One more proof of the Erdős–Turán inequality, and an error estimate in Wigner's law.

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Erdős and Turán [3] have proved the following inequality, which is a quantitative form of Weyl's equidistribution criterion.

Proposition 1 (Erdős – Turán)**.** *Let ν be a probability measure on the unit circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Then, for any $n_0 \geq 1$ and any arc $A \subset \mathbb{T}$,*

$$\left|\nu(A) - \frac{\operatorname{mes} A}{2\pi}\right| \leq K_1 \left\{ \frac{1}{n_0} + \sum_{n=1}^{n_0} \frac{|\widehat{\nu}(n)|}{n} \right\} , \qquad (1)$$

where

$$\widehat{\nu}(n) = \int_{\mathbb{T}} \exp(-in\theta) d\nu(\theta) ,$$

and $K_1 > 0$ is a universal constant.

A number of proofs have appeared since then, an especially elegant one given by Ganelius [5]. In most of the proofs, the indicator of A is approximated by its convolution with an appropriate (Fejér-type) kernel. We shall present another proof, based on the arguments developed by Chebyshev, Markov, and Stieltjes to prove the Central Limit Theorem (see Akhiezer [1, Ch. 3]). In this approach, the indicator of A is approximated from above and from below by certain interpolation polynomials. The argument does not use the group structure on $\mathbb{T}$, and thus works in a more general setting.

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In Section 1, we formulate a slightly different proposition and show that it implies Proposition 1. In Section 2 we reproduce the part of the arguments of Chebyshev, Markov, and Stieltes that we need for the sequel. For the convenience of the reader, we try to keep the exposition self-contained. In Section 3 we apply the construction of Section 2 to prove the Erdős–Turán inequality. In Section 4 we formulate another inequality that can be proved using the same construction. As an application to random matrices, we use an inequality from [4] and deduce a form of Wigner's law with a reasonable error estimate.

1 Introduction

Let the measure σ_1 on $\mathbb{R}$ be defined by

$$d\sigma_1(x) = \frac{1}{\pi}(1-x^2)_+^{-1/2}\, dx \ .$$

Let $T_n(\cos\theta) = \cos n\theta$ be the Chebyshev polynomials of the first kind; these are orthogonal with respect to σ_1. We shall prove the Erdős – Turán inequality in the following form:

Proposition 2. *Let μ be a probability measure on $\mathbb{R}$ [1]. Then, for any $n_0 \geq 1$ and any $x_0 \in \mathbb{R}$,*

$$\big|\mu[x_0,+\infty) - \sigma_1[x_0,+\infty)\big| \leq K_2 \left\{ \frac{1}{n_0} + \sum_{n=1}^{n_0} \frac{1}{n} \left| \int_{\mathbb{R}} T_n(x) d\mu(x) \right| \right\} \ . \qquad (2)$$

Proposition 2 implies Proposition 1. Let ν be a measure on $\mathbb{T}$, and let $A \subset \mathbb{T}$ be an arc. Rotate $\mathbb{T}$ (together with ν and A) moving the center of A to 0; this does not change the right-hand side of (1).

Denote $\nu_1(B) = \nu(B) + \nu(-B)$; ν_1 is a measure on $[0,\pi]$. The change of variables $x = \cos\theta$ pushes it forward to μ_1 on $[-1,1]$. Now apply Proposition 2 to μ_1, observing that

$$\int_{-1}^{1} T_n(x) d\mu_1(x) = \Re\, \widehat{\nu}(n) \ .$$

□

[1] We do not assume that $\operatorname{supp}\mu \subset [-1,1]$

2 The Chebyshev–Markov–Stieltjes construction

Let σ be a probability measure on $\mathbb{R}$ (with finite moments); let $S_0, S_1, \cdots$ be the orthogonal polynomials with respect to σ. For a probability measure μ on $\mathbb{R}$, denote

$$\varepsilon_n = \varepsilon_n(\mu) = \int_{\mathbb{R}} S_n(x) d\mu(x) \ , \quad n = 1, 2, 3, \cdots \ .$$

We shall estimate the distance between μ and σ in terms of the numbers ε_n.

Let $x_1 < x_2 < \cdots < x_{n_0}$ be the zeros of S_{n_0}. Construct the polynomials P, Q of degree $\leq 2n_0 - 2$, so that

$$P(x_k) = \begin{cases} 0, & 1 \leq k < k_0 \\ 1, & k_0 \leq k \leq n_0 \end{cases} ; \qquad P'(x_k) = 0 \quad \text{for } k \neq k_0;$$

$$Q(x_k) = \begin{cases} 0, & 1 \leq k \leq k_0 \\ 1, & k_0 < k \leq n_0 \end{cases} ; \qquad Q'(x_k) = 0 \quad \text{for } k \neq k_0 \ .$$

Lemma 3 (Chebyshev–Markov–Stieltjes)**.**

$$P \geq \mathbf{1}_{\left[x_{k_0}, +\infty\right)} \geq \mathbf{1}_{\left(x_{k_0}, +\infty\right)} \geq Q \ .$$

Proof. Let us prove for example the first inequality. The derivative P' of P vanishes at x_k, $k \neq k_0$, and also at intermediate points $x_k < y_k < x_{k+1}$, $k \neq k_0, n_0$. The degree of P' is at most $2n_0 - 3$, hence it has no more zeroes. Now, $P(x_{k_0}) > P(x_{k_0-1})$; hence P is increasing on (x_{k_0-1}, y_{k_0+1}). Therefore P' is decreasing on (y_{k_0+1}, x_{k_0+2}), increasing on (x_{k_0+2}, y_{k_0+3}), et cet. Thus $P(x) \geq 1$ for $x \geq x_{k_0}$. Similarly, $P(x) \geq 0$ for $x < x_{k_0}$. ☐

Let $P = \sum_{n=0}^{n_0} p_n S_n$, $Q = \sum_{n=0}^{n_0} q_n S_n$. Then

$$\begin{aligned} \mu[x_{k_0}, +\infty) &\leq \int_{\mathbb{R}} P(x) d\mu(x) = p_0 + \sum_{n=1}^{2n_0-2} \varepsilon_n p_n \\ &= q_0 + (p_0 - q_0) + \sum_{n=1}^{2n_0-2} \varepsilon_n p_n \\ &\leq \sigma(x_{k_0}, +\infty) + (p_0 - q_0) + \sum_{n=1}^{2n_0-2} |\varepsilon_n||p_n| \ . \end{aligned}$$

Similarly,

$$\mu(x_{k_0}, +\infty) \geq \sigma[x_{k_0}, +\infty) - (p_0 - q_0) - \sum_{n=1}^{2n_0-2} |\varepsilon_n||q_n| \ .$$

Therefore

$$\big|\mu[x_{k_0}, +\infty) - \sigma[x_{k_0}, +\infty)\big| \leq (p_0 - q_0) + \sum_{n=1}^{2n_0-2} |\varepsilon_n| \max(|p_n|, |q_n|) \ . \tag{3}$$

Thus we need to estimate $p_0 - q_0$, $|p_n|$, $|q_n|$. This can be done using the following observation (which we have also used in [8].) Let R be the Lagrange interpolation polynomial of degree $n_0 - 1$, defined by

$$R(x_k) = \delta_{kk_0} \ , \quad k = 1, 2, \cdots, n_0 \ .$$

Equivalently,

$$R(x) = \frac{S_{n_0}(x)}{S'_{n_0}(x_{k_0})(x - x_{k_0})} \ . \tag{4}$$

Lemma 4. *$P - Q = R^2$.*

Proof. The polynomial $P - Q$ has multiple zeroes at x_k, $k \neq k_0$. Therefore $R^2 \,|\, (P - Q)$. Also, $\deg R^2 = 2n_0 - 2 \geq \deg(P - Q)$, and

$$R^2(x_{k_0}) = 1 = P(x_{k_0}) - Q(x_{k_0}) \ .$$

□

Thus

$$p_0 - q_0 = \int_{\mathbb{R}} R^2(x) d\sigma(x) \tag{5}$$

and

$$\begin{aligned} |p_n| &= \left|\int_{\mathbb{R}} P(x) S_n(x) d\sigma(x)\right| \\ &\leq \left|\int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x)\right| + \left|\int_{\mathbb{R}} (P(x) - \mathbf{1}_{[x_{k_0}, +\infty)}(x)) S_n(x) d\sigma(x)\right| \\ &\leq \left|\int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x)\right| + \int_{\mathbb{R}} R^2(x) |S_n(x)| d\sigma(x) \ . \end{aligned} \tag{6}$$

Similarly,

$$|q_n| \le |\int_{x_{k_0}}^{\infty} S_n(x) d\sigma(x)| + \int_{\mathbb{R}} R^2(x)|S_n(x)| d\sigma(x) \ .$$

3 Proof of Proposition 2

We apply the framework of Section 2 to $\sigma = \sigma_1$, $S_n = T_n$. Let $x_{k_0} = \cos\theta_0$, $0 \le \theta_0 \le \pi/2$. Then

$$T'_{n_0}(\cos\theta_0) \cdot -\sin\theta_0 = -n_0 \sin n\theta_0 \ ,$$

and hence

$$|T'_{n_0}(x_0)| = \frac{n_0}{|\sin\theta_0|} = \frac{n_0}{\sqrt{1 - x_{k_0}^2}} \ .$$

Thus, according to (5),

$$\begin{aligned} p_0 - q_0 &= \int_{\mathbb{R}} \frac{T_{n_0}(x)^2}{T'_{n_0}(x_0)^2 (x - x_0)^2} d\sigma_1(x) \\ &= \frac{\sin^2\theta_0}{4\pi n_0^2} \int_0^{\pi} \frac{\cos^2 n_0\theta}{\sin^2\frac{\theta+\theta_0}{2} \sin^2\frac{\theta-\theta_0}{2}} d\theta \ . \end{aligned}$$

Now,

$$\int_0^{\theta_0/2} \le \int_0^{\theta_0/2} C_1 d\theta/\theta_0^4 \le C_1/\theta_0^3 \le C_2 n_0/\theta_0^2 \ ,$$

$$\int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \le C_3 \int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \frac{d\theta}{\theta_0^2(\theta - \theta_0)^2} \le \frac{C_4 n_0}{\theta_0^2} \ ,$$

and similarly

$$\int_{\theta_0 + \pi/(3n_0)}^{\pi} \le C_5 n_0/\theta_0^2 \ .$$

Finally,

$$|T'_{n_0}(\cos\theta)| = n_0 \frac{|\sin n_0\theta|}{\sin\theta} \ge n_0/(C_6\theta_0) \ge |T'_{n_0}(\cos\theta_0)|/C_7$$

for $|\theta - \theta_0| \le \pi/(3n_0)$, hence

$$\int_{\theta_0 - \pi/(3n_0)}^{\theta_0 + \pi/(3n_0)} \frac{T_{n_0}(\cos\theta)^2 d\theta}{T'_{n_0}(\cos\theta_0)^2 (\cos\theta - \cos\theta_0)^2} \le C_8/n_0 \ .$$

Therefore

$$p_0 - q_0 \le C/n_0 \ . \tag{7}$$

Next,

$$\int_{x_{k_0}}^{\infty} T_n(x) d\sigma_1(x) = \int_0^{\theta_0} \cos n\theta \, \frac{d\theta}{\pi} = \frac{\sin n\theta_0}{n\pi} \ ; \tag{8}$$

$$\begin{aligned} \int_{\mathbb{R}} R^2(x)|T_n(x)| d\sigma_1(x) &= \int_0^{\pi} \frac{\cos^2 n_0\theta}{\frac{n_0^2}{\sin^2\theta_0}(\cos\theta - \cos\theta_0)^2} |\cos n\theta| \frac{d\theta}{\pi} \\ &\le \frac{C_1\theta_0^2}{n_0^2} \int_0^{\pi} \frac{\cos^2 n_0\theta \, |\cos n\theta| \, d\theta}{\sin^2\frac{\theta+\theta_0}{2} \sin^2\frac{\theta-\theta_0}{2}} \ . \end{aligned}$$

Now,

$$\int_0^{\theta_0/2} \le C_2/\theta_0^3 \le C_3 n_0/\theta_0^2 \ ;$$

$$\int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \le C_4 \int_{\theta_0/2}^{\theta_0 - \pi/(3n_0)} \frac{d\theta}{\theta_0^2(\theta-\theta_0)^2} \le C_5 n_0/\theta_0^2 \ ,$$

and similarly

$$\int_{\theta_0 + \pi/(3n_0)}^{\pi} \le C_6 n_0/\theta_0^2 \ ;$$

$$\int_{\theta_0 - \pi/(3n_0)}^{\theta_0 + \pi/(3n_0)} \le (C_7/n_0)(n_0^2/\theta_0^2) = C_7 n_0/\theta_0^2 \ .$$

Therefore

$$\int_{\mathbb{R}} R^2(x)|T_n(x)| d\sigma_1(x) \le C_8/n_0 \ . \tag{9}$$

Combining (6), (8) and (9), we deduce:

$$|p_n| \le C/n \ . \tag{10}$$

Similarly, $|q_n| \le C/n$.

Proof of Proposition 2. Substitute (7) and (10) into (3), taking

$$m_0 = \lceil n_0/2 \rceil + 1$$

instead of n_0. We deduce that (2) holds when $x_0 = x_{k_0}$ is a non-negative zero of T_{m_0}. By symmetry, a similar inequality holds for negative zeroes. For a general $x_0 \in \mathbb{R}$, apply the inequality to the two zeroes of T_{m_0} that are adjacent to x_0 (one of them may formally be $\pm\infty$.) ☐

4 Another inequality, and an application to Wigner's law

Let the measure σ_2 on $\mathbb{R}$ be defined by

$$d\sigma_2(x) = \frac{2}{\pi}(1-x^2)_+^{1/2}\,dx\ .$$

Let $U_n(\cos\theta) = \cos n\theta$ be the Chebyshev polynomials of the second kind; these are orthogonal with respect to σ_2.

Proposition 5. *Let μ be a probability measure on $\mathbb{R}$. Then, for any $n_0 \geq 1$ and any $x_0 \in \mathbb{R}$,*

$$\begin{aligned}&\left|\mu[x_0,+\infty) - \sigma_2[x_0,+\infty)\right| \\ &\qquad\leq K_5 \left\{ \frac{\rho(x_0;n_0)}{n_0} + \rho(x_0;n_0)^{1/2} \sum_{n=1}^{n_0} n^{-1} \left| \int_{\mathbb{R}} U_n(x) d\mu(x) \right| \right\}\ , \qquad (11)\end{aligned}$$

where $\rho(x;n_0) = \max(1-|x|, n_0^{-2})$.

Observe that $\rho \leq 1$. Similar inequalities with 1 instead of ρ have been proved by Grabner [7] and Voit [9]. On the other hand, the dependence on x in (11) is sharp, in the following sense: for any x_0, there exists a probability measure μ on $\mathbb{R}$ such that $\int_{\mathbb{R}} U_n(x) d\mu(x) = 0$ for $1 \leq n \leq n_0$, and

$$\left|\mu[x_0,+\infty) - \sigma_2[x_0,+\infty)\right| \geq C^{-1}\rho(x_0;n_0)/n_0\ ,$$

where $C > 0$ is independent of n_0; cf. Akhiezer [1, Ch. 3].

The proof of Proposition 5 is parallel to that of Proposition 2: we apply the inequalities of Section 2 to the measure σ_2 and the polynomials U_n.

Grabner [7] and Voit [9] have applied their inequalities to estimate the cap discrepancy of a measure on the sphere. We present an application to random matrices.

Let A be an $N \times N$ Hermitian random matrix, such that

1. $\{A_{uv} \,|\, 1 \leq u \leq v \leq N\}$ are independent,
2. $\mathbb{E}|A_{uv}|^{2k} \leq (Ck)^k$, $k = 1, 2, \cdots$;
3. the distribution of every A_{uv} is symmetric, and $\mathbb{E}|A_{uv}|^2 = 1$ for $u \neq v$.

Let $\mu_A = N^{-1} \sum_{k=1}^N \delta_{\lambda_k(A)/(2\sqrt{N})}$ be the empirical measure of the eigenvalues of A (which is a random measure). By [4, Theorem 1.5.3],

$$0 \leq \mathbb{E} \int_{\mathbb{R}} U_n(x) d\mu_A(x) \leq Cn/N \ , \quad 1 \leq n \leq N^{1/3} \ .$$

Applying Proposition 5, we deduce the following form of Wigner's law:

Proposition 6. *Under the assumptions 1.-3.,*

$$\left| \mathbb{E}\,\# \left\{ k \,\middle|\, \lambda_k > 2\sqrt{N} x_0 \right\} - N\sigma_2(x_0, +\infty) \right| \\ \leq C \max\left(N^{2/3}(1 - |x_0|), 1 \right) \quad (12)$$

for any $x_0 \in \mathbb{R}$.

Better bounds are available for $x \in (-1+\varepsilon, 1-\varepsilon)$ (cf. Götze and Tikhomirov [6], Erdős, Schlein, and Yau [2]). On the other hand, for x very close to ± 1, the right-hand side in our bound is of order $O(1)$, which is in some sense optimal.

Remark 7. A similar method allows to bound the variance of the number of eigenvalues on a half-line:

$$\mathbb{V} \# \left\{ k \,\middle|\, \lambda_k > 2\sqrt{N} x_0 \right\} \leq C \max\left(N^{2/3}(1 - |x_0|), 1 \right)^{5/2} ;$$

therefore one can also bound the probability that $\# \left\{ k \,\middle|\, \lambda_k > 2\sqrt{N} x_0 \right\}$ deviates from $N\sigma_2(x_0, +\infty)$.

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