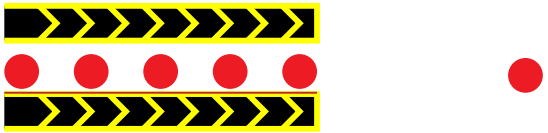
The Power of Two-Choices in Regulating Interval Partitions

Ohad N. Feldheim (Stanford)
Joint work with Ori Gurel-Gurevich (HUJI)

September 2016

Balls and bins model

Consider an online process in which N balls are randomly assigned, one by one to M bins.



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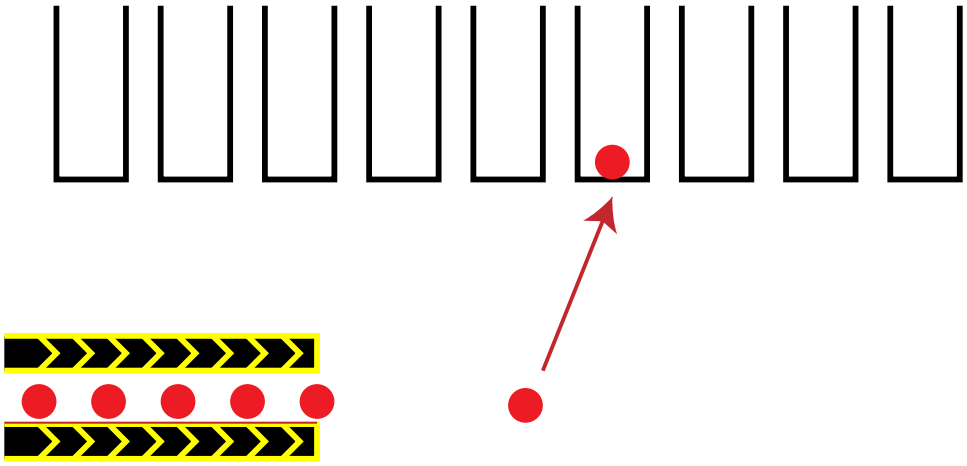

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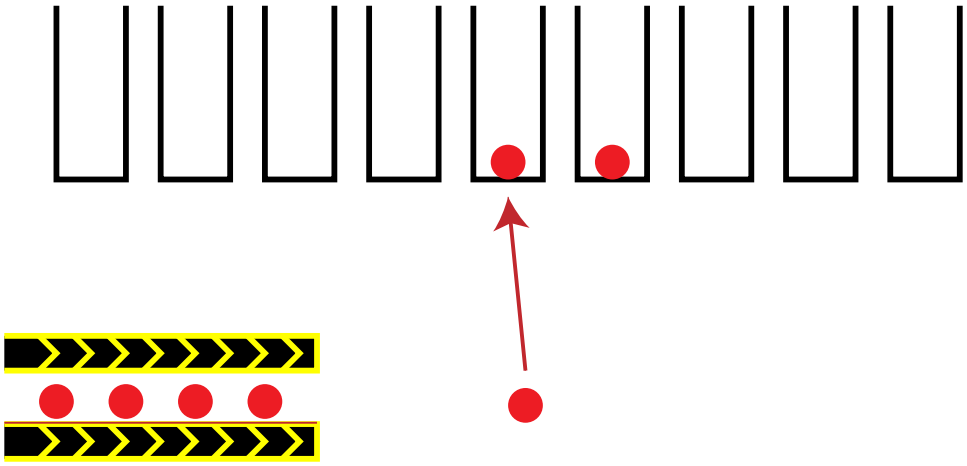

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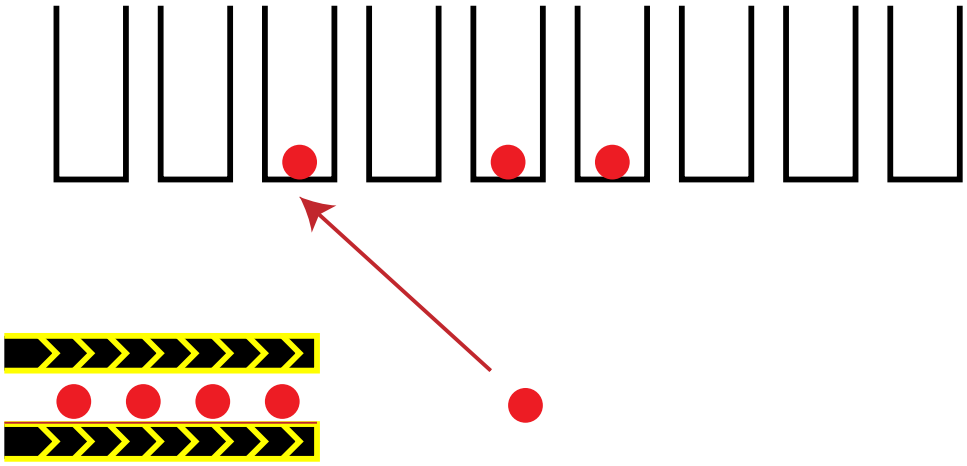

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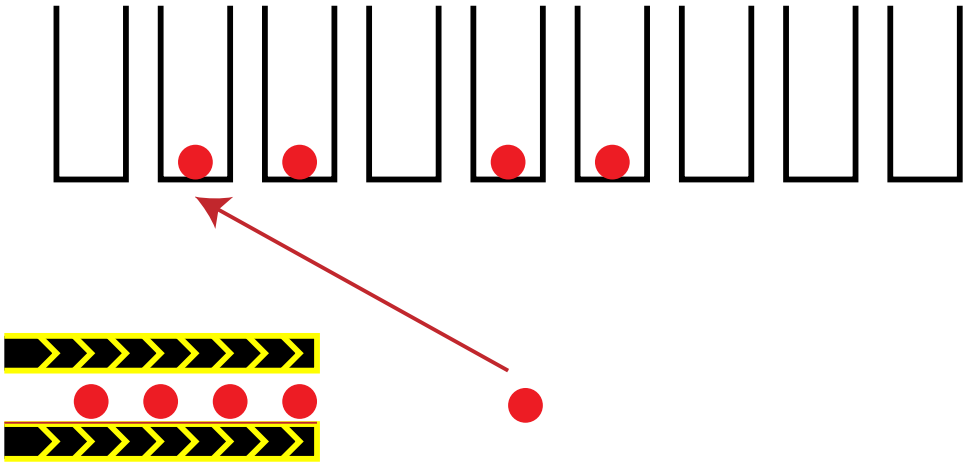

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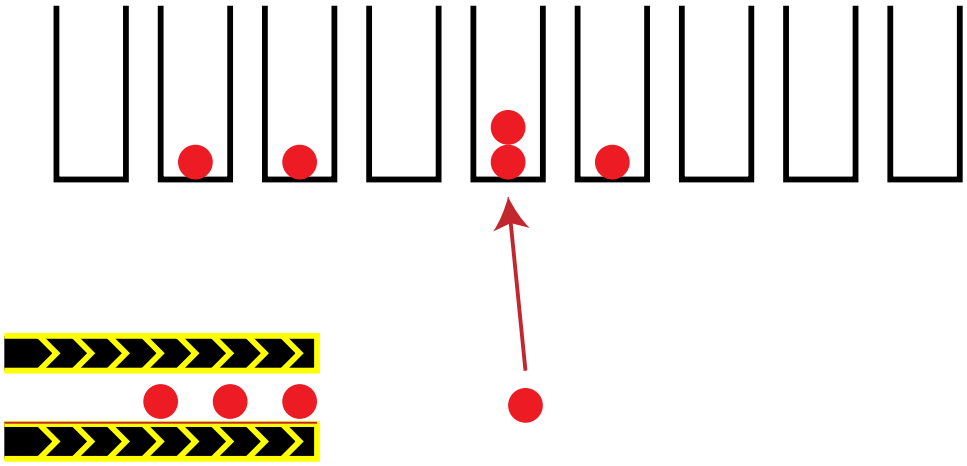

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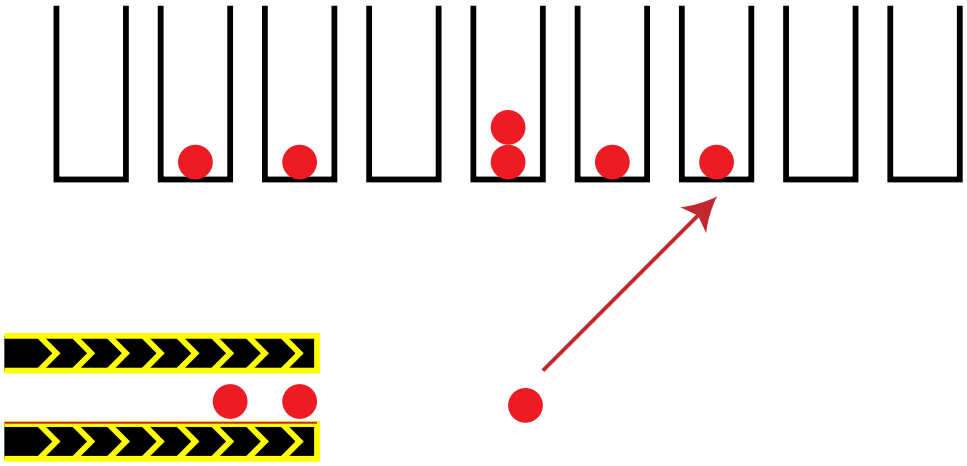

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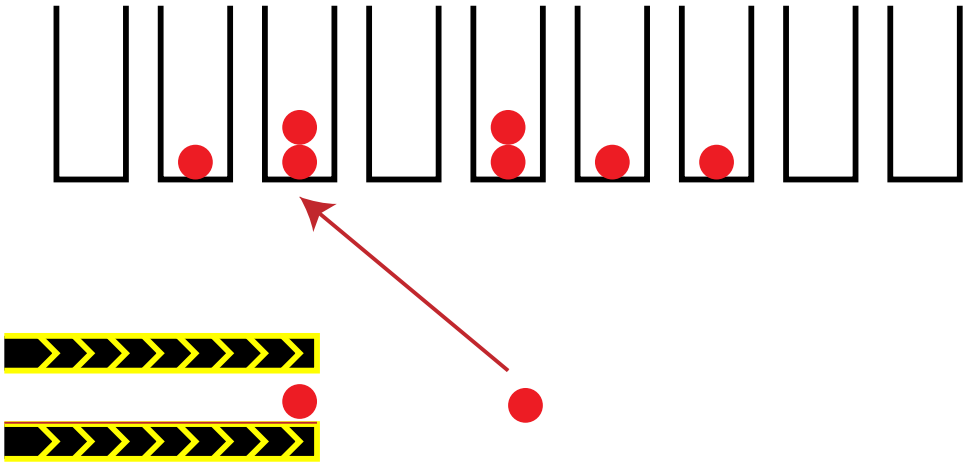

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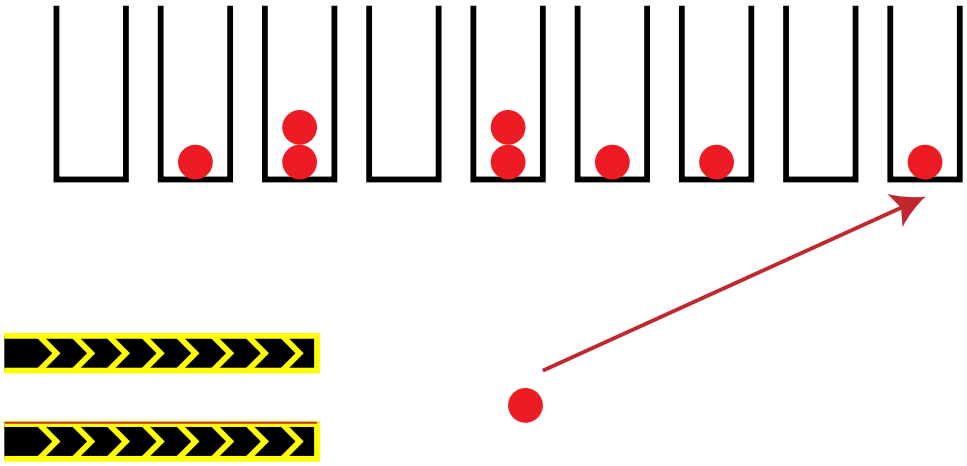

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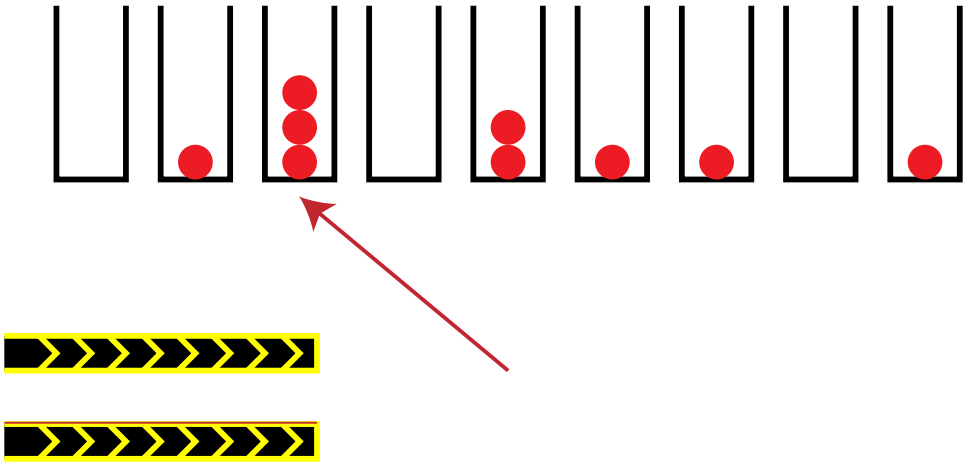

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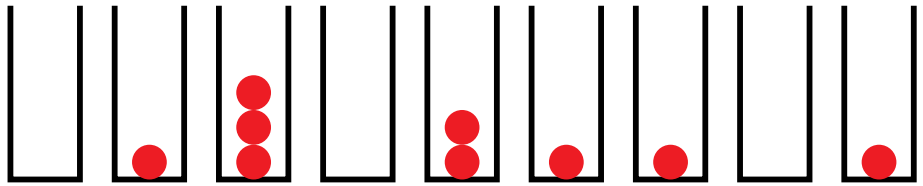

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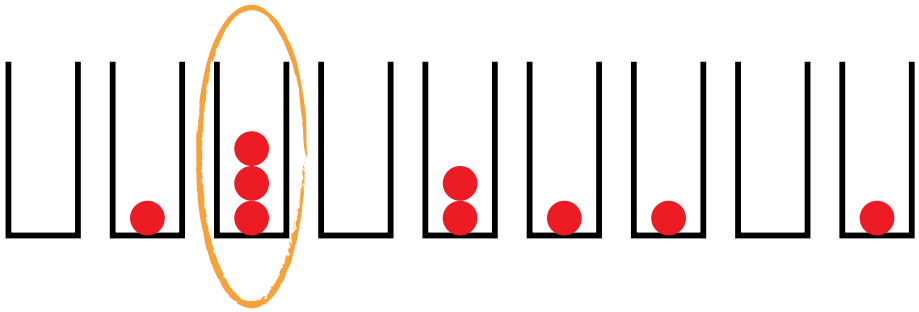

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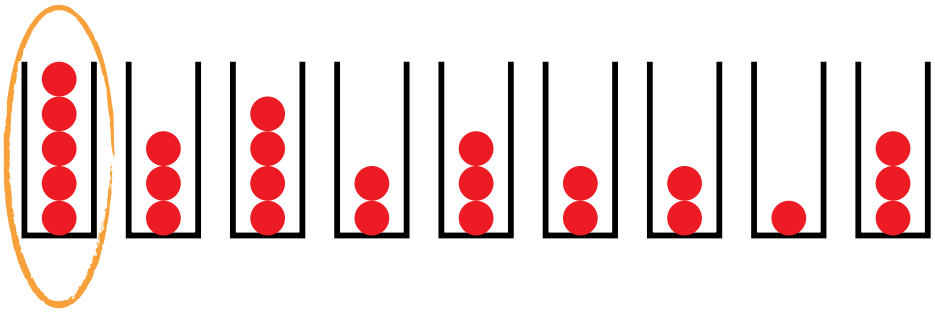

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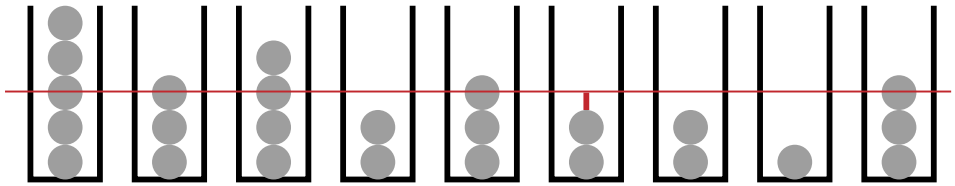

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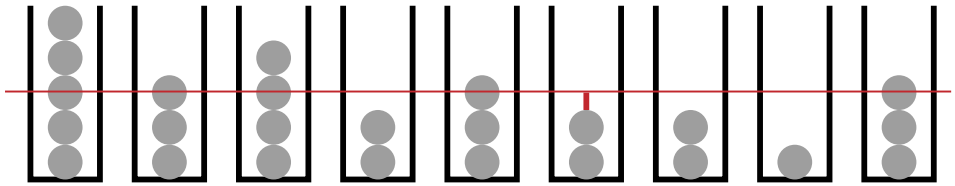

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- *Load balancing* is an effort to reduce these quantities. (possible with control over the distribution of the balls.)

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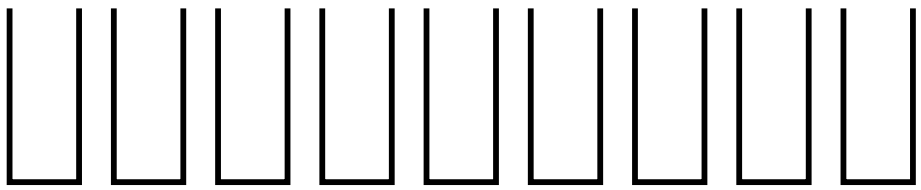
Azar, Broder, Karlin, and Upfal (1994): Very little choice is sufficient for rather balanced allocation - choice between two random cells per ball.



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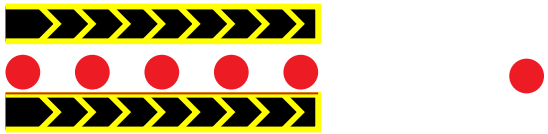

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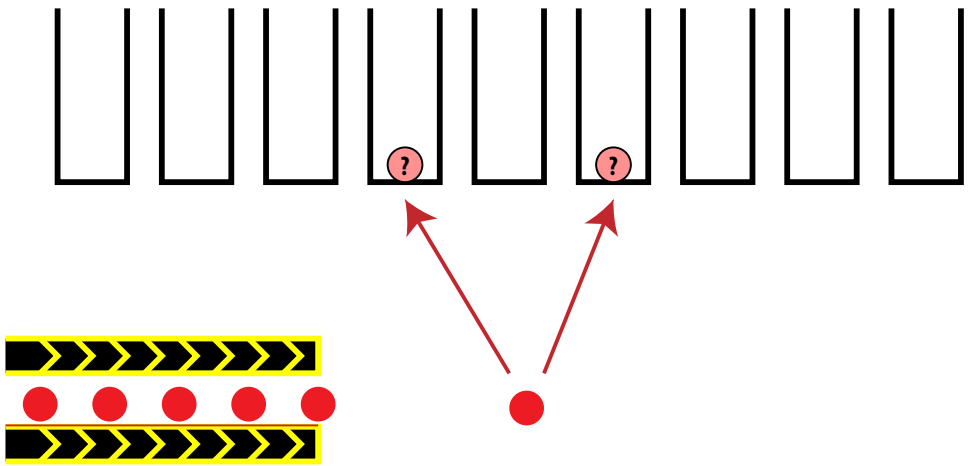

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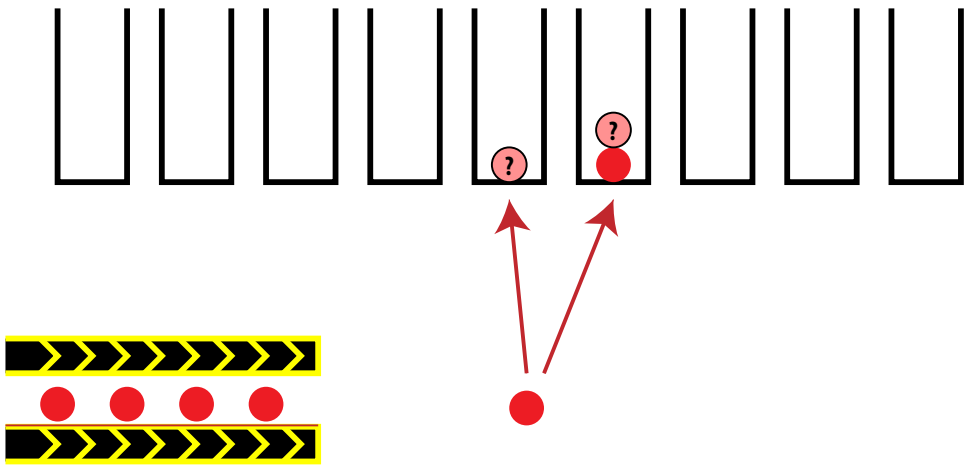

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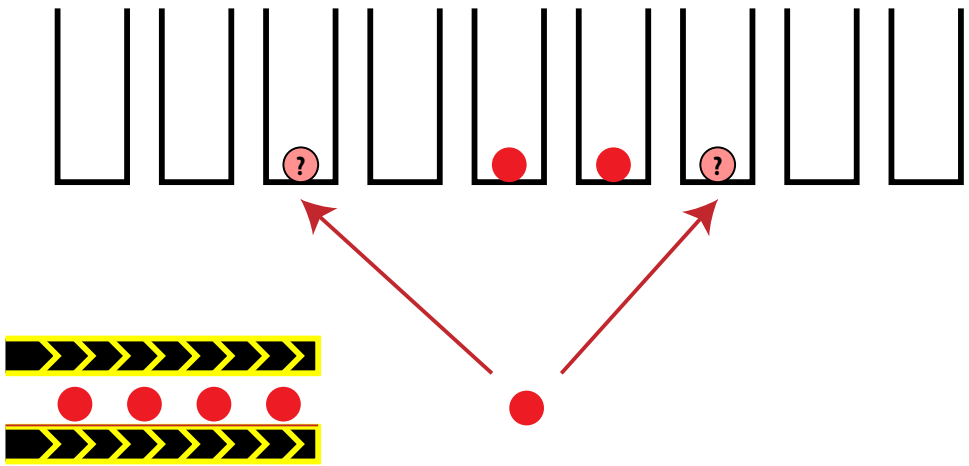

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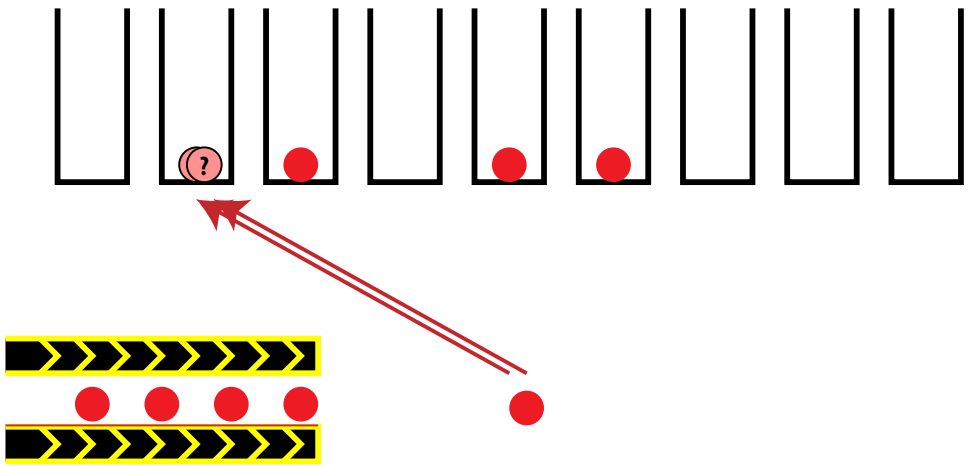

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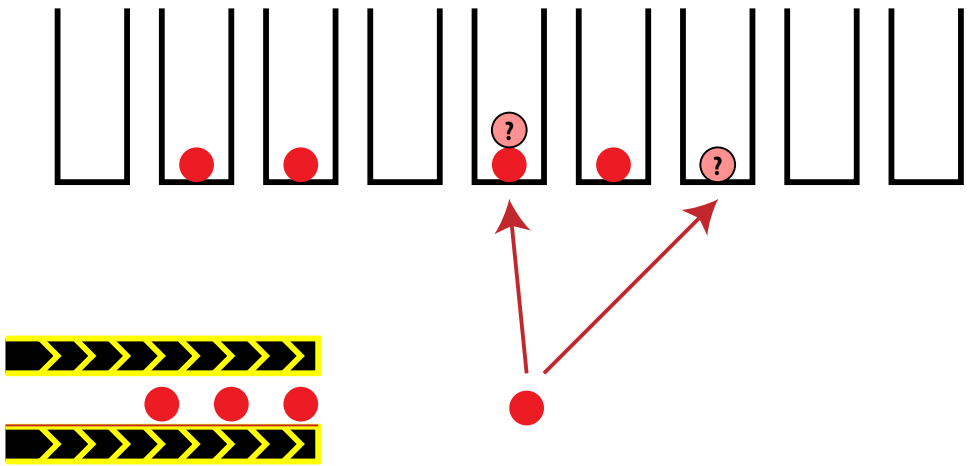

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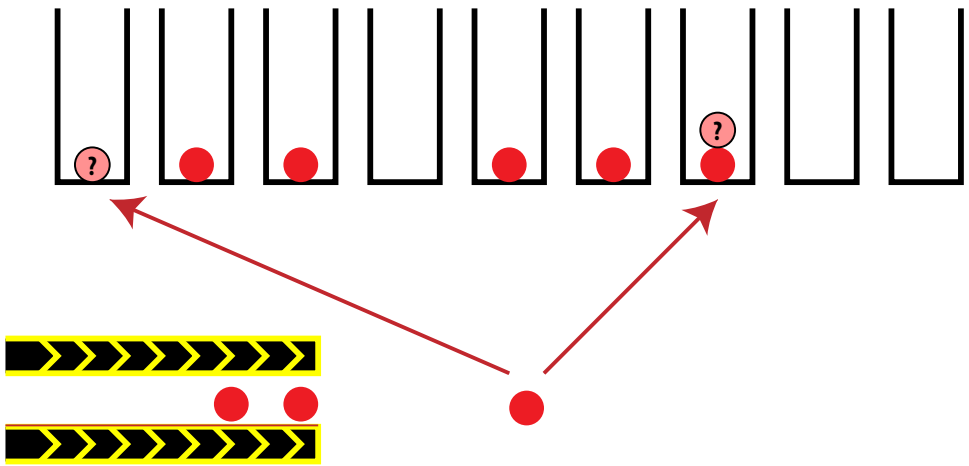

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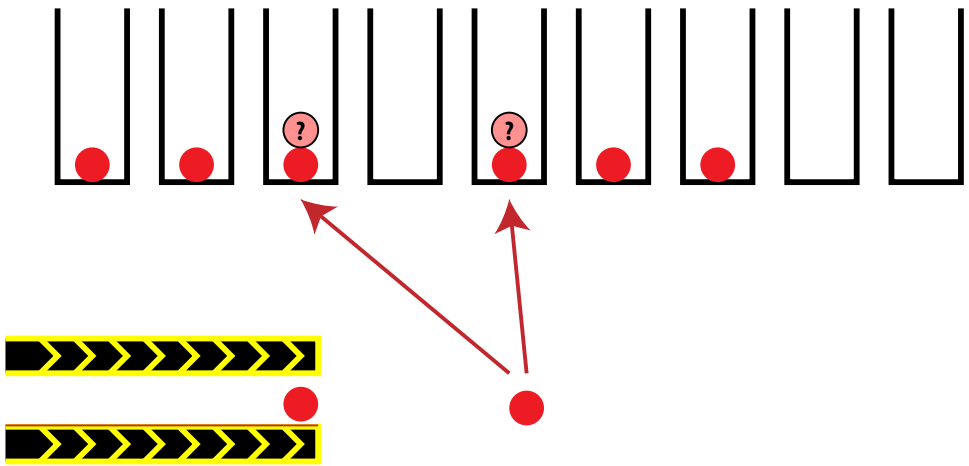

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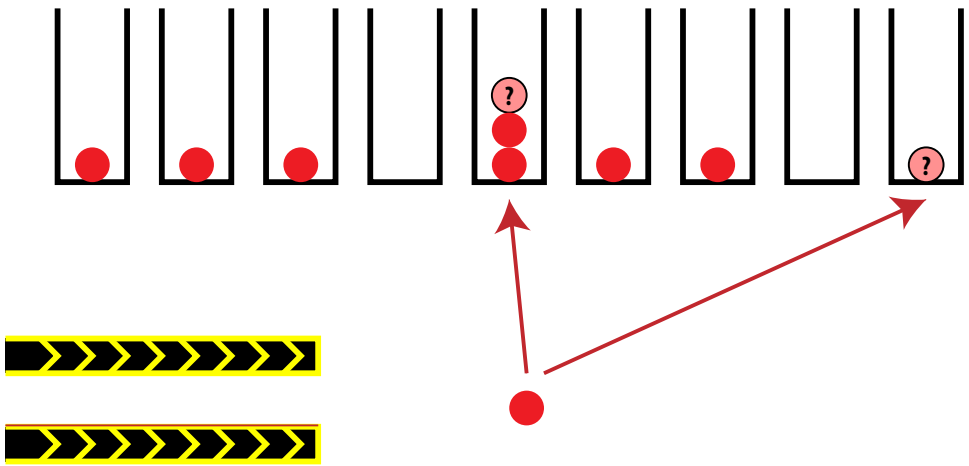

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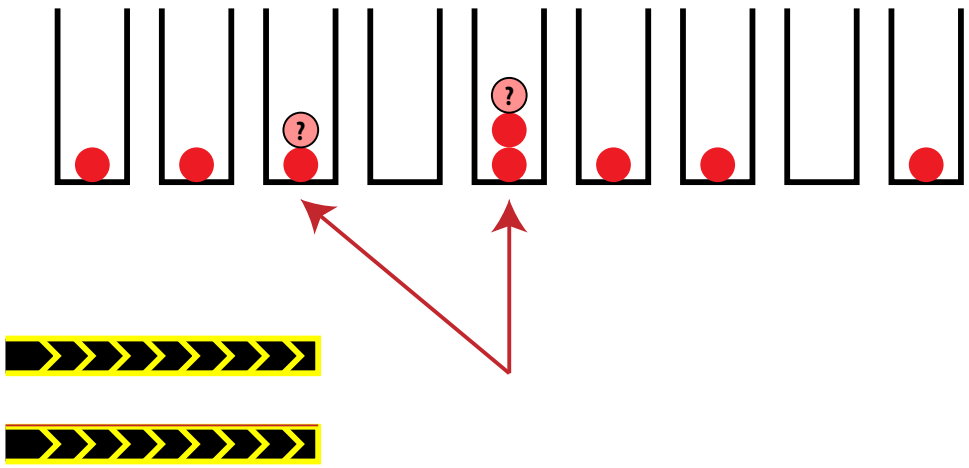

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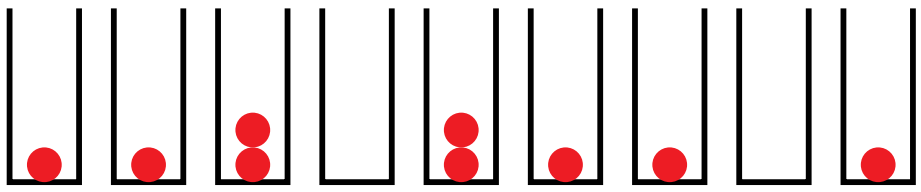

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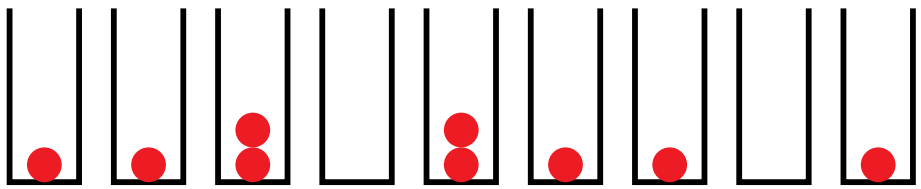

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Balls	no-choice max. dev.	2-choices max. dev.	no-choice typ. dev.	2-choices typ. dev.
M	$\frac{\log M}{\log\log M}$	$\frac{\log\log M}{2}$	O(1)	O(1)
$N \gg M$	$\Theta\left(\sqrt{\frac{N\log M}{M}}\right)$	$\Theta(\log M)$	$\Theta\left(\sqrt{\frac{N}{M}}\right)$	O(1)

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This observation had many applications

- Server load-balancing
- Distributed shared memory
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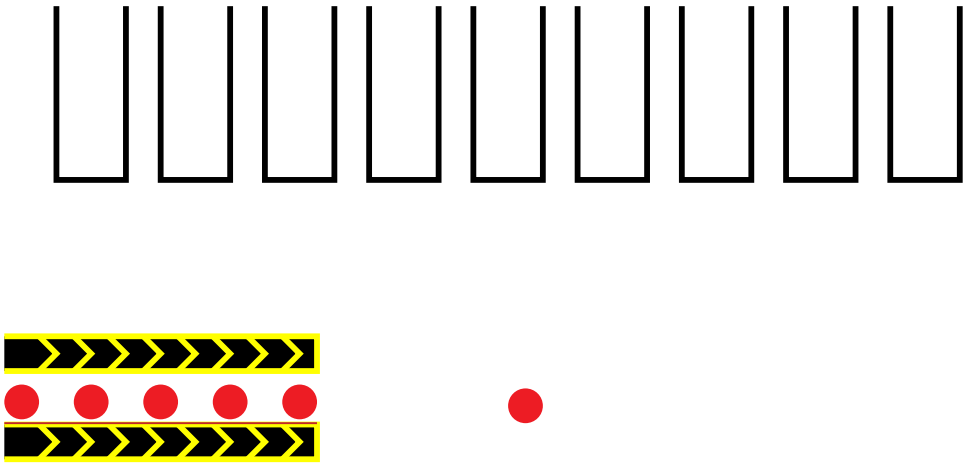
- More choice does not significantly reduce the maximum load.
- If balls keep appearing and dying at rate 1 the phenomenon persists (Luczak & McDiarmid '05)
- However if one can't keep track of the number of balls per bin (due to having $M^{1-\epsilon}$ bits of memory), then no asymptotic improvement over no-choice is possible (Alon, Gurel-Gurevich, Lubetzky '09)

Two-choices and One-retry

One-retry: A related intermediate setup. The chooser is only offered a chance to re-roll the target bin once per ball.



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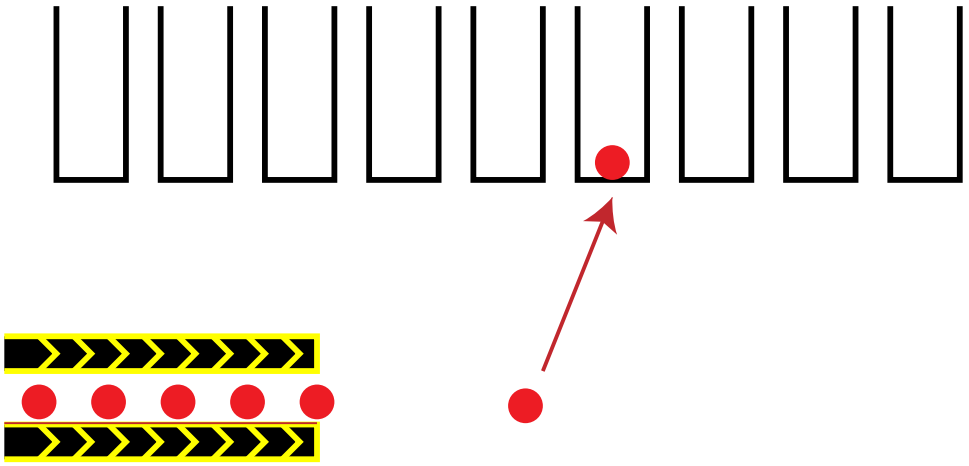

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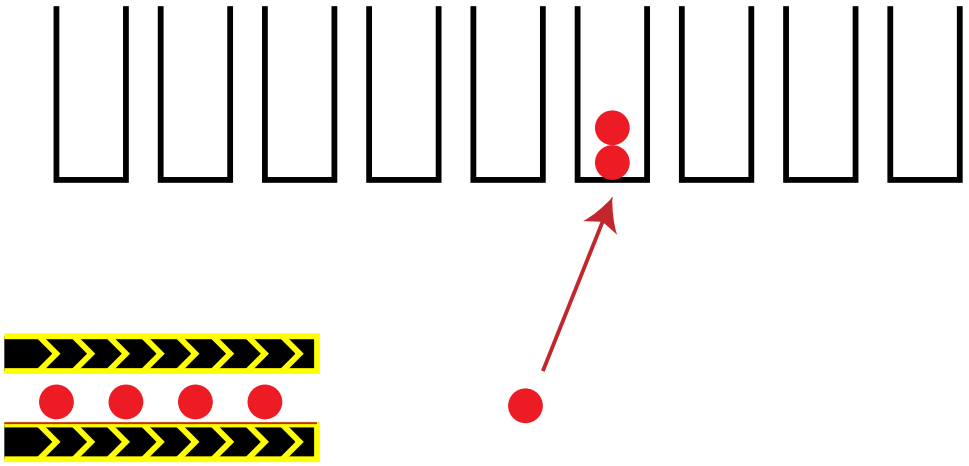

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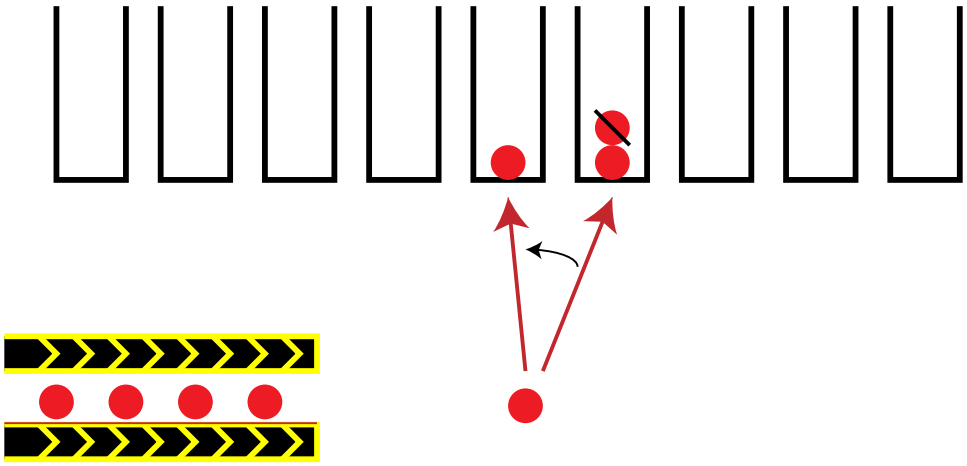

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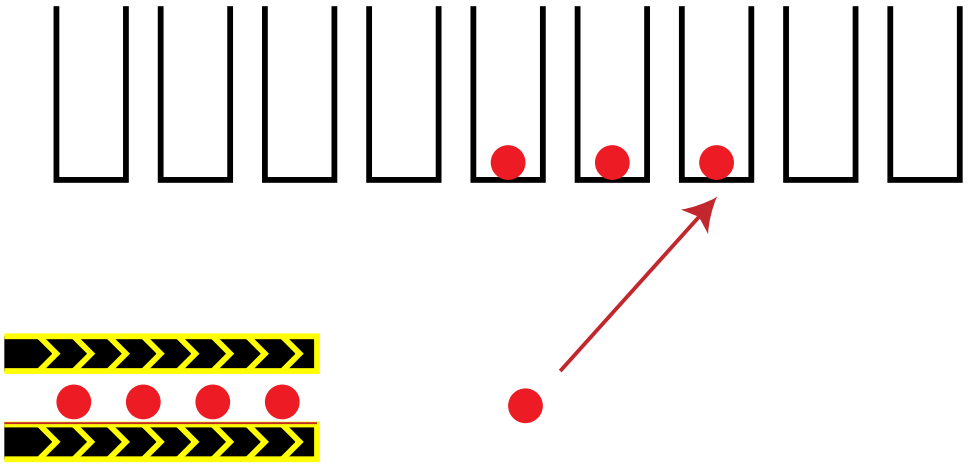

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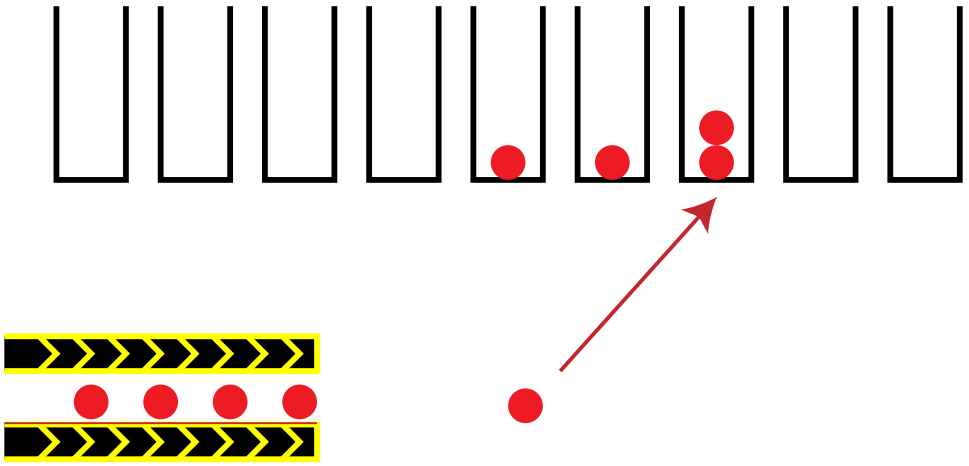

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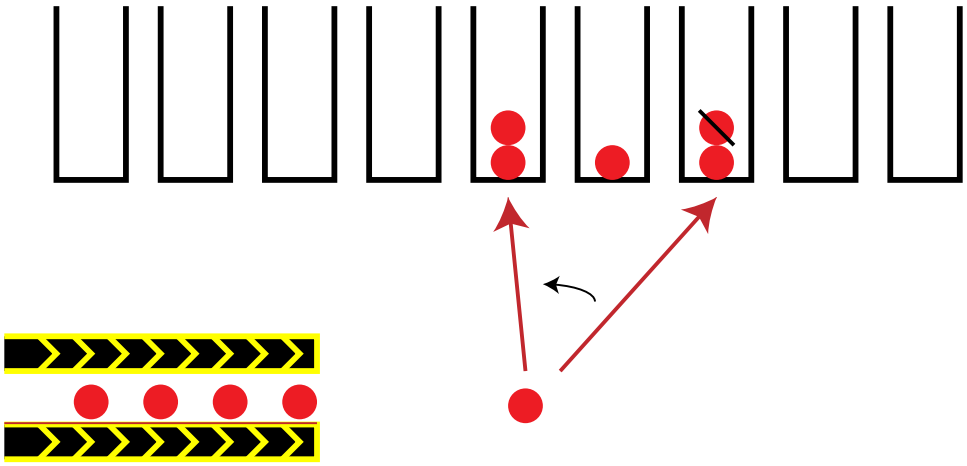

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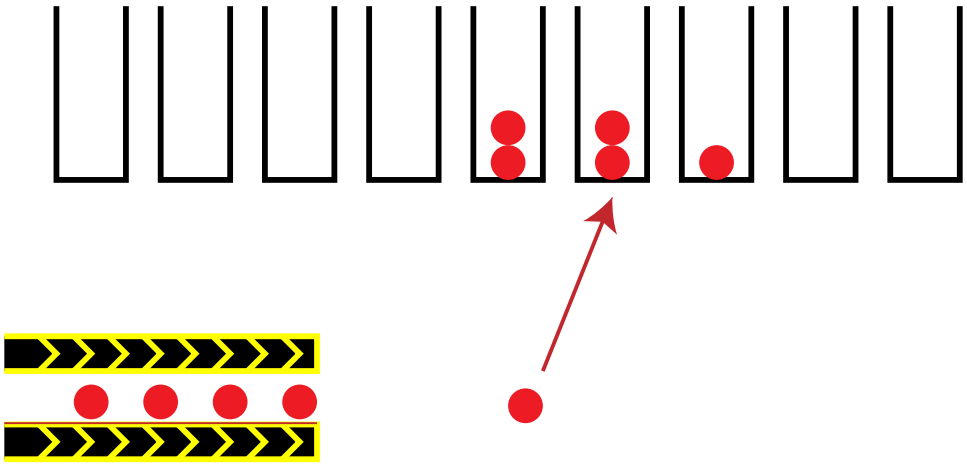

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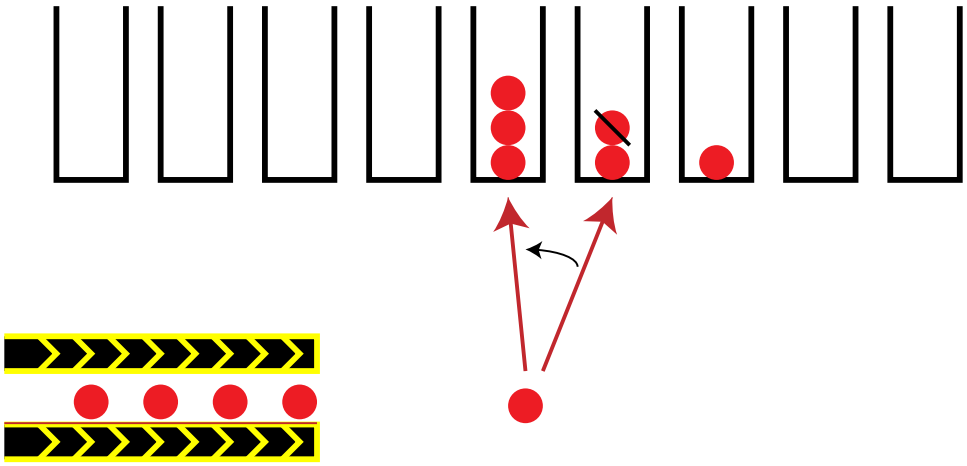

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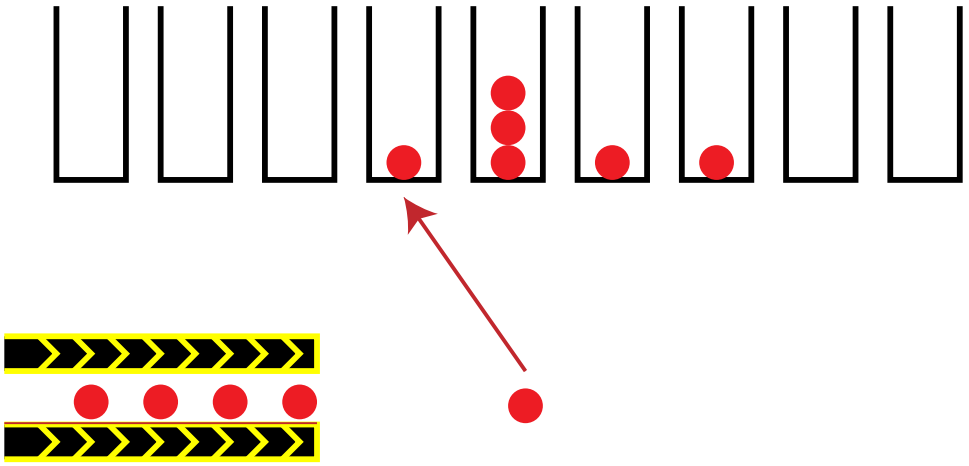

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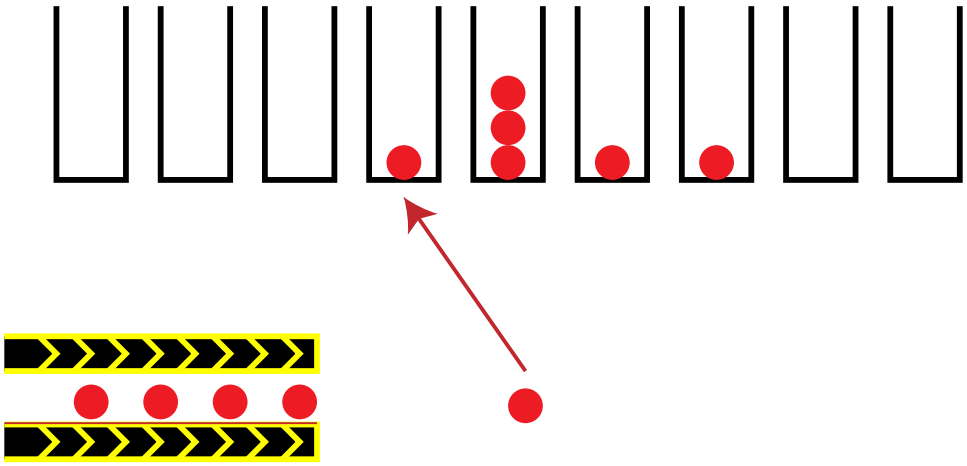

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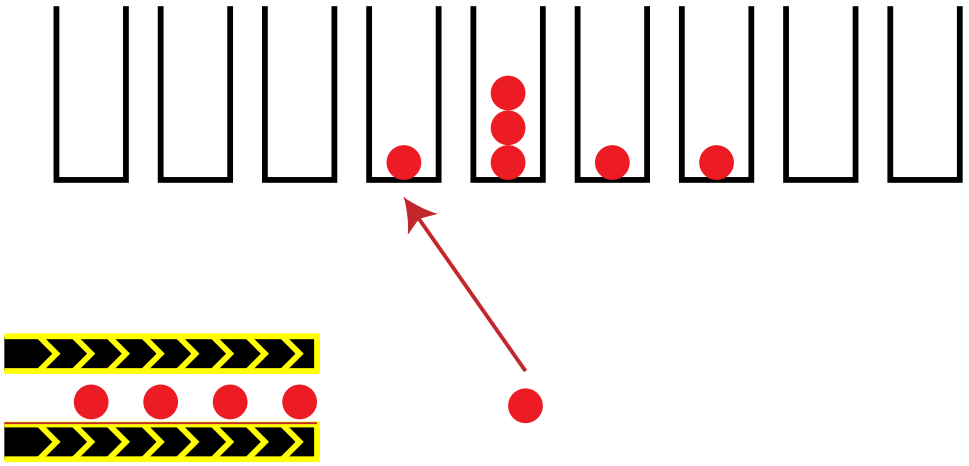

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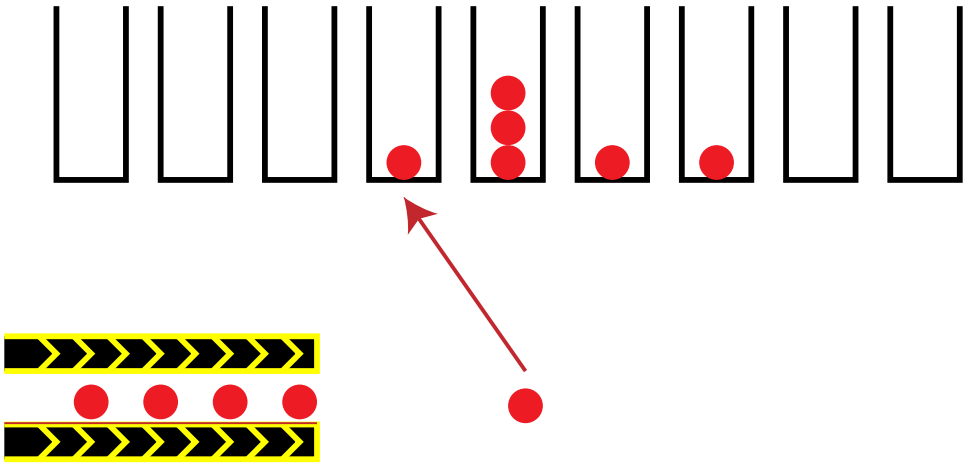

Equivalent to being oblivious to one of the two bins in the two choices setup. Asymptotic discrepancy like two-choices when $N \gg M$. When $N = M$, the discrepancy is $\sqrt{\log M / \log \log M}$.

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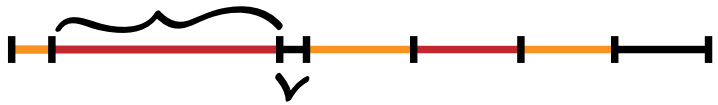

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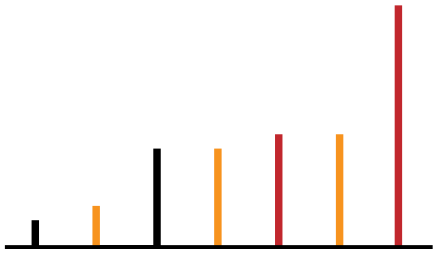

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Non-geometric:

- Empirical normalized interval distribution.

Two-choices and interval partitions

Benjamini: Can two choices regulate interval partitions?

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However, even Kakutani process offers merely a factor 2 improvement over the uniform process in terms of interval variation (Pyke-Zwet 2004).

Convergence of 2-Max interval partition process

Studying 2-Max is a rather difficult task:

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- Cf. lower bound of $\frac{C\log N}{N}$, no choice estimate $\sqrt{\frac{C\log N}{N}}$.
- There exists a single universal strategy which obtains this for all N.

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Discrete counterpart

For N balls on $\mathcal{U}([M])$, a probabilistic retry strategy obtains

$$\mathbb{P}\left(\max_{a<b\in[M]} |\mu^n([a,b]) - \mu([a,b])| > \Delta \log^3 M\right) \leq Ce^{-c\Delta}.$$

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Our result implies that by rejecting at most one of every two candidates this could be done.

Stochastic point of view on the power of one-retry

Stochastic Approach to Two-choices: Regulating Balls and Bins

Large family of one-retry distributions

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$$\frac{1 - f(x)}{M} + \frac{1}{2M} = g(x)$$

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i.e. for $\theta < 1$, such a process has typical fluctuation $O(\frac{-\log\theta}{\theta})$.

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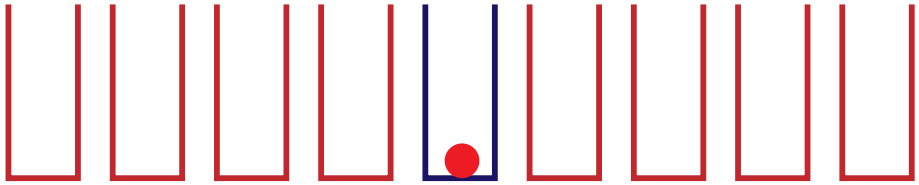

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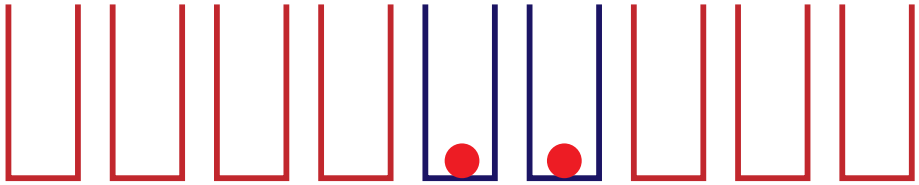

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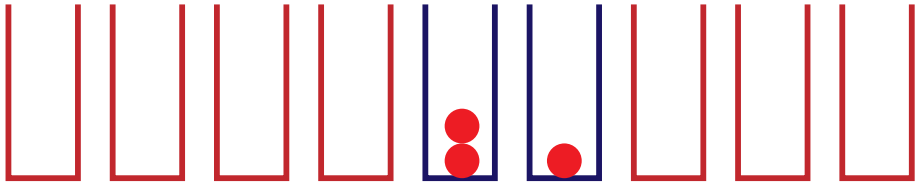

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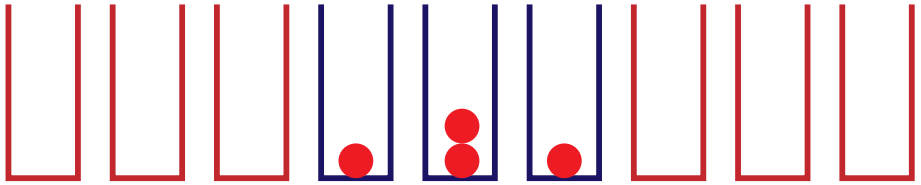

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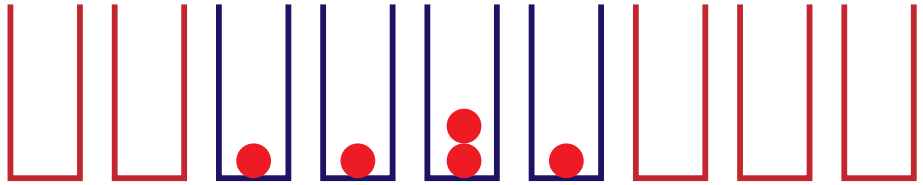

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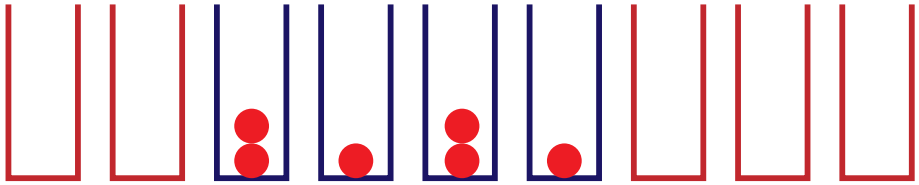

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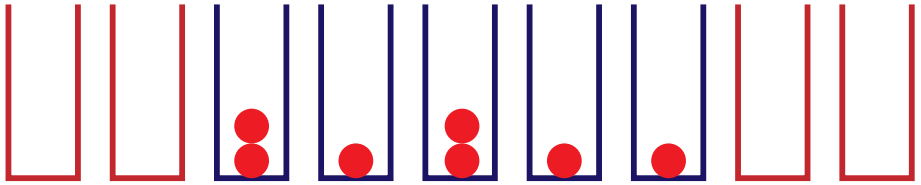

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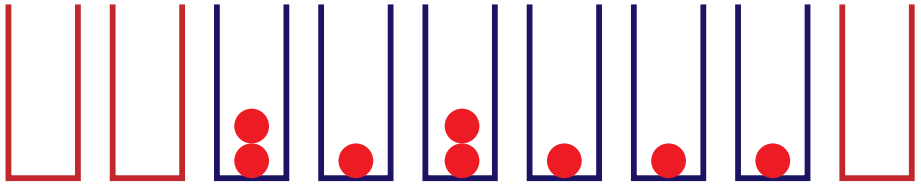

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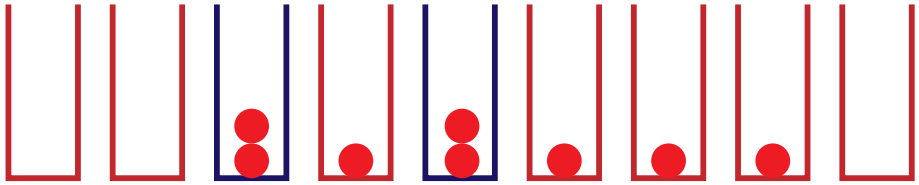

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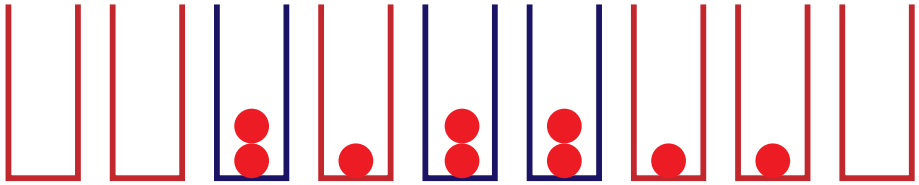

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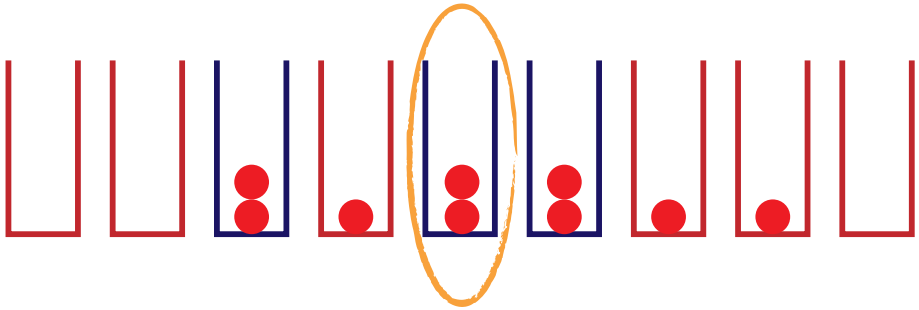

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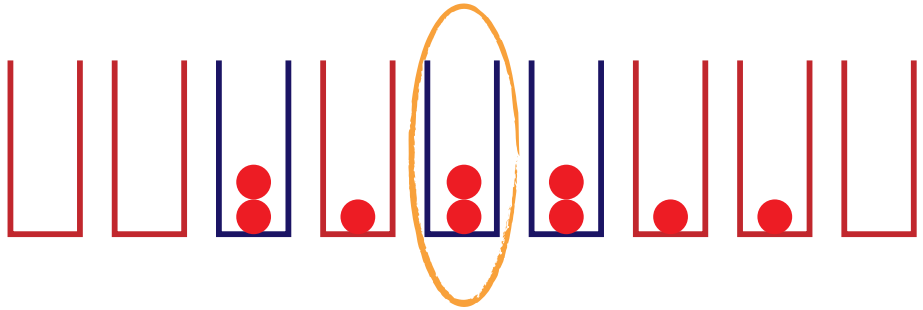

Proposition (repeat)

For such a process $\mathbb{P}\left(|X_t - t| > \frac{\Delta}{\theta}\right) < C\theta^{-2}e^{-\Delta/3}$

Among M such processes, for fixed θ, the extremal fluctuation is $O(\log M)$.

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We expect N balls, Maximal load of $N/M + \Theta(\log M)$.
All that is left is to show that the same holds after N balls were distributed. - i.e. show concentration of the stopping time.

Stochastic Approach to Two-choices: Regulating Interval Partition

Proving the main Theorem

We will illustrate the proof for discrepancy of $\log^4 N/N$, in a continuous setting. The proof in the paper for discrepancy of $\log^3 N/N$ is similar but requires working with expectations rather than probabilities.

Balancing poisson point process

Consider two point processes X_t^0, X_t^1 with changing causal intensities $\lambda^0(t), \lambda^1(t)$ (which may depend on other variables), which satisfy

$$\begin{aligned} \lambda_t^1, \lambda_t^0 &< 2 && \forall t \\ \lambda_t^0 - \lambda_t^1 &\geq \theta && X_t^0 \leq X_t^1 \\ \lambda_t^0 - \lambda_t^1 &\leq -\theta && X_t^0 > X_t^1 \end{aligned}$$

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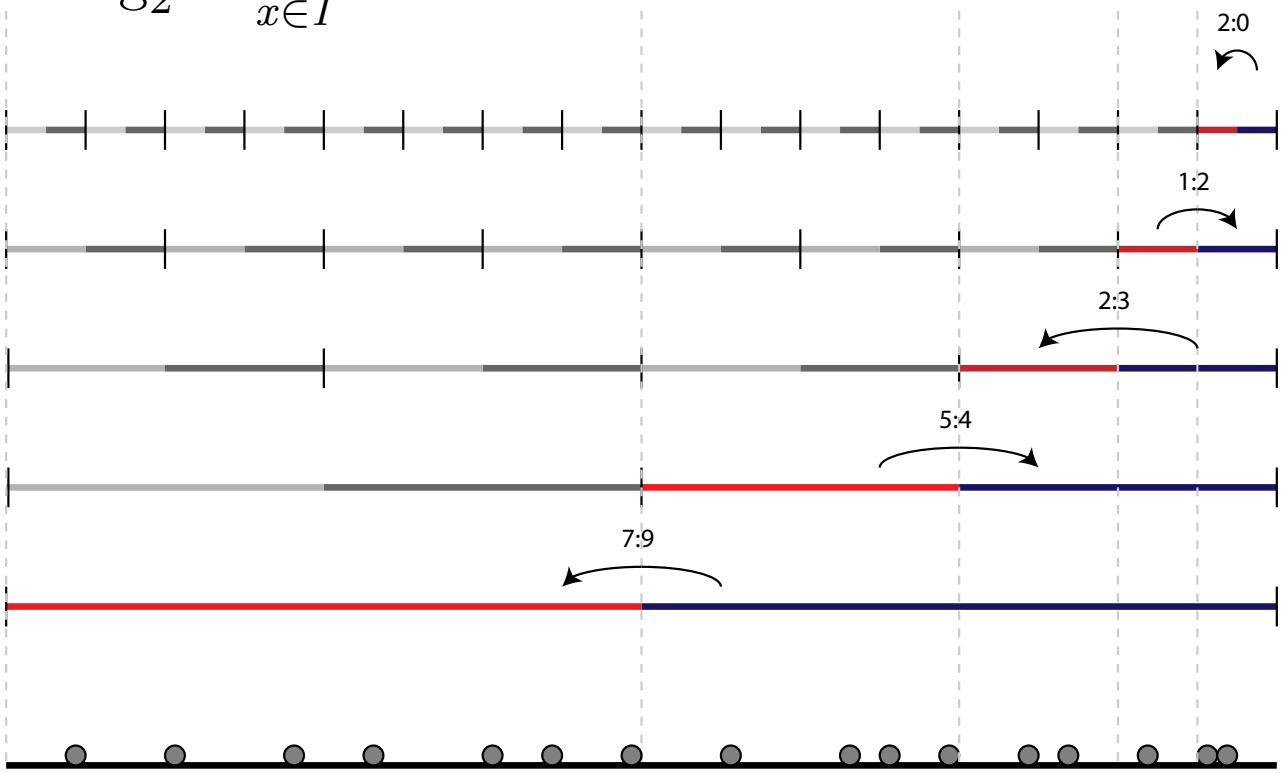
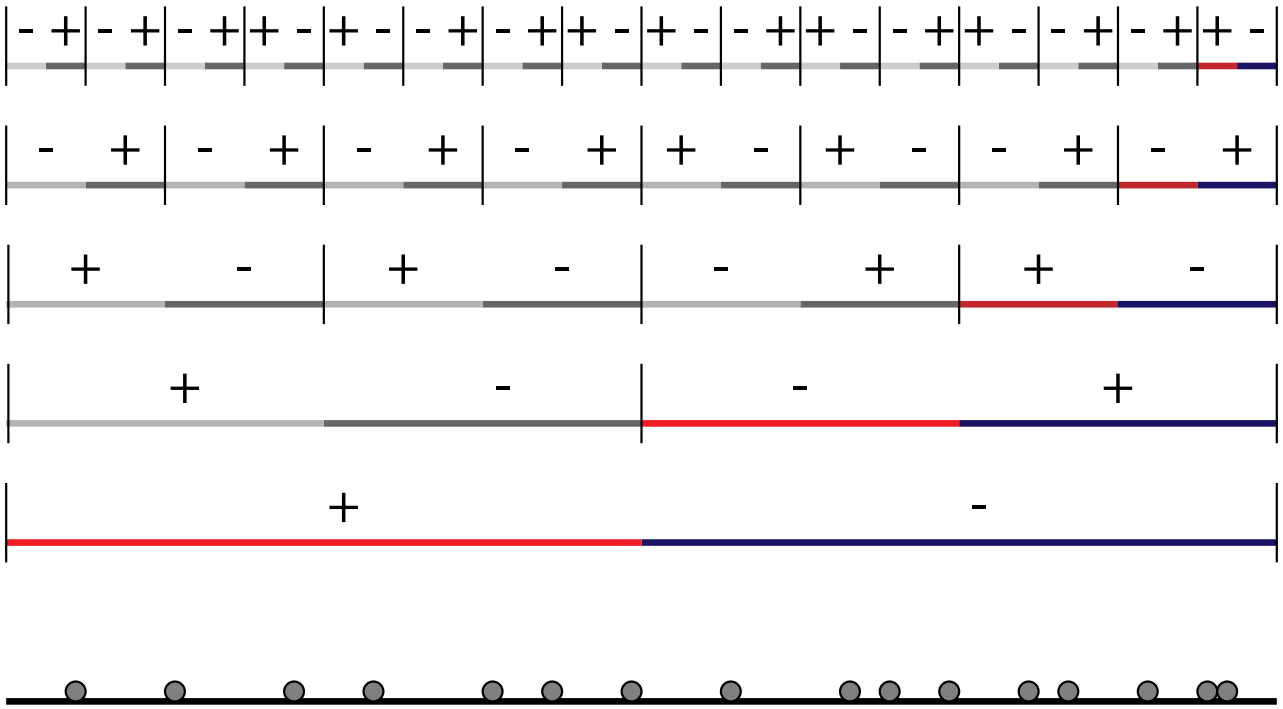


Hierarchy of drifts

We build a hierarchy of $\log_2 N$ drifts of strength $C/\log_2 N$, to control diadic discrepancies. The intensity of every point x is

drift hierarchy *rate regulating term*

$$1+\frac{C}{\log_2 N}\sum_{x\in I}(-1)^{\mathbf{1}\{x\in I_{\text{left}}\}}(-1)^{\mathbf{1}\{\mu_t(I_{\text{left}}\leq I_{\text{right}})\}}+c(-1)^{\mathbf{1}\{\mu_t([0,1])>t\}}$$



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This drift could be made between $1-\frac{1}{5}$ and $1+\frac{1}{5}$ so that it could be realized by a one-retry strategy as before.

Hierarchy of drifts - Ct.

Write $D(I) = \#\{\text{balls in } I_{\text{left}}\} - \#\{\text{balls in } I_{\text{right}}\}$. By the balancing processes lemma, $D(I)$ is typically $(\log_2 n)^2$.

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We now bound the discrepancy of a diadic interval I of size $\frac{1}{2^{j+1}}$ by

$$\sum_{i=0}^{j-1} \frac{1}{2^{j-i}} D(I^i)$$

Where I_n^i is an interval containing I of size 2^{-i}.

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Since any interval with diadic endpoint could be decomposed into at most $2\log_2 n$ diadic intervals - the theorem follows.

Future directions

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- Other measures of discrepancy.
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- Simpler algorithm?





THANK YOU.