbb{Z}$

In this section we conclude the proof of Theorem 1 by applying Propositions 4.2 and 4.3 to imitate the proof of [16, Theorem 1]. The key observation, which follows directly from (26), is that \mathcal{B}_t^i is always contained in the axis-aligned interval

$$\mathcal{L}_t^i := \rho_t^i \cdot \theta^i + [-\rho_t^{i-1}, \rho_t^{i+1}] \cdot \theta^{i+1}.$$

Throughout the proof, putting several superscripts in a function serves to represent summation over them, e.g. $\rho_t^{13} := \rho_t^1 + \rho_t^3$. Omitting a superscript serves to represent summation over all possible superscripts, e.g. $\rho_t := \sum_{i \in \mathcal{J}} \rho_t^i$. We start by proving several auxiliary claims.

Claim 4.4. $\phi_t^i + d_t^i \leq |\mathcal{L}_t^i|$.

Proof. Let $S \in \mathcal{S}_t(\mathcal{G}_t^i)$ such that $D^i(S) \cap \mathcal{L}_t^i \neq \emptyset$, where $D^i(S)$ is defined as the set $S \times \{t\}$, rotated to direction θ^i . If $D^i(S) \subset \mathcal{L}_t^i$, then, by Proposition 2.18, we have $\phi_t^i(S) + d_t^i(S) \leq h_t = |S| = |D^i(S)|$. Otherwise, by Proposition 2.20 and (13), $\phi_t^i(S) = b_t^i(S) \leq |D^i(S) \cap \mathcal{L}_t^i|$, and $d_t^i(S) = 0$. The claim follows. \square

By Proposition 4.3 we have $\Delta\Phi_t^i \geq 2\Delta\rho_t^i$ for every $t \in \mathbb{N}$. Summing this over times $1, \dots, t$ we obtain

$$\phi_t^i + d_t^i \geq \phi_0^i - f_t^i + 2\rho_t^i - 2\rho_0^i. \quad (28)$$

Recalling the definition of \mathcal{L}_t^i , observe that $|\mathcal{L}_t^i| = 1 + \rho_t^{i-1} + \rho_t^{i+1}$ and that ρ_t is the semi-perimeter of the rectangle bounding the occupied set. We prove two claims under the assumption $\rho_t - \rho_0 > f_t$, which will be established inductively in the course of the proof.

Claim 4.5. *If $\rho_t - \rho_0 \geq f_t$ then $\phi_t + d_t \geq \rho_t + \rho_0$.*

Proof. Summing (28) over all $i \in \mathcal{J}$, and observing that $\phi_0 = 2\rho_0$, yields

$$\phi_t + d_t \geq \phi_0 - f_t + 2\rho_t - 2\rho_0 \geq \phi_0 + \rho_t - \rho_0 = \rho_t + \rho_0. \quad \square$$

Claim 4.6. *If $\rho_t - \rho_0 \geq f_t$ then $b_t^{i,i+2} > 0$ for every $i \in \mathcal{J}$.*

Proof. Without loss of generality we assume that $i = 0$. Note that $\rho_t^{02} + \rho_t^{13} = \rho_t$. We thus consider two cases.

Case 1. $\rho_t^{13} \geq \frac{1}{2}\rho_t$. By Claim 4.4 we have

$$\phi_t^{13} + d_t^{13} \leq |\mathcal{L}_t^1| + |\mathcal{L}_t^3| = 2\rho_t^{02} + 2 = 2\rho_t - 2\rho_t^{13} + 2 \leq \rho_t + 2.$$

By Claim 4.5, we obtain

$$\phi_t^{02} + d_t^{02} = \phi_t + d_t - \phi_t^{13} - d_t^{13} \geq \rho_t + \rho_0 - \rho_t - 2 > 0,$$

from which we conclude, by Proposition 2.6, that $b_t^{02} > 0$.

Case 2. $\rho_t^{02} \geq \frac{1}{2}\rho_t$. We have

$$\phi_t^{02} + d_t^{02} \geq \phi_0^{02} - f_t^{02} + 2\rho_t^{02} - 2\rho_0^{02} \geq \phi_0^{02} - f_t + \rho_t - 2\rho_0^{02} \geq \phi_0^{02} + \rho_0 - 2\rho_0^{02} > 0,$$

where the first inequality is obtained by applying (28) for $i = 0, 2$, the second – by plugging in $f_t^{02} \leq f_t$ and $\rho_t^{02} \geq \frac{1}{2}\rho_t$, the third – by the assumption that $\rho_t - \rho_0 \geq f_t$, and the fourth is straightforward from the definitions. As before, by applying Proposition 2.6 we conclude that $b_t^{02} > 0$. \square

To control the size of the occupied set, we apply our method for avoiding dead-ends to the plane. We define B' , the modified strategy in the plane, as the strategy that follows the modified strategy of Proposition 3.2 in every game \mathcal{G}_t^i separately.

Recalling our notation $q(G, g)$ for the range of values q such that (G, q, g) is won by Container, we now establish the following proposition, from which Theorem 1 will be easily obtained.

Proposition 4.7. $3 \notin q(\mathbb{Z} \boxtimes \mathbb{Z}, O(h_t^6) + \frac{8t}{h_t})$.

Proof. Let $t \in \mathbb{N}, i \in \mathcal{J}$. By Proposition 3.2(a), following the modified strategy in \mathcal{G}_t^i which started at some $t_0 \in \mathbb{N}$, we are guaranteed to have

$$|B_t^i| \leq O(h_{t-t_0}^6) + \frac{2(t-t_0)}{h_{t-t_0}} + |B_{t_0}^i| \leq O(h_t^6) + \frac{2t}{h_t} + |B_{t_0}^i|,$$

where $B_{t_0}^i$ is the initial occupied set of \mathcal{G}_t^i . Using the fact that $|B_{t_0}^i| \leq 4$, and that at most four games are played in parallel at every time-step, we conclude that indeed $|B_t^i| \leq O(h_t^6) + \frac{8t}{h_t}$.

By Proposition 3.2(b), $B_t^i \neq \emptyset \iff B_t \neq \emptyset$, and hence to complete the proof it suffices to show that $B_t \neq \emptyset$. To this end, we show by induction on t that we have $\Delta\rho_t \geq 3$, so that the occupied set is never contained. In turn $t = 1$ exactly three deleted vertices are needed to block a front, hence at most one can be blocked, so that indeed $\Delta\rho_1 \geq 3$. Next, assume that this is true for all $t' \leq t$ so that $\rho_t - \rho_0 = \sum_{\tau=1}^t \Delta\rho_\tau \geq 3t$. By Proposition 4.2, we never count deleted vertices more than once towards f_t , so that $f_t \leq |\mathcal{F}_t| \leq 3t$, and hence $\rho_t - \rho_0 \geq f_t$.

Next, let $i \in \mathcal{J}$. By Claim 4.4, we have

$$\phi_t^{i,i+1} + d_t^{i,i+1} \leq |\mathcal{L}_t^i| + |\mathcal{L}_t^{i+1}| = \rho_t^{i-1} + \rho_t^{i+1} + \rho_t^i + \rho_t^{i+2} + 2 = \rho_t + 2.$$

However, by Claim 4.5 we know that $\phi_t + d_t \geq \rho_t + \rho_0$, from which we deduce that $\phi_t^{i+2,i+3} + d_t^{i+2,i+3} > 0$. By Proposition 2.6 this implies that $b_t^{i+2,i+3} > 0$. On the other hand, by Claim 4.6 we have $b_t^{i,i+2} > 0$. As $i \in \mathcal{J}$ was arbitrary, we conclude that no pair of fronts, whether adjacent or opposite can be blocked at time t . All in all, at least three fronts must contain occupied vertices, so that $\Delta\rho_{t+1} \geq 3$, as required. \square

Proof of Theorem 1 for $\mathbb{Z} \boxtimes \mathbb{Z}$. Setting $h_t = C_1 t^{\frac{1}{7}}$ for sufficiently large C_1 so that h satisfies (1), we conclude from Proposition 4.7 that $3 \notin q(\mathbb{Z} \boxtimes \mathbb{Z}, C_2 t^{6/7})$ for all sufficiently large C_2 . From monotonicity over the strength of Con-

tainer, this implies that $q(\mathbb{Z} \boxtimes \mathbb{Z}, C_2 t^{6/7}) \subset (3, \infty)$. On the other hand, by [16] we have $q(\mathbb{Z} \boxtimes \mathbb{Z}, \infty) = (3, \infty)$. Thus, by monotonicity over the spreading function, we get $q(\mathbb{Z} \boxtimes \mathbb{Z}, C_2 t^{6/7}) \supset (3, \infty)$. The theorem follows. \square

Remark 5. By setting $h \equiv \lceil \frac{1}{8\alpha} \rceil$, this proof establishes a linear analog of Theorem 1 for $\mathbb{Z} \boxtimes \mathbb{Z}$, as per Remark 1.

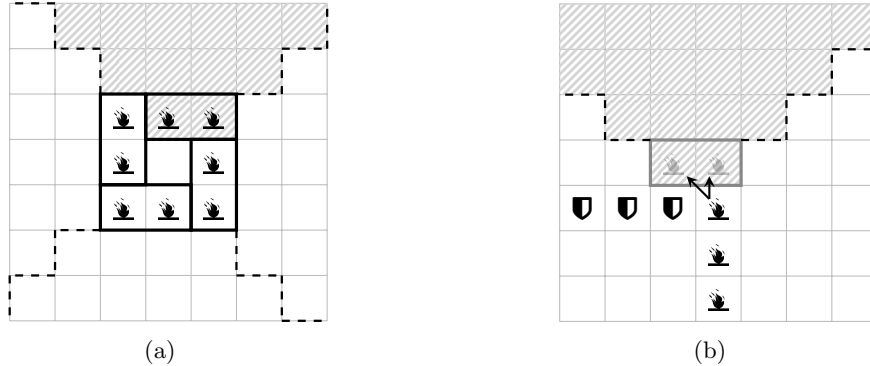


Figure 4: (4a) illustrates the initial game in the plane. The four fronts are outlined by rectangles, and the four disjoint infinite trapezoids are outlined by dashed lines. The infinite trapezoid of front 0 is highlighted by a filling pattern. In (4b) front 0 is re-ignited by front 1. The initial occupied set of the re-initialised game at front 0 is depicted in gray, and the infinite trapezoid is highlighted by a filling pattern.

5 Upper bounds

This section is dedicated to the proof of Theorem 2. We first reduce the $\mathbb{Z} \boxtimes \mathbb{Z}$ case of the theorem to that of \mathbb{Z}_{\uparrow}^2 , and then prove the latter.

5.1 A reduction to the eighth plane

Here we reduce the $\mathbb{Z} \boxtimes \mathbb{Z}$ case of Theorem 2 to the following proposition, which is a slightly stronger version of the \mathbb{Z}_{\uparrow}^2 case.

Proposition 5.1. *Container wins the game $(\mathbb{Z}_{\uparrow}^2, c\sqrt{t}, 1)$, for any $c < \frac{1}{6}$, by deleting the vertices of a single row, whose position depends only on the initial occupied set.*

Proof of Theorem 2. The winning Container strategy in $(\mathbb{Z} \boxtimes \mathbb{Z}, g, 3)$ is obtained by handling each front separately, containing the occupied set in an axis-parallel rectangle. This description is accompanied by Figure 5.

Let $r_0 > 0$ such that the initial occupied set is contained in $[-r_0, r_0] \times [-r_0, r_0]$. r_1, r_2, r_3 and r_4 are suitably chosen large constants, implicitly defined in the course of the proof.

Step 1. At times $[0, r_1 - 1]$ Container deletes the horizontal segment $[-r_1, r_1] \times \{r_1\}$, restricting the occupied set to $\{(x, y) : y < r_1\}$. Here r_1 is selected so that this could be achieved for an initial set of radius r_0 .

Step 2. At times $[r_1, 2r_1 + r_2 - 1]$ Container uses 1 deletion per turn to eliminate the eastern edge of the rectangle, deleting the vertical segment $\{r_2\} \times [-r_2, r_1]$, by applying the \mathbb{Z}_{\uparrow}^2 Container strategy of Proposition 5.1, rotated 90° clockwise, against an initial occupied set contained in a ball of radius $r_0 + r_1$. Denote the constant height at which this strategy contains the occupied set by r_2 .

Meanwhile, the remaining 2 deletions per turn are used as follows. One maintains the northern edge of the deleted rectangle westwards, making sure that the occupied set is restricted to $\{(x, y) : y < r_1\}$. The other deletion firstly maintains the same northern edge eastwards until it hits the line $x = r_2$, and then prolongs the eastern edge

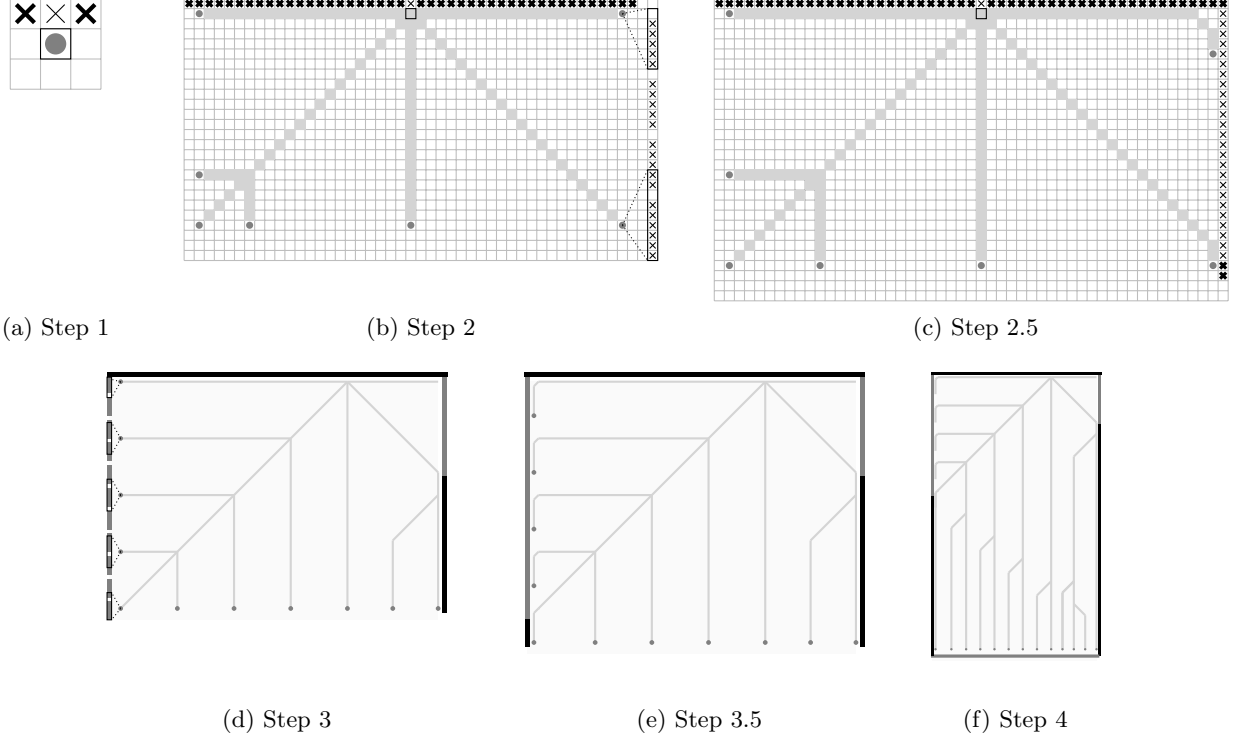


Figure 5: A depiction of the winning Container strategy against Spreader with a single initial occupied vertex in $\mathbb{Z} \boxtimes \mathbb{Z}$, as described in Section 5.1. The current occupied set is depicted by gray circles, while the history of the occupied set's movement – by lighter gray squares. In (5b) and (5c) the vertices deleted to maintain an existing front are marked by thick X's, and those deleted by the implementation of the eighth plane strategy are marked by thin X's. In (5d), (5e), and (5f), the same distinction is made via black and gray rectangles. In (5b) and (5d) dashed lines and black-outlined rectangles depict the regions on the front reachable by Spreader prior to the time-step in which Container completely blocks the front.

southwards, deleting the vertical segment $\{r_2\} \times [-2r_1 - r_2, -r_2]$. At time $2r_1 + r_2$ the occupied set is thus restricted to $\{(x, y) : y < r_1, x < r_2\}$.

Step 3. At times $[2r_1 + r_2, 3r_1 + r_2 + r_3]$, Container uses a single deletion per turn to create the western edge of the rectangle, deleting the vertical segment $\{-r_3\} \times [-r_3, r_1]$, by applying the \mathbb{Z}_{ν}^2 strategy, rotated 90° counter-clockwise, against an initial occupied set of radius bounded by $r_0 + 2r_1 + r_2$. Denote the constant height at which this strategy contains the occupied set by r_3 . Meanwhile, one additional deletion per turn maintains the eastern front southwards, while the other firstly maintains the northern front westwards until it reaches the line $x = -r_3$, and then prolongs the western front southwards, deleting the vertical segment $\{-r_3\} \times [-3r_1 - r_2 - r_3, -r_3]$. At time $3r_1 + r_2 + r_3$ the occupied set is thus restricted to $\{(x, y) : y < r_1, -r_3 < x < r_2\}$.

Step 4. Finally, Container uses two deletions per turn to maintain the eastern and western fronts southwards, restricting the occupied set to $\{(x, y) : y < r_1, -r_3 < x < r_2\}$. The last deletion is used to delete the horizontal segment $[-r_3, r_2] \times \{-r_4\}$ for some large enough r_4 . At the end of the step, the occupied set is restricted to the rectangle $\{(x, y) : -r_4 < y < r_1, -r_3 < x < r_2\}$ and Container wins. \square

The remainder of the section is dedicated to proving Proposition 5.1.

5.2 Container win in the eighth plane – proof of Proposition 5.1

This section is dedicated to the proof of Proposition 5.1. The proof is accompanied by Figure 6.

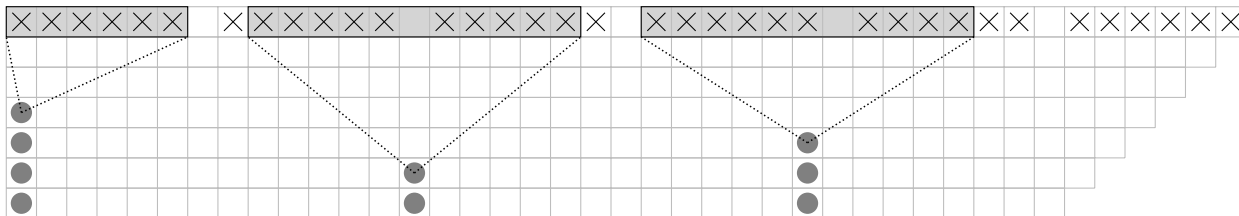


Figure 6: Illustration of the Container strategy in the eighth plane at time $H - (h + r_0)$ for $h = 2$. Circles mark the occupied vertices and \times mark deleted vertices; the set $D \subset L_H$ is shaded. The contribution of various elements of $\mathcal{B}_{H-(h+r_0)}$ to D is marked by dotted lines and black rectangles in L_H . Note that there are five vacant vertices in L_H (i.e. $|L_H \setminus \mathcal{F}_t| = 5$), while there are only two such vertices within D , these will be occupied by Container before time $H - r_0$. The occupation of L_H will follow in the remaining r_0 turns.

Throughout this section, let $c < \frac{1}{6}$ as per Proposition 5.1, and let $r_0 > 0$ be such that the initial occupied set is contained in $[0, r_0] \times [0, r_0]$. Hence, throughout the game, for all $t > 0$

$$\mathcal{B}_t \subset \mathbb{Z} \times [0, t + r_0]. \quad (29)$$

Let $h, H \in \mathbb{N}$ be a pair of constants to be specified later. We shall delete the horizontal segment $L_H := [0, H] \times \{H\}$, Restricting the occupied set to the lower triangle bounded by it. Our strategy follow three steps.

Phase 1 - Leaving out a regular sieve. Under the restriction that H is divisible by $h + r_0$ denote by

$$X := \left\{ \left(i \cdot \frac{H}{h+r_0}, H \right) : 0 \leq i \leq h+r_0-1 \right\},$$

a set of vertices on L_H regularly spaced at $\frac{H}{h+r_0}$ intervals. During the first phase of our strategy, spanning across times $[0, H - (h + r_0) - 1]$, Container deletes the vertices of $L_H \setminus X$ in an arbitrary order.

Phase 2 - Deleting the most imminent bottleneck through which spreader may cross L_H .

During the second phase of our strategy, spanning across times $[H - (h + r_0), H - 2r_0 - 1]$, Container deletes, at an arbitrary order, those remaining vertices of the sieve, which are of distance less than $h + r_0$ from an occupied vertex, along with arbitrary additional vertices of X which were not yet deleted, as much as the deletion power allows. We denote these nearby vertices by $D := \{u \in L_H : d(u, \mathcal{B}_{H-(h+r_0)}) \leq h + r_0\}$. To see that D could be deleted on time, we require the following lemma, whose proof is postponed to section 5.2.1 below.

Lemma 5.2. *D is contained in $2c\sqrt{H}$ horizontal segments of total length at most $3(h + r_0) \cdot c\sqrt{H}$.*

Since every two vertices in X are at distance at least $\frac{H}{h+r_0}$ from each other, for every $d \in \mathbb{N}$, a horizontal segment of length d can intersect at most $\left\lceil \frac{d}{H/(h+r_0)} \right\rceil \leq d \cdot \frac{h+r_0}{H} + 1$ vertices in X . Hence,

$$|D \cap X| \leq 2c\sqrt{H} + (3(h + r_0) + 2)c\sqrt{H} \cdot \frac{h+r_0}{H} = c\sqrt{H} \left(2 + 2\frac{h+r_0}{H} + \frac{3(h+r_0)^2}{H} \right).$$

To verify that this is indeed less or equal to $h - r_0$, observe that, writing $\bar{h} := \frac{h+r_0}{\sqrt{H}}$, which will be chosen to satisfy $\bar{h} \leq \sqrt{H}$, it would suffice to show that

$$3c\sqrt{H}(1 + \bar{h}^2) \leq \bar{h}\sqrt{H} - 2r_0.$$

Isolating c , it would suffice to show that we may chose H and \bar{h} that satisfy that

$$c \leq \frac{\bar{h}}{3(1 + \bar{h}^2)} - \frac{r_0}{2\sqrt{H}(1 + \bar{h}^2)}. \quad (30)$$

Indeed, using our assumption $c < \max_{d \geq 0} \left\{ \frac{\bar{h}}{3(1+h^2)} \right\} = \frac{1}{6}$ and the fact that we may pick arbitrarily large H , a choice of such \bar{h}, H satisfying all previous requirements is indeed possible.

Phase 3 - Deleting the remaining vertices of L_H . Finally, during the third phase of our strategy, spanning across times $[H - 2r_0, H]$, we delete the remaining vertices of L_H in an arbitrary order, taking advantage of the fact that, as these are not in D , they are too far to be occupied by Spreader in this time-frame. By time H , Container will have deleted all $H + 1$ vertices of L_H .

5.2.1 Bounding D – proof of Lemma 5.2

This section consists of the proof of Lemma 5.2. Denote by $z \rightsquigarrow x$ the transitive closure of the relation

$$(\exists s \leq H - (h + r_0) : z \in \mathcal{B}_{s-1}, x \in \Delta \mathcal{B}_s, d(z, x) = 1).$$

Write $\tilde{B} := \mathcal{B}_{H-h-r_0} \setminus \mathcal{B}_{H-2h-3r_0-1}$ and let $x \in \tilde{B}$. Denote $a(x) := \min\{z \in \mathcal{B}_{H-2h-3r_0-1} : z \rightsquigarrow x\}$, where minimum is taken with respect to lexicographical order, and write

$$A := \{a \in \mathcal{B}_{H-2h-3r_0-1} : \exists x \in \tilde{B} \text{ s.t. } a = a(x)\} \quad \text{and} \quad \Sigma(a) := \{x \in \tilde{B} : a(x) = a\}.$$

Given $B' \subset \mathcal{B}_t$ for some $t \in \mathbb{N}$, define $D(B') := \{u \in L_H : d(u, B') \leq h + r_0\}$. Observe that $D(\mathcal{B}_{H-2h-3r_0-1}) = \emptyset$ and $D = D(\mathcal{B}_{H-(h+r_0)}) = D(\tilde{B})$, since $d(\mathcal{B}_{H-2h-3r_0-1}, L_H) > h + r_0$. Because \tilde{B} is the disjoint union of $\Sigma(a)$ for $a \in A$, in order to establish the lemma, it would suffice to show that

$$|D(\Sigma(a))| \leq 2|\Sigma(a)|, \tag{31}$$

for every $a \in A$. Indeed, using (31) and recalling that $|\Delta \mathcal{B}_t| = c\sqrt{t}$, we can deduce that

$$|D| \leq 2|\tilde{B}| = 2 \sum_{k=H-2h-3r_0}^{H-h-r_0} |\Delta \mathcal{B}_k| \leq 2(h + 2r_0 + 1)|\Delta \mathcal{B}_{H-(h+r_0)}| \leq 3(h + r_0)c\sqrt{H}.$$

Moreover, by observing that for every $u \in \tilde{B}$ the set $D(\{u\})$ is either empty, or a horizontal segment of length $2(h + r_0) + 1$, the lemma would follow.

To prove (31), let $a \in A$ and denote by $v_0 = (x_0, y_0)$ and $v_1 = (x_1, y_1)$, the leftmost and rightmost elements of $\Sigma(a)$ whose distance from L_H is at most $h + r_0$, respectively. Let P_0 and P_1 be the associated paths from a to v_0 and to v_1 respectively. Observing that

$$|D(\Sigma(a))| \leq |[x_0 - (h + r_0), x_1 + (h + r_0)] \times \{H\}| \leq x_1 - x_0 + 2(h + r_0) + 1,$$

and that $P_0, P_1 \subset \Sigma(a)$, it would suffice to show that

$$2|P_0 \cup P_1| \geq x_1 - x_0 + 2(h + r_0) + 1. \tag{32}$$

To see this, assume, without loss of generality, that $|P_0| \leq |P_1|$. Observe that since $d(v_0, L_H) \leq h + r_0$ and $d(a, L_H) \geq d(\mathcal{B}_{H-2h-3r_0-1}, L_H) \geq 2(h + r_0) + 1$ (by (29)), we have $|P_0| > h + r_0$. Next, observe that

$$x_1 - x_0 \leq 2d(v_1, P_0 \cap P_1) \leq 2|P_1 \setminus P_0|,$$

where the first inequality uses $|P_0| \leq |P_1|$ and the triangle inequality, and the second inequality uses discrete continuity of P_1 . Putting all of these together, (32) follows. \square

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6 Table of Notations

Notation	Description
Section 1	
$\mathbb{Z} \boxtimes \mathbb{Z}$	Plane. \mathbb{Z}^2 with strong connectivity.
\mathbb{Z}_{\nearrow}^2	Eighth plane. Upper half of the positive quadrant of $\mathbb{Z} \boxtimes \mathbb{Z}$.
$g(t)$	Spreading function. Function controlling the size of the occupied set.
$q(G, g)$	$q(G, g) := \{g : (G, q, g) \text{ is Container win for every finite } \mathcal{B}_0\}$.
\mathcal{F}_t	Set of deleted vertices up to time t .
G_t	Induced graph after the deletion of the vertices of \mathcal{F}_t .
\mathcal{B}_t	Occupied set at time t .
Section 2	
\mathcal{S}_t	Segments. $\mathcal{S}_t := \{\{0, \dots, h_t - 1\} + h_t k : k \in \mathbb{Z}\}$.
\mathbb{N}_i	$\mathbb{N}_i := \{t \in \mathbb{N} : h_t/h_{t-1} = i\}$.
h_t and H_t	h_t is the size of the segments at time t . $H_t := 4q^2 h_t^2$.
$A_s(S)$	$A_s(S) := \{S' \in \mathcal{S}_s : S' \subset S\}$.
$B_t(S)$ and $b_t(S)$	$B_t(S)$ is the set of occupied vertices in $S \times \{t\}$. $b_t(S) := B_t(S) $. See (2).
(t, ℓ) -path	An upwards path in G_t of length ℓ , starting in \mathcal{B}_t .
B_t^ℓ	Set of vertices from which a (t, ℓ) -path starts.
$\widehat{b}_t^\ell(R)$	Size of largest collection of disjoint (t, ℓ) -paths starting in R .
Ω_t	Disrupted Segments. $\Omega_t := \left\{ S \in \mathcal{S}_t : (\mathcal{F}_t \setminus \mathcal{F}_{t-1}) \cap (S \times [t, t + h_t - 1]) \neq \emptyset \right\}$.
$I_s^\ell(S)$ and $I_t(S)$	$I_s^\ell(S)$ is the minimal interval around S with many disjoint (s, ℓ) -paths. $I_t(S) := I_t^{\widetilde{H}_t}(S)$. See (3).
$\chi_t(S)$	Indicator for the spreading status of S at time t . $\chi_t(S) := \mathbb{1}\{\tau_t(S) > 0\}$. See (4).
\widetilde{H}_t and \widetilde{h}_t	Functions similar to H_t and h_t , taking doubling times into consideration. See (6)
$\tau_t(S)$	Consolidation timer. Number of turns until S becomes simulative. See (5).
$p_t(S)$	Pivot. The single occupied vertex in a simulative segment S . See (7) and (8).
$\beta_t^\ell(y)$	Leftmost starting vertex of a (t, ℓ) -path ending in $y \in \mathcal{B}_{t+\ell}$.
$\ell_t(S)$	Look ahead. Vertical distance between the front and the simulated fire in S . See (10).
$L_t(S)$	Look ahead region. $L_t(S) := [t, t + \ell_t(S)]$. See (10).
$F_t(S)$ and $f_t(S)$	$F_t(S)$ is the set of deleted vertices associated with S up to time t . $f_t(S) := F_t(S) $. See (11).
$r_t(S)$	Range. Number of endpoints of $(t, \ell_t(S))$ -paths contained in $S \times L_t(S)$.
$\phi_t(S)$	The size of the simulated fire in S at time t . See (12).
$\Phi_t'(S)$	Pre-potential. $\Phi_t'(S) := \phi_t(S) + f_t(S)$.
$\lambda_t(S)$	Nearest segment left of S with a positive change in Φ_t' . See (14)
$d_t(S)$	Debt. The debt S holds towards the desired evolution of ϕ at time t . See (15).
$\Phi_t(S)$	Potential. $\Phi_t(S) := \phi_t(S) + f_t(S) + d_t(S)$.
Section 3	
\mathbb{Z}_{\nearrow}^2	Directed half plane. Upper half plane with edges directed upwards and diagonally.
\mathcal{A}_t^S	Game area. $\bigcup_{t'=0}^t \mathcal{B}_{t'} \cup F_{t'}$.
$\mathcal{T}(D)$	Infinite Trapezoid. Convex hull in \mathbb{Z}^2 of $D + \{k(1, 1), k(-1, 1) : k \in \mathbb{N}\}$.
Section 4	
θ^i and $\theta^{i,i+1}$	Cardinal and secondary directions in \mathbb{Z}^2 .
L_t^i and $L^i(d)$	Front line. $L_t^i := L^i(\rho_t^i)$, where $L^i(d) := d\theta^i + \mathbb{Z}\theta^{i+1}$. See (26).
ρ_t^i	Occupation radius. $\rho_t^i := \min \{r : L^i(r) \cap \bigcup_{t'=0}^{t-1} \mathcal{B}_{t'} = \emptyset\}$ See (26).
\mathcal{G}_t^i	Half-plane game played in direction i at time t employed by the plane strategy.
Section 5	
L_H	$L_H := [0, H] \times \{H\}$.
D	$D := \{u \in L_H : d(u, \mathcal{B}_{H-(h+r_0)}) \leq h + r_0\}$.