Multi-layered planar firefighting

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Abstract

Consider a model of fire spreading through a graph; initially some vertices are burning, and at every given time-step fire spreads from burning vertices to their neighbors. The firefighter problem is a solitaire game in which a player is allowed, at every time-step, to protect some non-burning vertices (by effectively deleting them) in order to contain the fire growth. How many vertices per turn, on average, must be protected in order to stop the fire from spreading infinitely?

Here we consider the problem on $\mathbb{Z}^2 \times [h]$ for both nearest neighbor adjacency and strong adjacency. We determine the critical protection rates for these graphs to be $1.5h$ and $3h$, respectively. This establishes the fact that using an optimal two-dimensional strategy for all layers in parallel is asymptotically optimal.

1 Introduction

lLet $G = (V, E)$ be an infinite graph. The firefighter problem on G is the following solitaire combinatorial game, introduced by Hartnell [12]. The game starts with a finite starting set of *burning* vertices $B(0) \subset V$. At every turn $t \in \mathbb{N}$, the player chooses an arbitrary finite collection of non-burning vertices and *protects* them permanently, subject to the constraint that at most $\mathsf{f}(t)$ vertices are protected by time t. Then, the unprotected neighbors of burning vertices become burning. The goal of the game is to ensure that eventually no new burning vertices are generated. If this goal is achieved, we say that the player *contained* the fire. We say that $C(G)$ is the *critical protection rate* if for $\mathsf{f}(t) = C(G)t$, there exists a starting set of burning vertices for which there is no play strategy that allows the player to contain the fire, while for $\mathsf{f}(t) = (C(G) + \varepsilon)t$ for any $\varepsilon > 0$, such a strategy exists for all finite $B_0 \subset V$.

Denote the set of all positive integers by $\mathbb{N}$ and the set of h smallest positive integers by $[h] := \{1, 2, \ldots, h\}$, for $h \in \mathbb{N}$. Write $\square$ (resp., $\boxtimes$) for the Cartesian product (resp., the strong Cartesian product) of graphs. We study the firefighter problem on $G_1 = G_1^h := (\mathbb{Z} \square \mathbb{Z}) \square [h]$ and $G_2 = G_2^h := (\mathbb{Z} \boxtimes \mathbb{Z}) \square [h]$. These two infinite graphs have the same set $\mathbb{Z} \times \mathbb{Z} \times [h]$ of vertices, which we think of as h vertically-stacked copies of the horizontal plane $\mathbb{Z}^2$. The difference is that the degree of a typical vertex in G_1 is 6 (four horizontal neighbors, one above and one below), versus degree 10 in G_2 (8 horizontal neighbors, one above and one below).

Our main result is the following.

Theorem 1. *For every $h \in \mathbb{N}$ and for $q \in \{1, 2\}$, we have $C(G_q) = \frac{3}{2}qh$.*

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The special case $h = 1$ of Theorem 1 has been obtained by the second and third authors in [7]. In light of this, Theorem 1 can be interpreted as a "parallel repetition" statement: in the multi-layered setting, the player cannot asymptotically improve upon the simple strategy of repeating the same two-dimensional strategy h times. In fact, in the course of the proof we will see that the bound $3qh$ is also viable for the graph $(\mathbb{Z}\boxtimes\mathbb{Z})\boxtimes[h]$. We believe that this observation holds in much greater generality (see discussion in Section 1.5).

1.1 Background

The firefighter problem is a model for the spread of a phenomenon at the face of an effort to contain it. The model is best suited to describe the spread of an epidemic (or a false rumor) in a population while a vaccine (or clear contradictory evidence) are administered to prevent it.

The problem was introduced by Hartnell [12] in 1995, formulated with one firefighter per time-step and a single vertex as initial fire and was generalized to the version described above. The problem could be seen as an earlier variant of Conway's celebrated *angel problem* [5], in which the angel has power $k = 1$, the devil is oblivious to the location of the angel and must be certain to catch it in order to win the game.

On finite graphs, the most natural challenges are to reduce the number of burning vertices as the process terminates and to reduce the speed at which the fire is contained, see [9, Chapter 5]. The problem has been also studied from an algorithmic point of view. MacGillivray and Wang [16] show that on bipartite graphs finding a strategy which minimizes the number of burning vertices is NP-complete. This has been extended to trees and cubic graphs in [8, 15]. On trees, however, the solution could be approximated up to a constant factor in polynomial time (See [1, 4, 11, 13, 14]). For a survey on the problem see [9] and references therein.

On infinite, vertex-transitive graphs with planar-type growth, the most natural question is to recover the critical protection rate. Currently, the only known values of $C(G)$ are $C(\mathbb{Z}\,\square\,\mathbb{Z}) = \frac{3}{2}$ and $C(\mathbb{Z}\boxtimes\mathbb{Z}) = 3$, obtained in [7] (following earlier results by Wang and Moller [18], Ng and Raff [19] and Messinger [17]). These rely on an isoperimetric argument developed by Fogarty in her thesis [10]. On the 6-regular hexagonal lattice we have $1 \le C(\mathbb{Z} \hexagon \mathbb{Z}) \le 2$ [10], while on the cubic triangular lattice $\frac{1}{2} \le C(\mathbb{Z} \bigtriangleup \mathbb{Z}) \le 1$ (unpublished, could be obtained in a similar way). Computing the critical protection rate of these lattices remains an open problem.

Develin and Hartke [6] generalized Fogarty's argument to show that $2d-1$ firefighters are required to contain a single-source fire in $\mathbb{Z}^{\square d}$ and conjectured that for $\mathsf{f}(t) \ll t^{d-2}$, there exists an initial fire such that any strategy using $\mathsf{f}(t)$ firefighters fails to contain it. A new result by Amir, Baldasso and Kozma [2] use a Fogarty-type argument together with an isoperimetric tool to prove a generalized version of Develin and Hartke's conjecture, namely that in Cayley graphs with polynomial growth, any strategy with $\mathsf{f}(t) \ll t^{d-2}$ cannot contain a large enough initial fire.

Only when G has a planar growth rate, constant protection rate should suffice to contain the fire and $C(G)$ is well defined. Thus, in a constant protection rate setting, the multi-layered variant studied here is a natural three-dimensional analog of the problem. We further believe that our methods could be applied to nearly any local connectivity structure between multiple layers of $\mathbb{Z}\,\square\,\mathbb{Z}$ and $\mathbb{Z}\boxtimes\mathbb{Z}$, so that whenever a three-dimensional layered graph has its layers connected along a sub-lattice, playing identically on each layer is asymptotically optimal. We also see the new tools presented here as a stepping stone for tackling other graphs, and hopefully also a continuous variant (see Section 1.5 below).

1.2 Outline of the proof for the lower bound

Firefronts and levels. The analysis in [7] of the fire evolution in the planar lattice considered four *firefronts*, forming a rectangle that bounded the fire at every time step. To generalize this to the multi-layered setting, we separately treat the horizontal components of each firefront, which we call *levels*. Each level is a horizontal segment of vertices, and the four levels in the each layer form a rectangle. The firefronts are defined in a way which guarantees that each front will always be adjacent to many burning vertices.

Much like in [7], the orientation of the levels differs by $45°$ between G_1 and G_2: levels are parallel to the horizontal axes in G_2, while in G_1 they are diagonal to them. In each front, the horizontal distances between the origin and each level may differ; however, a front cannot have "vertical jumps"; i.e., the horizontal distance to the origin differs by at most one between vertically adjacent levels of the same front. We call this structure the *fronts structure* (see Figure 3).

Movement of the levels. The horizontal distance between the origin and a level can never decrease, only increase by one or stay the same. When we decide to increase this distance in a certain direction, we say that that level *advances*, and otherwise that is it *still*. We say that a level is *burning* (at some time) if at least one vertex on it is burning. A burning level that advances is called *active*. The only possible way for a non-burning level to advance is when "pulled" by an adjacent advancing level (burning or not) in order to maintain the "no vertical jumps" property. To show that the fire cannot be contained, we will prove that the fronts structure advances indefinitely. In previous works, burning levels would always advance (see [7]). In this work, we may decide to keep a burning level still for some time. This new approach allows us to better control the interplay between the the number of burning vertices on a level and the length of horizontally adjacent levels.

Fierity and Potential. We keep count of protected and burning vertices encountered by the fronts structure. This could be thought of as a time change of the original process which is non-homogeneous in space, in the sense that different fronts undergo a different time change. We call the number of burning vertices on the k-th level of the i-th front its *fierity* and denote it by φ_i^k. When a level is burning, surely in the next time step the horizontal neighbors of its burning vertices are either burning or protected. An argument of Fogarty [10] uses this observation to show that sum of the fierity φ_i^k of a burning level and the number f_i^k of newly counted protected vertices increases by at least q (when playing on G_q). In our situation vertices may shift from one firefront to an adjacent front; to keep track of these transitions we count the difference between vertices shifted into the k-th level of the i-th front and out of it, which we denote by p_i^k.

Putting these together we define μ_i^k, the *potential* of the k-th level of the i-th front, as the sum of a level's fierity, protected and shifted vertices. A variant of Fogarty's argument is used to show that for an active level this number increases by at least q.

Much of the effort would have been spared if we could have just followed the footsteps of [7] and inductively prove that at any given time, there are at least $3h$ active levels. Via the corresponding potential increase, it would show that the fire cannot be contained. Unfortunately, this is not always the case; indeed, there could be times at which there are fewer than $3h$ active levels. We show, that when such a time occurs, this gap is compensated within $2h$ time-steps. Therefore, the average increase in the number of burning vertices on the front structure is still $3h$.

Overflow. The source of additional growth in the fierity of a level comes from vertical *overflow*. That is a situation in which the fire spreads from burning vertices above or below some level to that level. Overflow occurs when an active level is pulling another level. In order to guarantee sufficient overflow, we must allow a level to be active only when it has sufficiently many vertices, so that if it overflows to a neighbor, it will *significantly* increase its fierity.

Remark. We strive to provide a unified proof for $q = 1, 2$, instead of duplicating the proof with minor changes. In [7] a proof for $q = 2$ sufficed to imply the case $q = 1$ using the observation that $\mathbb{Z} \boxtimes \mathbb{Z}$ is contained in the square of $\mathbb{Z} \square \mathbb{Z}$, so playing on $\mathbb{Z} \boxtimes \mathbb{Z}$ is at least as easy for the player as playing on $\mathbb{Z} \square \mathbb{Z}$ with twice as many firefighters. For $h > 1$, however, the connection between G_1 and G_2 is not as obvious, since the square of G_1 also allows double vertical steps.

Upon a first reading, the reader is advised to focus on G_2 and ignore the case $q = 1$.

1.3 Outline of the paper

Section 2 consists of the definitions required to formulate the firefighter problem and the evolution of the firefronts structure given an activity criterion, concluding with a reduction of Theorem 1 to a lower bound on the potential of the firefronts structure (Proposition 5). In Section 3 we define a finer notion of potential, which treats each firefront separately. We further establish key inequalities controlling the growth of this potential (Proposition 6 and Proposition 7). Section 4 is dedicated to the proof of Proposition 5 and auxiliary lemmata.

1.4 Discussion

In this paper we equip $\mathbb{Z}^2 \times [h]$ with vertical connectivity of a Cartesian product, which is weaker than the strong Cartesian product. Hence we have $C(\mathbb{Z} \square \mathbb{Z} \boxtimes [h]) = \frac{3}{2}h$ and $C(\mathbb{Z} \boxtimes \mathbb{Z} \boxtimes [h]) = 3h$ as an immediate corollary of Theorem 1. Indeed, the statement of Theorem 1 only implies a lower bound on the critical protection rate, but the same containment strategy used to prove Theorem 1, which is repeating a horizontal containment strategy in each layer, would work here too (and also for any intermediate vertical connectivity). We believe that one can use the same methods to obtain the same critical protection rates for any weaker regular connectivity, that is, a connectivity in which in every $k \times k$ horizontal square there is a vertex connected to the vertex above it (except in the topmost layer). For example, one can consider a connectivity in which $(x_1, y_1, z_1), (x_2, y_2, z_2)$ with $z_1 \neq z_2$ are neighbors, if and only if $|z_1 - z_2| = 1, (x_1, y_1) = (x_2, y_2)$ and $x_1 + y_1$ is divisible by 3. We also tend to believe that even if this connectivity is further weakened by subdividing every vertical edge into a path constant length, the same result holds. To deal with weaker connectives one needs to change the definition of a front structure and the activity of the fronts in order to guaranteed overflow. We have decided not to do to, in order to increase the readability of our work.

The main technical innovation of the paper, is that the analysis of the fire is not done by estimating the number of burning vertices on the rectangular boundary of the burning region at every time-step, but rather on the rectangular boundary of an artificially chosen smaller domain. This allows the classical techniques from potential theory, to ignore thin stretches of burning vertices which reduce the number of burning site on the rectangular boundary of the burning region.

For $h = 1$, the lower bound of [7] is quite tight: for $\mathsf{f}(t) \leq \frac{3}{2}qt$ it is impossible to contain even the smallest possible initial fire — one burning vertex. Clearly, this no longer makes sense for $h > 1$. Indeed, the maximal degree in G_q is $2 + 4q$, so it is possible to contain, in a single time-step, any initial fire of size

$$\frac{1}{4}h \leq \frac{\frac{3}{2}qh}{2 + 4q}.$$

Furthermore, it is possible to have an initial fire of size at least $\frac{1}{32}qh^3$ that can be contained in h time-steps for $\mathsf{f}(t) = \frac{3}{2}qht$. Assume for simplicity that h is divisible by $8/q$, and set $r = \frac{3}{4}h - 1$. The pyramid-shaped construction (see Figure 1) goes as follows: in the bottom layer we have a square (resp., diamond) of side (resp., diagonal) length r, which forms the base of the pyramid. The choice of r makes it possible to horizontally surround the base in a single time-step. Above the base we have a smaller square/diamond, of side/diagonal length $r - 2$, which grows to a square/diamond of side/diagonal length r in the second time-step. Again, there are just enough firefighters to horizontally surround the fire on layer 2, by placing them directly above the layer 1 firefighters. By time h, the cylinder of firefighters will reach the top layer, and the fire will be contained.

It is quite easy to see that for $q = 2$ the initial fire consists of

$$r^2 + (r-2)^2 + (r-4)^2 + \cdots = \frac{r(r+1)(r+2)}{6} = \frac{(r+1)^3}{6} - \frac{r+1}{6} = \frac{9}{128}h^3 - \frac{1}{8}h \geq \frac{1}{16}h^3$$

burning vertices, and for $q = 1$ there are

$$\frac{1+r^2}{2} + \frac{1+(r-2)^2}{2} + \cdots + \frac{1}{2} = \frac{r(r+1)(r+2)}{12} + \frac{r+1}{4} > \frac{(r+1)^3}{12} = \frac{9}{256}h^3 > \frac{1}{32}h^3$$

burning vertices.

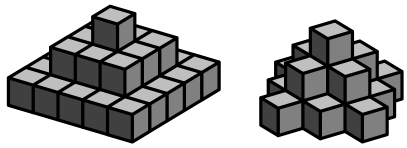


Figure 1: An initial fire in G_q of size at least $qh^3/32$ that can be contained with $\frac{3}{2}qh$ firefighters per turn. Here $h = 8$, $r = 5$, and indeed $5^2 + 3^2 + 1^2 = 35 > 8^3/16$ for $q = 2$ and $13 + 5 + 1 = 19 > 8^3/32$ for $q = 1$.

1.5 Open problems

In the section we present what we see as the most interesting problems concerning the firefighter problem on lattices with planar type growth, which we hope that this new technique could help tackling.

Problem 1. *Given any local connectivity for $\mathbb{Z} \times \mathbb{Z}$, compute the critical protection rate. In particular, compute $C(\mathbb{Z} \hexagon \mathbb{Z})$ and $C(\mathbb{Z} \triangle \mathbb{Z})$.*

For $\mathbb{Z} \hexagon \mathbb{Z}$, at first sight the problem seems similar to $\mathbb{Z} \boxtimes \mathbb{Z}$, and indeed several authors claimed to have solved it only to find an error in their arguments. Even the second and third authors have claimed in [7] that this problem seems approachable using the same techniques. However, the fact that the total length of three adjacent hexagonal fronts is not necessarily half of the circumference makes the current analysis fail. It appears that part of the reason that current methods fall short of tackling the hexagonal lattice, is the fact that they player can create a situation in which there are many burning vertices on the convex boundary of the burning domain, but not on the bounding hexagon containing it. It seems worthwhile to try and extend the techniques presented here to take into account these burning vertices.

Continuous variant. In 2007 Bressan [3] introduced the following continuous space-time variant of the problem, which we present here as a scaling limit of a continuous space-discrete time process. Consider a starting fire $F(0)$ that is a path-connected set in $\mathbb{R}^2$ containing the origin. At every ε time-step, the player adds to the set of protected paths $P(t)$, initialized as the empty set, a segment of length $c \cdot \varepsilon$ (in some metric d_{protect}), disjoint from the fire. After this, the fire $F(t+\varepsilon)$ is increased to be the connected component of origin in $F(t) + B_\varepsilon \setminus P(t)$, where B_ε is the ε ball in some metric d_{fire}. Taking ε to 0, a continuous variant of the firefighter problem arises where $C(d_{\text{protect}}, d_{\text{fire}})$ is the infinimum of the set of c for which the fire could be contained. For example, $C(L_\infty, L_1) = 1.5$ and $C(L_1, L_\infty) = 3$ by taking scaling limit over the results of [7].

In the natural setting, when B_ε is the Euclidean ball and segment length is measured in Euclidean metric, the answer is conjectured by Bressan to be 2 and a tight upper bound is provided. We conjecture that the same holds for any L_p metric.

Problem 2. *In the continuous firefighter problem on L_p, compute $C(L_p, L_p)$.*

We hope that progress towards Problem 1 would render it possible, through analogy and through scaling limits, to obtain new results for Problem 2.

2 Preliminaries

Notation and conventions. The sub-lattice $\mathbb{Z}^2 \times \{k\}$ is called the k-th *layer* of our graph (either G_1 or G_2). We think of it as being horizontal. Throughout the paper, we follow the convention that superscripts refer to layers (and levels), subscripts refer to directions (and fronts), and the argument of functions is time. Moreover, we reserve the indices i and j for directions/fronts, the indices k and ℓ for levels/layers, and the variables s, t and τ for time. Addition and subtraction in subscript indices is henceforth always taken modulo 4.

The notation $A \sqcup B$ refers to the union $A \cup B$ of two sets, A and B, assumed to be *disjoint*.

For a function g on the integers (resp., on sets), denote by $\Delta g(t)$ its discrete derivative $\Delta g(t) := g(t) - g(t-1)$ (resp., $\Delta g(t) := g(t) \setminus g(t-1)$). Sometimes it would be more convenient to directly define $\Delta g(t)$, and obtain $g(t)$ inductively as $g(t) := g(t-1) + \Delta g(t)$ for some base case $g(0)$ (resp., as $g(t) := g(t-1) \sqcup \Delta g(t)$).

Omitting an index (superscript or subscript) in a notation serves to represent summation (or union) over this index; e.g., $f_i(t) := \sum_{k\in[h]} f_i^k(t)$. Additionally, writing max (resp., min) for an index represents taking the maximum (resp., minimum) over it; e.g., $r_i^{\min}(t) := \min_{k\in[h]} r_i^k(t)$.

Given a set of vertices $U \subset \mathbb{Z}^2 \times [h]$, denote by U^+ its closed neighborhood, namely vertices in U or adjacent to some $u \in U$.

Evolution of the process, given a player strategy. Given an initial set $B(0) \subset \mathbb{Z}^2 \times [h]$ of burning vertices, and an increasing function $\mathsf{F}(t)_{t\in\mathbb{N}}$ of sets of vertices that the player would like to protect by time t, we now specify the evolution of the process, by defining the set $B(t)$ of vertices burning at time t.

To avoid the question of precise timing (i.e., what happens first at time t), we highlight the timeline of the game in the following table:

Time	Event	Updated quantity
$-\frac{1}{3}$	The initial fire is created	$B(0)$ is set
$t-\frac{2}{3}$	Additional vertices are protected	The protected set becomes $\mathsf{F}(t) \setminus B(t-1)$
$t-\frac{1}{3}$	Fire spreads to unprotected adjacent vertices	$B(t)$ is determined
t	Nothing	

We inductively define $B_{\mathsf{F}}(t) = B(t)$ for all $t \in \mathbb{N}$ by

$$B(t) = B(t-1) \cup \Big(B(t-1)^+ \setminus \mathsf{F}(t)\Big). \tag{1}$$

Observe that here we allow $\mathsf{F}(t)$ to overlap with $B(t-1)$; however, such overlap has no effect on the evolution of $B(t)$. The reader may think of $\Delta\mathsf{F}(t)$ as orders given by the player concerning which vertices to protect at time t, so that "illegal" orders (i.e., orders to protect vertices that are already burning) are simply ignored. Hence, $\mathsf{F}(t) \setminus B(t-1)$ are the vertices protected just before the fire spreads further at time $t - \frac{1}{3}$.

In Section 2.2 we will define $F(t)$, which is a refinement of $\mathsf{F}(t)$, consisting only of effectively protected vertices; that is, vertices that would have been burned up to time t, but were protected.

Directions. First we define, in a clockwise fashion, the horizontal directions for the firefronts in G_1 and G_2 (see Section 1). For G_1, these are

$$\theta_0 := (+\tfrac{1}{2}, +\tfrac{1}{2}, 0), \qquad \theta_1 := (+\tfrac{1}{2}, -\tfrac{1}{2}, 0),$$
$$\theta_2 := (-\tfrac{1}{2}, -\tfrac{1}{2}, 0), \qquad \theta_3 := (-\tfrac{1}{2}, +\tfrac{1}{2}, 0);$$

while for G_2, these are

$$\theta_0 := (0, +1, 0), \qquad \theta_1 := (+1, 0, 0),$$
$$\theta_2 := (0, -1, 0), \qquad \theta_3 := (-1, 0, 0).$$

We denote by $\mathcal{I} := \{0, 1, 2, 3\}$ the index set of horizontal directions. Recall that we use these always modulo 4; in particular we write $|i - j| = 1$ if i and j are consecutive modulo 4 (e.g., $i = 3, j = 0$).

In addition, let $\phi := (0, 0, 1)$ denote the vertical unit vector.[1]

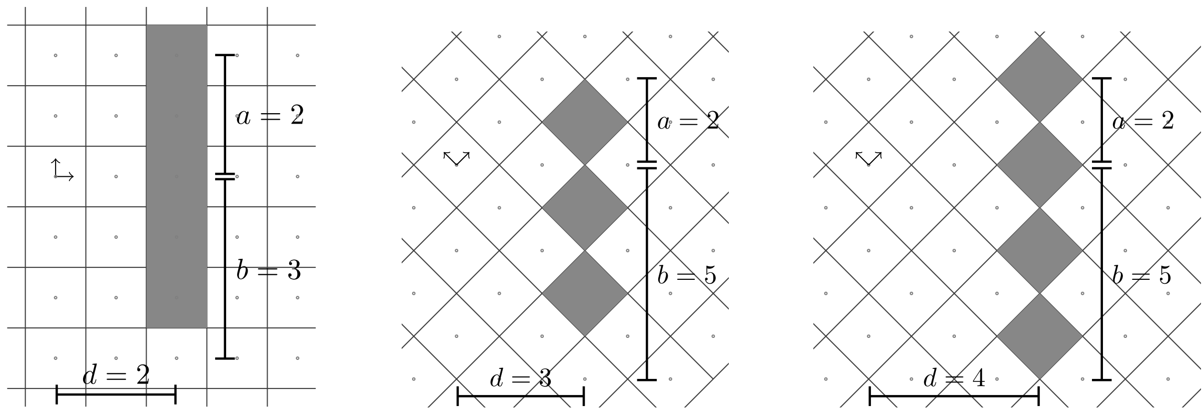
The (infinite) line of distance d from the origin, perpendicular to the i-th direction, in the k-th layer, can be written in terms of $\{\theta_j : j \in \mathcal{I}\}$ and ϕ as

$$d\theta_i + \mathbb{R}\theta_{i+1} + k\phi = \{d\theta_i + m\theta_{i+1} + k\phi : m \in \mathbb{R}\}.$$

We will be concerned with vertices on finite segments of these lines, as the following notation captures. For $a, b, d \in \mathbb{N}$, $i \in \mathcal{I}$ and $k \in [h]$, let

$$L^k_{i,d}(a, b) := \left\{d\theta_i + m\theta_{i+1} + k\phi \;:\; m \in [-a, b)\right\} \cap \mathbb{Z}^3.$$

For G_2 we simply have $|L^k_{i,d}(a, b)| = a + b$, but for G_1 the cardinality depends on the parities



(a) $q{=}2, d{=}2, a{=}2, b{=}3$ (b) $q{=}1, d{=}3, a{=}2, b{=}5$ (c) $q{=}1, d{=}4, a{=}2, b{=}5$

Figure 2: Illustration of $L^k_{1,d}(a, b)$ (in dark gray) for various values of q, d, a and b. The origin and the Cartesian axes are marked by short arrows. In each subfigure, θ_0 points upwards, θ_1 — to the right, θ_2 — downwards and θ_3 — to the left.

of $a + d$ and $b + d$ (see Figure 2) and we have

$$|L^k_{i,d}(a, b)| = \frac{1}{2}(a + b) + \frac{1}{4}(-1)^{a+d} - \frac{1}{4}(-1)^{b+d} \leq \frac{1}{2}(a + b + 1).$$

For simplicity we use the unified bound

$$|L^k_{i,d}(a, b)| \leq \frac{q}{2}(a + b + 2 - q) \leq \frac{q}{2}(a + b + 1). \tag{2}$$

2.1 The fronts structure

Given two vectors $\vec{x} = (x^1, \ldots, x^h), \vec{y} = (y^1, \ldots, y^h)$, we say that $\vec{x}$ *dominates* $\vec{y}$ if $y^k \leq x^k$ for all $k \in [h]$. Call a vector $\vec{x} \in \mathbb{N}^h$ *Lipschitz* if $|x^{k+1} - x^k| \leq 1$ for all $k \in [h-1]$. Given a vector $\vec{x}$, denote by $\mathrm{Lip}(\vec{x})$ the minimal (with respect to domination) Lipschitz vector that dominates $\vec{x}$.

[1] Note the visual cue provided by the letter θ (resp., ϕ) to the vector's horizontal (resp., vertical) orientation.

Observe that for any Lipschitz vector $\vec{x} \in \mathbb{N}^h$ and for any Boolean vector $\vec{y} \in \{0,1\}^h$ we have $\mathrm{Lip}(\vec{x}+\vec{y}) - \vec{x} \in \{0,1\}^h$, since $\vec{x}+\vec{\mathbf{1}}$ is Lipschitz and dominates $\vec{x}+\vec{y}$, where $\vec{\mathbf{1}} \in \mathbb{N}^h$ is the all-ones vector.

We now define the firefronts structure corresponding to four Lipschitz vectors $\vec{\rho}_0, \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3$ (see Figure 3).

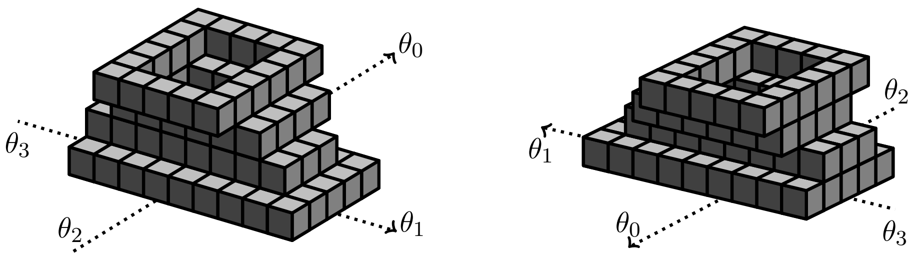


Figure 3: Two isometric projections of the firefronts structure corresponding to the Lipschitz vectors $\vec{\rho}_0 = (2,1,2,3), \vec{\rho}_1 = (5,4,3,2), \vec{\rho}_2 = (2,1,1,2), \vec{\rho}_3 = (3,3,2,2)$.

Definition 2. Given four Lipschitz vectors $\rho = (\vec{\rho}_0, \vec{\rho}_1, \vec{\rho}_2, \vec{\rho}_3)$, where $\vec{\rho}_i = (r_i^k)_{k\in[h]}$, define the *fronts structure* $(L_0(\rho), L_1(\rho), L_2(\rho), L_3(\rho))$ of ρ to be $L_i(\rho) = \bigsqcup_{k\in[h]} L_i^k(\rho)$, where

$$L_i^k(\rho) = L_{i,r_i^k}^k(r_{i-1}^k, r_{i+1}^k) = \left\{ r_i^k\theta_i + m\theta_{i+1} + k\phi \ : \ m \in [-r_{i-1}^k, r_{i+1}^k) \right\} \cap \mathbb{Z}^3.$$

Note that $L_i^k(\rho) \cap L_j^\ell(\rho) = \varnothing$ for $i \neq j \in \mathcal{I}$ and for all $k, \ell \in [h]$.

2.2 Evolution of the fronts structure

As explained in Section 1.2, we inductively define $\vec{\rho}_i(t) = \big(r_i^1(t), \ldots, r_i^h(t)\big)$, a non-decreasing sequence of Lipschitz vectors whose fronts structure we wish to analyze.

Let $\vec{\rho}(0)$ be some initial quadruple of Lipschitz vectors, and assume without loss of generality that $r_i^k(0) \geq 1$ for all $i \in \mathcal{I}$ and $k \in [h]$. Given the initial fronts structure $\big(L_i(\vec{\rho}(0))\big)_{i\in\mathcal{I}}$, define

$$\rho_i(t) := \mathrm{Lip}\big(\vec{\rho}_i(t-1) + \vec{\alpha}_i(t)\big), \tag{3}$$

where the Boolean vector $\vec{\alpha}_i(t) = (a_i^1(t), \ldots, a_i^h(t)) \in \{0,1\}^h$ consists of the *activity indicators* for the levels of the i-th front. That is, we say that the k-th level of the i-th front is *active* if and only if $a_i^k(t) = 1$. The precise definition of $\vec{\alpha}_i(t)$ is delicate, and we postpone this until Section 4.1 (see (17)); for now, it suffices to state that it only depends on the history of the process up until time $t-1$, and that it satisfies two conditions related to the notion of fierity to be defined below (see (5)).

Note that $\vec{\rho}_i(t)$ dominates $\vec{\rho}_i(t-1) + \vec{\alpha}_i(t)$, and thus $\Delta r_i^k(t) \geq a_i^k(t)$. Moreover, $\vec{\rho}_i(t-1)$ is Lipschitz and $\alpha_i(t)$ is Boolean so that $\Delta r_i^k(t) \in \{0,1\}$. In fact, summing over $k \in [h]$ we have

$$\Delta r_i(t) - a_i(t) = \Big\| \vec{\rho}_i(t) - \big(\vec{\rho}_i(t-1) + \vec{\alpha}_i(t)\big) \Big\|_1. \tag{4}$$

It may be useful for the reader to think of the change in this quantity caused by the Lip operations as "the k-th level of a front being *pulled forward* by some other active level of that front". Consider, for instance, $\vec{\rho}_1(t-1) = (5,4,3,2)$ of Figure 3 and suppose that $\vec{\alpha}_1(t) = (1,0,0,0)$. In this case, only the bottom level is active and yet $\vec{\rho}_1(t) = (6,5,4,3)$ and $\Delta r_1(t) - a_1(t) = h-1$.

Henceforth, denote by $L_i^k(t) := L_i^k(\rho(t))$ the k-th level of the i-th front at time t.

Fierity. Next we define the *fierity* $\varphi_i^k(t)$ of the k-th level of the i-th front at time t.

$$\Phi_i^k(t) := L_i^k(t) \cap B(t),$$
$$\varphi_i^k(t) := \Big| \Phi_i^k(t) \Big|.$$

This quantity measures the number of burning vertices on a particular level.

The definition of activity indicators in Section 4.1 will guarantee the following two fierity-related conditions.

$$a_i^k(t) = 0 \quad \text{whenever} \quad \varphi_i^k(t-1) = 0, \tag{5a}$$

$$a_i^k(t) = 1 \quad \text{whenever} \quad \varphi_i^k(t-1) > 4qh^4 - 2. \tag{5b}$$

Next, we inductively define $\Delta F_i^k(t)$, the hitherto uncounted protected vertices on the k-th level of the i-th front at time t. Setting $F_i^k(0) := \varnothing$, this defines the cumulative counterpart $F_i^k(t)$.

$$\Delta F_i^k(t) := \Big((\mathsf{F}(t) \cap L_i^k(t)) \setminus B(t) \Big) \setminus F^k(t-1);$$

$$\Delta f_i^k(t) := |\Delta F_i^k(t)|.$$

Recall that $F^k(t) = \bigsqcup_{i \in \mathcal{I}} F_i^k(t)$ and $F(t) = \bigsqcup_{k \in [k]} F^k(t)$, by our index omitting convention. Observing that $F(t) \subseteq \mathsf{F}(t)$, we have $f(t) \le \mathsf{f}(t)$. Hence, we may replace the assumption $\mathsf{f}(t) = ct$ by the bound $f(t) \le ct$. The reader needs not be alarmed by the visual similarity between f and f (or between F and F) since f and F are no longer required for the lower bound and will thus play no role in Sections 3 and 4 (except for a brief appearance in the proof of Proposition 12 in Section 3.2).

Remark. The task of selecting $\vec{\alpha}_i(t)$ could be thought of as a solitaire sub-game, in which the player takes the challenge of finding a sequence of $\vec{\alpha}_i(t)$ satisfying (5) such that $a(t) > 0$ infinitely often.

2.3 Proof of the main theorem

Theorem 1 naturally consists of an upper bound and a lower bound on $C(G_q)$. In light of the result of [7], we henceforth assume $h \ge 2$. The upper bound, which is nothing more than repeating a two-dimensional strategy in each of the h layers simultaneously, is set by the following claim.

Claim 3. For $q \in [2]$ and for every $h \ge 2$ we have $C(G_q) \le \frac{3}{2} qh$.

Proof. We need to show that for any $\varepsilon > 0$ and any $\mathsf{f}(t) = (\frac{3}{2}qh + \varepsilon)t$, there exists a strategy that allows containment of the fire. Fix $\varepsilon > 0$ and let $\varepsilon' := \varepsilon/h$. Let $S \subset G_q$ be the initial set of burning vertices. By taking the union of horizontal projections, we can find a finite $S' \subset \mathbb{Z}^2$ such that $S \subseteq S' \times [h]$. In [7, Section 3], the second and third authors show the existence of a strategy capable of containing any finite-source fire in $\mathbb{Z} \square \mathbb{Z}$ (resp., $\mathbb{Z} \boxtimes \mathbb{Z}$) for $\mathsf{f}(t) \ge (\frac{3}{2} + \varepsilon')t$ (resp., $\mathsf{f}(t) \ge (3 + \varepsilon')t$). We refer to this strategy, applied to the initial set S', as the layer strategy. To achieve the upper bound for G_q, apply the layer strategy in each layer in parallel to obtain a containment strategy on G_q for

$$\mathsf{f}(t) \ge \left(\frac{3qh}{2} + h\varepsilon' \right) t = \left(\frac{3qh}{2} + \varepsilon \right) t. \qquad \square$$

The lower bound is expressed by the somewhat technical Lemma 4. To state it, we require the following constant. Let

$$\lambda := h + q \max\{r_0(0) + r_2(0), r_1(0) + r_3(0)\}.$$

In light of (2), λ is an upper bound on the total length of any two initial firefronts.

Lemma 4. *Let $q \in [2]$, and suppose that*

$$\varphi(0) \ge \lambda + 55h^5 \tag{6}$$

and that

$$f(t) \leq \frac{3qh}{2} t \text{ for all } t \in \mathbb{N}. \tag{7}$$

Then $\varphi(t) + f(t) \geq 3qht$ *for all* $t \in \mathbb{N}$.

Reduction of the lower bound in Theorem 1 to Lemma 4. Set $r_i^k(0) = 30h^4$ for all $k \in [h]$ and $i \in \mathcal{I}$, and pick the initial burning set to be $B(0) = L(0) = \sqcup_{i \in \mathcal{I}} L_i(0)$. Observe that $r_i(0) = 30h^5$ and $\varphi_i(0) = |L_i(0)| = 30qh^5$ for all $i \in \mathcal{I}$, so indeed

$$120qh^5 = \varphi(0) \geq \lambda + 55h^5 = h + 60qh^5 + 55h^5.$$

Set $c = \frac{3}{2}qh$ and apply Lemma 4 to deduce that $\varphi(t) + f(t) \geq 2ct$ for all $t \in \mathbb{N}$. Now $\varphi(t) \geq ct$ since $f(t) \leq ct$, and by the definition of $\varphi(t)$ we obtain $|B(t)| \geq \varphi(t) \geq ct$. Thus the fire expands indefinitely and is never contained. □

To prove Lemma 4 we use an inductive argument. The induction hypothesis is

$$\mathcal{H}(t): \quad \varphi(t) + f(t) \geq 3qht + \lambda + 52h^5.$$

The base case $\mathcal{H}(t)$ for $t \leq 2h$ is implied by (6) since $3h^5 > 3qht$ for all $t \leq 2h$. It thus remains to prove the following proposition, which is the inductive step. This will be done in Section 4.3.

Proposition 5. *Fix* $t > 2h$. *Assuming (6) and (7), if* $\mathcal{H}(s)$ *holds for all* $s < t$ *then* $\mathcal{H}(t)$ *holds.*

3 Fire growth

In this section we bound from below the increase in the number of burning vertices on $L_i^k(t)$.

3.1 Shifted vertices and the potential of the fronts structure

Towards defining the potential $\mu_i^k(t)$, we need to keep track of vertices that shifted from the i-th firefront to an adjacent front $j = i \pm 1$. To see how such a change might occur, consult Figure 5.

For $i \in \mathcal{I}$, $k \in [h]$ and $t \in \mathbb{N}$ let

$$V_{i,i-1}^k(t) := (L_{i-1}^k(t) \setminus L_{i-1}^k(t-1)) \cap L_i^k(t-1)$$
$$V_{i,i+1}^k(t) := (L_{i+1}^k(t) \setminus L_{i+1}^k(t-1)) \cap \left(L_i^k(t-1)\right)^+$$

The reader should keep in mind that each of $V_{i,i-1}^k(t)$ and $V_{i,i+1}^k(t)$ always consists of at most one vertex; formally, this will be established as part of Proposition 9.

We may now define $p_i^k(t)$, which counts the total change in the number of vertices on the k-th level of the i-th front as the result of transition of vertices between this front and adjacent fronts.

First, define $p_{i,j}^k(t)$ for $i, j \in \mathcal{I}$ satisfying $|i - j| = 1$ as follows. Set $p_{i,j}^k(0) := 0$ and let $p_{i,j}^k(t) := p_{i,j}^k(t-1) + \Delta p_{i,j}^k(t)$, where

$$\Delta p_{i,j}^k(t) := \left| V_{i,j}^k(t) \cap \left(B(t) \cup F(t)\right) \right| - \left| V_{j,i}^k(t) \cap \left(B(t) \cup F(t)\right) \right|.$$

Next, let

$$\begin{aligned} p_i^k(t) &:= p_{i,i+1}^k(t) + p_{i,i-1}^k(t); \\ \Delta p_i^k(t) &:= \Delta p_{i,i+1}^k(t) + \Delta p_{i,i-1}^k(t). \end{aligned}$$

The skew-symmetry in the definition of $\Delta p^k_{i,j}(t)$ leads to cancellations when summing over all directions $i \in \mathcal{I}$. Indeed,

$$\begin{aligned}\Delta p^k(t) &= \sum_{i\in\mathcal{I}}\Big(\Delta p^k_{i,i+1}(t) + \Delta p^k_{i,i-1}(t)\Big) = \sum_{i\in\mathcal{I}} \Delta p^k_{i,i+1}(t) + \sum_{i\in\mathcal{I}} \Delta p^k_{i,i-1}(t) \\ &= \sum_{i\in\mathcal{I}} \Delta p^k_{i,i+1}(t) + \sum_{i\in\mathcal{I}} \Delta p^k_{i+1,i}(t) = 0, \end{aligned} \tag{8}$$

and thus also $\Delta p(t) = 0$ and $p(t) = 0$, for all $t \in \mathbb{N}$.

We now have all the ingredients to define the *potential* of the k-th level of the i-th front as

$$\mu^k_i(t) := \varphi^k_i(t) + f^k_i(t) + p^k_i(t). \tag{9}$$

Note that $\mu(t) = \varphi(t) + f(t)$ by the skew-symmetry observation above, and that for $t = 0$ we simply get $\mu_i(0) = \varphi_i(0)$.

We now present two potential-related bounds, whose proofs are postponed to Section 3.3. The first states that $\Delta\mu^k_i(t)$ is always non-negative, and grows by q whenever the level is active.

Proposition 6. *For all $t \in \mathbb{N}$, $k \in [h]$ and $i \in \mathcal{I}$ we have $\Delta\mu^k_i(t) \geq q a^k_i(t)$.*

In particular, this implies

$$\mu^k_i(t) \geq \mu^k_i(s) \text{ for all } t \geq s. \tag{10}$$

The second bound states that the potential of each firefront exceeds its fierity minus a constant.

Proposition 7. *For all $t \in \mathbb{N}$ and $i \in \mathcal{I}$ we have $\mu_i(t) > \varphi_i(t) - 4qh^5$.*

3.2 Local fire growth bounds

Growth within layers. The following bound is a slight generalization of a result of Fogarty [10, Theorem 1]. To better comprehend this variant, please see Figure 4.

Proposition 8. *Let $q \in \{1, 2\}$, $i \in \mathcal{I}$, $k \in [h]$, and suppose that $\left|L^k_{i,d}(a, b) \cap B(t-1)\right| > 0$ in G_q for some $a, b, d \in \mathbb{N}$. Then, for $\delta_-, \delta_+ \in \{0, 1\}$ we have*

$$\left|\Big(L^k_{i,d+1}(a+\delta_-, b+\delta_+) \sqcup V_- \sqcup V_+\Big) \cap \big(B(t) \cup \mathsf{F}(t)\big)\right| \geq \left|L^k_{i,d}(a, b) \cap B(t-1)\right| + q,$$

where

$$V_- = \begin{cases} \big(\{d\theta_i - a\theta_{i+1} + k\phi\} \cap \mathbb{Z}^3\big) & \delta_- = 0, \\ \varnothing & \delta_- = 1, \end{cases}$$

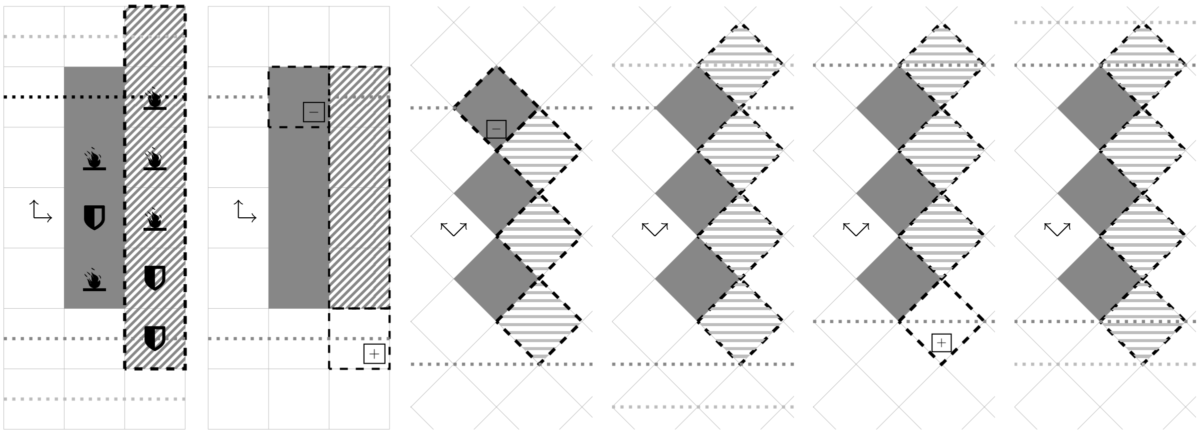
$$V_+ = \begin{cases} \Big(\big\{(d+1)\theta_i + b\theta_{i+1} + k\phi\big\} \cap \mathbb{Z}^3\Big) & \delta_+ = 0, \\ \varnothing & \delta_+ = 1. \end{cases}$$

Proof. Write $A = L^k_{i,d}(a, b) \cap B(t-1)$ and observe that $A \neq \varnothing$ by our assumption. Denote by $x_{\max}$ (resp., $x_{\min}$) the element $d\theta_i + m\theta_{i+1} + k\phi \in A$ with the maximal (resp., minimal) value of m. Let

$$N = (\mathbb{Z}^2 \times \{k\}) \cap A^+ \cap L^k_{i,d+1}(a+1, b+1)$$

and note that $\theta_i \pm \theta_{i+1} \in \mathbb{Z}^3$ for both $q = 1$ and $q = 2$, while $\theta_i \in \mathbb{Z}^3$ only for $q = 2$. It follows that $|N| \geq |A| + q$, since

$$N \supseteq \{x + \theta_i + \theta_{i+1} \;:\; x \in A\} \sqcup \{x_{\min} + \theta_i - \theta_{i+1}\} \sqcup \Big(\{x_{\min} + \theta_i\} \cap \mathbb{Z}^3\Big).$$



(a) $q{=}2,a{=}2,b{=}2,$ $\delta_-{=}\delta_+{=}1$ (b) $q{=}2,a{=}2,b{=}2,$ $\delta_-{=}\delta_+{=}0$ (c) $q{=}1,a{=}3,b{=}3,$ $\delta_-{=}\delta_+{=}0$ (d) $q{=}1,a{=}3,b{=}3,$ $\delta_-{=}\delta_+{=}1$ (e) $q{=}1,a{=}4,b{=}2,$ $\delta_-{=}\delta_+{=}0$ (f) $q{=}1,a{=}4,b{=}2,$ $\delta_-{=}\delta_+{=}1$

Figure 4: The transition from $L^k_{1,1}(a,b)$ to $L^k_{1,2}(a+\delta_-, b+\delta_+)$ for different values of q, a, b, δ_- and δ_+. Actual spread of burning vertices is illustrated in sub-figure (4a). $L^k_{1,1}(a,b)$ vertices are colored with a darker shade, while $L^k_{1,2}(a+\delta_-, b+\delta_+)$ bear a lighter striped color. The origin and the Cartesian axes are marked by short arrows, and when V_- (resp., V_+) is non-empty, its vertex is tagged by ⊟ (resp., ⊞). Dark dotted lines mark $-a\theta_0+\mathbb{R}\theta_1+k\phi$ and $b\theta_0+\mathbb{R}\theta_1+k\phi$, while light dotted lines mark $-(a+\delta_-)\theta_0+\mathbb{R}\theta_1+k\phi$ and $(b+\delta_+)\theta_0+\mathbb{R}\theta_1+k\phi$. Observe, for $q=1$, how the parity of $d+a$ and $d+b$ affects the shape of $L^k_{1,2}(a+\delta_-, b+\delta_+)$, and the cardinality of V_- and V_+.

This resolves the case $\delta_- = \delta_+ = 1$.

If $x_{\min} \neq d\theta_i - a\theta_{i+1} + k\phi$, then $N \subseteq L^k_{i,d+1}(a, b+1)$ (see Figures 4a, 4d, 4e and 4f); otherwise, $V_- \subseteq A \subseteq B(t)$ (see Figures 4b and 4c). This resolves the case $\delta_- > \delta_+$. Similarly, if $x_{\max} \neq d\theta_i + (b-1)\theta_{i+1} + k\phi$, then $N \subseteq L^k_{i,d+1}(a+1, b)$ (see Figures 4a, 4c, 4d and 4f); otherwise, V_+ is non-empty and its sole member is adjacent to $x_{\max}$ so that $V_+ \subseteq B(t) \cup F(t)$ (see Figures 4b and 4e). This resolves the case $\delta_- < \delta_+$. Since $L^k_{i,d+1}(a, b+1) \cap L^k_{i,d+1}(a+1, b) = L^k_{i,d+1}(a,b)$, which resolves the last case $\delta_- = \delta_+ = 0$. □

To apply Proposition 8 for the fronts structure we require some technical computations, provided in the following proposition. Please see Figure 5 for a more intuitive perspective.

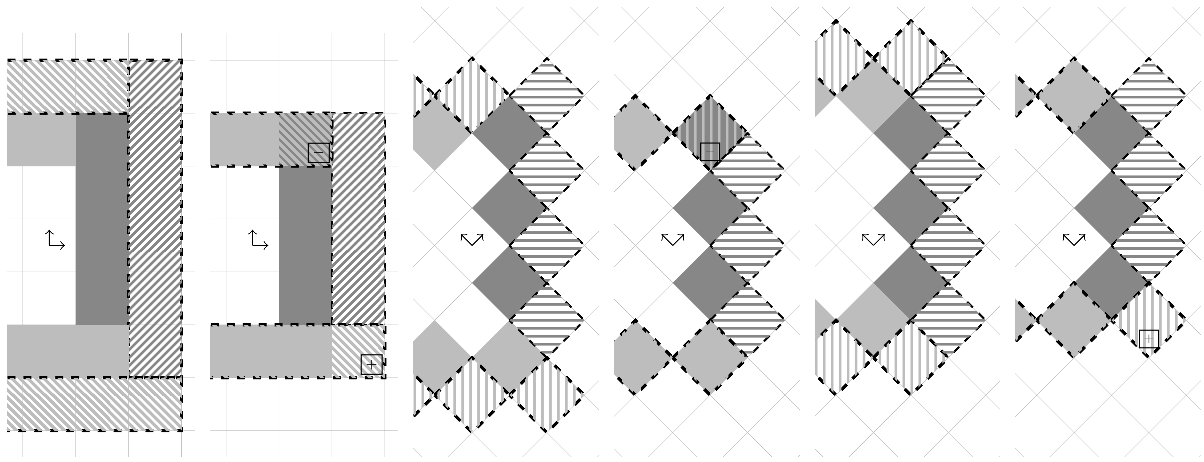
Proposition 9. *For $d = r^k_i(t-1)$, $d_\pm = r^k_{i\pm1}(t-1)$, $\delta = \Delta r^k_i(t)$, $\delta_\pm = \Delta r^k_{i\pm1}(t)$ we have*

$$L^k_{i,d}(d_-, d_+) = L^k_i(t-1), \qquad L^k_{i,d+\delta}(d_- + \delta_-, d_+ + \delta_+) = L^k_i(t). \tag{11}$$

$$V^k_{i,i-1}(t) = \begin{cases} \left(\left\{d\theta^{(q)}_i - d_-\theta^{(q)}_{i+1} + k\phi\right\} \cap \mathbb{Z}^3\right) & \delta(1-\delta_-) = 0, \\ \varnothing & \delta(1-\delta_-) = 1, \end{cases} \tag{12}$$

$$V^k_{i,i+1}(t) = \begin{cases} \left(\left\{(d+1)\theta^{(q)}_i + d_+\theta^{(q)}_{i+1} + k\phi\right\} \cap \mathbb{Z}^3\right) & \delta(1-\delta_+) = 1, \\ \varnothing & \delta(1-\delta_+) = 0. \end{cases} \tag{13}$$

Proof. First, observe that (11) is immediate from Definition 2, recalling that $L^k_i(t) = L^k_i(\rho(t))$. Next, to see (12) and (13), observe that if $\Delta r^k_{i-1}(t) = \delta_- = 1$ then $d_- < r^k_{i-1}(t)$, so that $L^k_{i-1}(t) \cap L^k_i(t-1) = \varnothing$ (see Figures 5a, 5c and 5e). If $\Delta r^k_{i-1}(t) = 0$, then, by definition, an



(a) q=2, $r_{i-1}(t-1)$=2, $r_{i+1}(t-1)$=2, $\Delta r_{i\pm1}(t)$=1

(b) q=2, $r_{i-1}(t-1)$=2, $r_{i+1}(t-1)$=2, $\Delta r_{i\pm1}(t)$=0

(c) q=1, $r_{i-1}(t-1)$=3, $r_{i+1}(t-1)$=3, $\Delta r_{i\pm1}(t)$=1

(d) q=1, $r_{i-1}(t-1)$=3, $r_{i+1}(t-1)$=3, $\Delta r_{i\pm1}(t)$=0

(e) q=1, $r_{i-1}(t-1)$=4, $r_{i+1}(t-1)$=2, $\Delta r_{i\pm1}(t)$=1

(f) q=1, $r_{i-1}(t-1)$=4, $r_{i+1}(t-1)$=2, $\Delta r_{i\pm1}(t)$=0

Figure 5: The interaction between L^k_{i-1}, L^k_{i+1} and L^k_i for time $t-1$ and t, as a result of different parities of $r^k_i(t-1) + r^k_{i-1}(t-1)$, $r^k_i(t-1) + r^k_{i+1}(t-1)$ and different values of $\Delta r^k_{i-1}(t)$, $\Delta r^k_{i+1}(t)$. $L^k_i(t-1)$ and $L^k_i(t)$ are indicated by dark squares and stripes, respectively, while $L^k_{i\pm1}(t-1)$ and $L^k_{i\pm1}(t)$ are indicated by light squares and stripes. The origin and the Cartesian axes are marked by short arrows. Observe that $L^k_{i\pm1}(t-1)$ and $L^k_i(t)$ follow a different striped pattern. Since $\Delta r^k_i(t-1) = 1$ in all sub-figures, $\Delta p^k_{i,i\pm1}(t)$ are always non-negative and $\Delta p^k_{i\pm1,i}(t)$ are non-positive. When $\Delta p^k_{i,i\pm1}(t)$ take non-zero value, the vertex of $V^k_{i,i\pm1}(t-1)$ is tagged by ⊟ or ⊞.

element is in both $L^k_{i-1}(t)$ and $L^k_i(t-1)$ if and only if it is of the form

$$\begin{aligned} r^k_i(t-1)\theta_i - r^k_{i-1}(t)\theta_{i+1} + k\phi &= r^k_i(t-1)\theta_i - \Big(r^k_{i-1}(t-1) + \Delta r^k_{i-1}(t)\Big)\theta_{i+1} + k\phi \\ &= d\theta_i - d_-\theta_{i+1} + k\phi, \end{aligned}$$

and $\delta_- = 0$ (see Figures 5b and 5d). Because $L^k_{i-1}(t)$ and $L^k_i(t)$ are disjoint, we must have $d\theta_i - d_-\theta_{i+1} + k\phi \notin L^k_{i-1}(t-1)$.

Similarly, if $\Delta r^k_{i+1}(t) = \delta_+ = 1$ then $d_+ < r^k_{i+1}(t)$, so that $L^k_{i+1}(t) \cap L^k_i(t-1) = \varnothing$, whereas if $\Delta r^k_{i+1}(t) = 0$ then an element is in both $L^k_{i+1}(t)$ and $\big(L^k_i(t-1)\big)^+$ if it is of the form

$$\Big(r^k_i(t-1) + \beta\Big)\theta_i + r^k_{i+1}(t-1)\theta_{i+1} + k\phi = (d+\beta)\theta_i + d_+\theta_{i+1} + k\phi$$

with $\beta \in \{0, \pm1\}$ (see Figures 5b and 5f). However, we have

$$\begin{aligned} &\big\{(d+\beta)\theta_i + d_+\theta_{i+1} + k\phi \,:\, \beta \in \{0,\pm1\}\big\} \setminus L^k_{i+1}(t-1) \\ &\qquad\qquad = \begin{cases} \big\{(d+1)\theta_i + d_+\theta_{i+1} + k\phi\big\} & \Delta r^k_i(t) = 1, \\ \varnothing & \Delta r^k_i(t) = 0, \end{cases} \end{aligned}$$

and the proposition follows. □

Proposition 9 implies the following, which serves a role in the proof of Proposition 6.

Corollary 10. *Let $t \in \mathbb{N}$, $k \in [h]$ and $i, j \in \mathcal{I}$ such that $|i-j| = 1$. Then $\Delta r^k_i(t) = 0$ implies $V^k_{i,j}(t) = \varnothing$, while $\Delta r^k_i(t) = 1$ implies $V^k_{j,i}(t) = \varnothing$.*

In particular, for all $t \in \mathbb{N}$, $k \in [h]$ and $i \in \mathcal{I}$ we have

$$|\Delta p_i^k(t)| \leq 2 \tag{14}$$

since $\Delta p_{i,i\pm1}^k(t) \in \{-1, 0, 1\}$. Furthermore, by Corollary 10 we obtain the following, which is used to prove Proposition 7.

Corollary 11. *Let $t \in \mathbb{N}$, $k \in [h]$ and $i \in \mathcal{I}$. Then $\Delta r_i^k(t) = 1$ implies $\Delta p_i^k(t) \geq 0$.*

Vertical growth. The inter-layer graph connectivity of G_q implies the following. For $\ell = k \pm 1$, we have

$$\left| L_{i,d}^\ell(a,b) \cap \big(B(t) \cup \mathsf{F}(t)\big) \right| \geq \left| L_{i,d}^k(a,b) \cap B(t-1) \right|. \tag{15}$$

Proposition 12. *Let $i \in \mathcal{I}$ and $k, \ell \in [h]$ such that $|k - \ell| = 1$, and assume that $\Delta r_i^\ell(t) = 1$ and $r_i^k(t-1) = r_i^\ell(t-1) + 1$. Then $\varphi_i^\ell(t) + \Delta f_i^\ell(t) \geq \varphi_i^k(t-1) - 2$.*

Proof. We have $L_i^k(t-1) = L_{i,d}^k(d_-, d_+)$, where $d = r_i^k(t-1) = r_i^\ell(t)$ and $d_\pm = r_{i\pm1}^k(t-1)$. Moreover, by the Lipschitz property of $\vec{\rho}_{i\pm1}(t-1)$ we have

$$d_\pm = r_{i\pm1}^k(t-1) \leq 1 + r_{i\pm1}^\ell(t-1) \leq 1 + r_{i\pm1}^\ell(t),$$

so that

$$L_i^\ell(t) \supseteq L_{i,d}^\ell(d_- - 1, d_+ - 1)$$

and thus

$$\begin{aligned}
\varphi_i^\ell(t) + \Delta f_i^\ell(t) &= \left| \Phi_i^\ell(t) \sqcup \Delta F_i^\ell(t) \right| \\
&= \left| L_i^\ell(t) \cap \big(B(t) \cup \mathsf{F}(t)\big) \right| \\
&\geq \left| L_{i,d}^\ell(d_- - 1, d_+ - 1) \cap \big(B(t) \cup \mathsf{F}(t)\big) \right| \\
&\geq \left| L_{i,d}^\ell(d_-, d_+) \cap \big(B(t) \cup \mathsf{F}(t)\big) \right| - 2 \\
&\geq \varphi_i^k(t-1) - 2,
\end{aligned}$$

where the last inequality is by (15). $\square$

3.3 Potential bounds' proofs

Proof of Proposition 6. We consider two cases separately. First, if $\Delta r_i^k(t) = 0$ then $a_i^k(t) = 0$, so we need to show that $\Delta \mu_i^k(t) \geq 0$. Now

$$\varphi_i^k(t) + \Delta f_i^k(t) = \left| \Phi_i^k(t) \sqcup \Delta F_i^k(t) \right| = \left| \Phi_i^k(t-1) \right| + \left| \Delta \Phi_i^k(t) \sqcup \Delta F_i^k(t) \right|,$$

since $\Phi_i^k(t-1) \subseteq \Phi_i^k(t)$. Moreover, $V_{i,i-1}^k(t) = V_{i,i+1}^k(t) = \varnothing$ by Corollary 10, so

$$\Delta p_i^k(t) = \Delta p_{i,i-1}^k(t) + \Delta p_{i,i+1}^k(t) = -\left| \Big(V_{i-1,i}^k(t) \sqcup V_{i+1,i}^k(t)\Big) \cap \big(B(t) \sqcup F(t)\big) \right|.$$

Putting these together we obtain

$$\begin{aligned}
\Delta \mu_i^k(t) &= \Delta \varphi_i^k(t) + \Delta f_i^k(t) + \Delta p_i^k(t) \\
&= \left| \Delta \Phi_i^k(t) \sqcup \Delta F_i^k(t) \right| - \left| \Big(V_{i-1,i}^k(t) \sqcup V_{i+1,i}^k(t)\Big) \cap \big(B(t) \sqcup F(t)\big) \right| \geq 0,
\end{aligned}$$

since $V_{i,i\pm1}^k(t) \subseteq L_{i\pm1}^k(t) \setminus L_{i\pm1}^k(t-1)$.

Next, if $\Delta r_i^k(t) = 1$ we have $V_{i-1,i}^k(t) = V_{i+1,i}^k(t) = \varnothing$ by Corollary 10, so that

$$\begin{aligned}\varphi_i^k(t) + \Delta p_i^k(t) &= |L_i^k(t) \cap B(t)| + \Delta p_{i,i+1}^k(t) + \Delta p_{i,i-1}^k(t) \\ &= \left|\left(L_i^k(t) \sqcup \left(V_{i,i-1}^k(t) \sqcup V_{i,i+1}^k(t)\right)\right) \cap B(t)\right| + \left|\left(V_{i,i-1}^k(t) \sqcup V_{i,i+1}^k(t)\right) \cap F(t)\right|.\end{aligned}$$

Note that as $\Delta r_i^k(t) = 1$, we have $\Delta F_i^k(t) = (L_i^k(t) \setminus B(t)) \cap F(t)$ so that showing

$$\Delta\mu_i^k(t) = \Delta\varphi_i^k(t) + \Delta p_i^k + \Delta f_i^k \geq q a_i^k(t)$$

reduces to proving that

$$\left|\left(L_i^k(t) \sqcup \left(V_{i,i-1}^k(t) \sqcup V_{i,i+1}^k(t)\right)\right) \cap \left(B(t) \sqcup F(t)\right)\right| = \varphi_i^k(t) + \Delta p_i^k + \Delta f_i^k \geq q a_i^k(t) + \varphi_i^k(t-1).$$

When $\varphi_i^k(t-1) = 0$ we have $a_i^k(t) = 0$ by (5a), and this is straightforward; otherwise, this follows from Proposition 8, using Proposition 9. □

Proof of Proposition 7. It suffices to show that for all $k \in [h]$

$$p_i^k(t) + f_i^k(t) = \mu_i^k(t) - \varphi_i^k(t) > -4qh^4.$$

Fix $t \in \mathbb{N}$ and $k \in [h]$. If $\Delta r_i^k(\tau) = 1$ for all $\tau \in [t]$, then

$$p_i^k(t) = \sum_{\tau \in [t]} \Delta p_i^k(\tau) \geq 0$$

by Corollary 11, so $p_i^k(t) + f_i^k(t) \geq 0$ and we are done; otherwise, let

$$s = \max\{\tau \in [t] \;:\; \Delta r_i^k(\tau) = 0\}.$$

By the definition of s we have $\Delta r_i^k(s) = 0$, so that $a_i^k(s) = 0$ and, by (5b), $\varphi_i^k(s-1) \leq 4qh^4 - 2$.

Next we verify the following inequalities.

$$\begin{aligned} f_i^k(t) &\geq f_i^k(s-1), \\ p_i^k(t) &\geq p_i^k(s), \\ \mu_i^k(s-1) &> 0. \end{aligned}$$

The first follows from the fact that $\Delta f_i^k(\tau) \geq 0$ for all $\tau \in \mathbb{N}$; the second — from the fact that $\Delta p_i^k(\tau) \geq 0$ for $s < \tau \leq t$ which, in turn, follows from Corollary 11 and the choice of s; the last follows from (10) and the fact that $\mu_i^k(0) = \varphi_i^k(0) > 0$.

Putting these three inquealities together, we obtain

$$\begin{aligned} f_i^k(t) + p_i^k(t) &\geq f_i^k(s-1) + p_i^k(s) \\ &= \mu_i^k(s-1) - \varphi_i^k(s-1) + \Delta p_i^k(s) \\ &\geq 0 - (4qh^4 - 2) - 2 = -4qh^4, \end{aligned}$$

where the last inequality is by (14). □

4 Proof of the main technical statement

4.1 Precise evolution of the fronts structure

Finally we are ready to present the definition of the activity indicators $a_i^k(t)$, which concludes the exact description of the evolution of the fronts structure. This is done using an auxiliary indicator function $g_i(t)$. The role of $g_i(t)$ is to indicate whether the i-th front have experienced a period of reduced growth of its potential while having significant fierity (in comparison with the natural qh growth per time-step, guaranteed by Proposition 6 when $a_i^k(t) = 1$ for all $k \in [h]$). This definition guarantees that leaving such a period (that is, to have $g_i(t) = 0$ and $g_i(t-1) = 1$) requires either an increase of $2qh^2$ in the potential, or a reduction of the fierity of the front below a threshold.

Define

$$g_i(t) := \begin{cases} 1 & \Big(\varphi_i(t) > 4qh^5\Big) \text{ and } \Big(\Delta\mu_i(t) < qh\big(1 - g_i(t-1)\big\} + 2qh^2 g_i(t-1)\Big) \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Using this, we define $a_i^k(t)$ as

$$a_i(t) := \begin{cases} 1 & \varphi_i^k(t-1) > \Big(4h\big(1 - g_i(t-1)\big) + 4qh^3 g_i(t-1)\Big)\Big(r_i^k(t-1) - r_i^{\min}(t-1)\Big) \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

It is easy to verify that both (5a) and (5b) hold.

Equipped with the precise definition of $a_i^k(t)$, we state three auxiliary lemmata, whose proofs we postpone to Section 4.4. Lemma 13 establishes that whenever a level is "pulled", its potential growth is typical. Lemma 14 establishes that the duration of a period of reduced potential growth is at most $2h$ time steps. Finally, Lemma 15 establishes that a long period of high fierity can be subdivided into small pieces, such that during each piece (except the last), the average potential growth is typical.

Lemma 13. *Let $t \in \mathbb{N}$, $i \in \mathcal{I}$, and suppose that $\Delta r_i(t) > a_i(t)$. Then $g_i(t) = 0$ and $\Delta\mu_i(t) \geq qh$.*

Lemma 14. *Let $t \in \mathbb{N}$, $i \in \mathcal{I}$, and suppose that $\Delta\mu_i(t+1) < qh$ and $g_i(t) = 0$. Then there exists $t+1 \leq t' \leq t+2h$ with $g_i(t') = 0$.*

Lemma 15. *Let $s, t \in \mathbb{N}$ such that $s < t$ and let $i \in \mathcal{I}$. Suppose that $g_i(s) = 0$ and that for all $s \leq \tau \leq t+2h$ we have $\varphi_i(\tau) > 4qh^5$. Then there exists $t \leq t' \leq t+2h$ such that $g_i(t') = 0$ and*

$$\mu_i(t') - \mu_i(s) \geq qh(t' - s).$$

The following proposition extends Proposition 6; it relates the potential growth with the expansion of the fronts structure.

Proposition 16. *For all $t \in \mathbb{N}$ and $i \in \mathcal{I}$ we have $\Delta\mu_i(t) \geq q\Delta r_i(t)$.*

Proof. Fix $i \in \mathcal{I}$ and $t \in \mathbb{N}$. If $\Delta r_i(t) = a_i(t)$ then we are done by Proposition 6. Otherwise, we have $\Delta r_i(t) > a_i(t)$ and thus $\Delta\mu_i(t) \geq qh$ by Lemma 13. But $qh \geq q\Delta r_i(t)$ since $\Delta r_i^k(t) \in \{0, 1\}$ for all $k \in [h]$. □

Note that $\Delta r_i(t) \geq a_i(t)$ and thus Proposition 16 is indeed stronger than what is obtained by summing Proposition 6 over all $k \in [h]$. It implies the following.

Corollary 17. *For all $t \in \mathbb{N}$ and $i \in \mathcal{I}$ we have*

$$2\varphi_i(t) \leq \lambda - \varphi_{i-1}(0) - \varphi_{i+1}(0) + \mu_{i-1}(t) + \mu_{i+1}(t) \tag{18}$$

Proof. Apply Proposition 16 for $j = i \pm 1$, and sum over $\tau \in [t]$. We get

$$q(r_j(t) - r_j(0)) \leq \mu_j(t) - \mu_j(0) = \mu_j(t) - \varphi_j(0).$$

Next, observe that $\Phi_i^k(t) \subseteq L_i^k(t)$ and use (2) to obtain

$$2\varphi_i^k(t) \leq 1 + q\Big(r_{i-1}^k(t) + r_{i+1}^k(t)\Big).$$

Summing over $k \in [h]$ and plugging in the definition of λ establishes the result. □

4.2 The fierity of two firefronts

The following proposition, which is reminiscent of [7, Lemma 9], establishes that, under suitable assumptions, the sum of the fierity of any two firefronts at two nearby times is somewhat significant.

Proposition 18. *Let $i, j \in \mathcal{I}$ such that $i \neq j$, let $s \leq t \leq s + 2h$ and suppose that (6) and (7) hold, and $\mathcal{H}(s)$ is satisfied. Then $\varphi_i(s) + \varphi_j(t) > 8qh^5$.*

Proof. We establish the proposition using the following inequality, which we verify below.

$$\varphi(s) - f(s) < \lambda + 8qh^5 + 2\varphi_i(s) + 2\varphi_j(s). \tag{19}$$

Indeed, $\mathcal{H}(s)$ implies

$$\varphi(s) + f(s) \geq \lambda + 52h^5 + 3qhs,$$

and since

$$52h^5 \geq 24qh^5 + 32h^2 > 24qh^5 + 8h^2 + 3qh(t - s),$$

we have

$$\begin{aligned}\varphi(s) - f(s) &\geq \lambda + 24qh^5 + 8h^2 + 3qht - 2f(s)\\ &\geq \lambda + 24qh^5 + 8h^2 + 2f(t) - 2f(s),\end{aligned}$$

where the last inequality is by (7). Using (19), we get

$$\varphi_i(s) + \varphi_j(s) > 8qh^5 + 4h^2 + f(t) - f(s) \geq 8qh^5 + 4h^2 + f_j(t) - f_j(s).$$

Thus it remains to show that

$$\varphi_j(t) + f_j(t) + 4h^2 \geq \varphi_j(s) + f_j(s).$$

This is implied by (10) and (14) since

$$\Delta\varphi_j(\tau) + \Delta f_j(\tau) = \Delta\mu_j(\tau) - \Delta p_j(\tau) \geq -2h$$

for all $s < \tau \leq t$ and thus

$$\varphi_j(t) + f_j(t) \geq \varphi_j(s) + f_j(s) - 2h(t - s) \geq \varphi_j(s) + f_j(s) - 4h^2.$$

Now we verify (19) for the two possible cases.
Adjacent fronts. Without loss of generality, assume $i = 0$ and $j = 1$. Use (18) on the third and fourth fronts to obtain

$$2\varphi_2(s) + 2\varphi_3(s) \leq 2\lambda - \varphi(0) + \mu(s) \leq \lambda + \mu(s),$$

where $\varphi(0) \geq \lambda$ by (6). Subtracting both sides from $2\varphi(s) + \lambda$ we get

$$\lambda + 2\varphi_0(s) + 2\varphi_1(s) = \lambda + 2\varphi(s) - \big(2\varphi_2(s) + 2\varphi_3(s)\big) \geq 2\varphi(s) - \mu(s) = \varphi(s) - f(s),$$

which implies (19).

Opposing fronts. Without loss of generality assume $i = 0$ and $j = 2$. Use (18) for the second and fourth fronts to obtain

$$\varphi_1(s) + \varphi_3(s) \le \lambda - \varphi_0(0) - \varphi_2(0) + \mu_0(s) + \mu_2(s) \le \lambda + \mu_0(s) + \mu_2(s),$$

since $\varphi_0(0), \varphi_2(0) \ge 0$. Next use Proposition 7 for the second and fourth fronts to obtain

$$\varphi_1(s) + \varphi_3(s) < \mu_1(s) + \mu_3(s) + 8qh^5.$$

Altogether,

$$2\varphi_1(s) + 2\varphi_3(s) < \lambda + \mu(s) + 8qh^5,$$

which implies (19) also in this case. □

4.3 Proof of the induction step

Proof of Proposition 5. The proof relies on an induction on the following property. We say that (s_0, s_1, s_2, s_3) *control* time s if these satisfy $s \le s_i \le s + 4h$ and $g_i(s_i) = 0$ for all $i \in \mathcal{I}$, $|s_i - s_j| \le 2h$ for all $i, j \in \mathcal{I}$, and

$$\sum_{i\in\mathcal{I}} \mu_i(s_i) \ge \varphi(0) - qhs_{\min} + \sum_{i\in\mathcal{I}} qhs_i. \tag{20}$$

First note that if time $t - 2h$ is controlled by some (t_0, t_1, t_2, t_3) then $\mathcal{H}(t)$ holds. Indeed, by (10) and (20) we have

$$\begin{aligned}\mu(t) = \sum_{i\in\mathcal{I}} \mu_i(t) \ge \sum_{i\in\mathcal{I}} \mu_i(t_i) \ge \varphi(0) - qht_{\min} + \sum_{i\in\mathcal{I}} qht_i \ge \varphi(0) + 3qht_{\min}\\ \ge \varphi(0) + 3qh(t - 2h) > \varphi(0) + 3qht - 2h^5 > \lambda + 3qht + 52h^5,\end{aligned}$$

where the last inequality is due to (6). Thus $\mathcal{H}(t)$ holds.

We prove by induction that each time $y \le t-2h$ is controlled by some (y_0, y_1, y_2, y_3). Clearly time $y = 0$ is controlled by $(0, 0, 0, 0)$; it remains to show that if $\mathcal{H}(s)$ holds for all $s < t$ and time y is controlled, then time $y + 1$ is also controlled. To do so, we show that if (y_0, y_1, y_2, y_3) control time y, either they control time $y + 1$ too, or there exist (s_0, s_1, s_2, s_3) with $s_i \ge y_i$ and at least one i such that $s_i > y_i$ that control either time y or time $y+1$. By iterating this finitely many times, the result will follow.

To this end, assume that (y_0, y_1, y_2, y_3) control time y. If $y < y_{\min}$, then (y_0, y_1, y_2, y_3) also control $y + 1$. Assume thus, without loss of generality, that $y_0 = y = y_{\min}$.

If $\Delta\mu_0(y_0 + 1) \ge qh$, then it is easy to verify that $g_0(y_0 + 1) = 0$ and $\{y_0 + 1, y_1, y_2, y_3\}$ satisfy (20), so $\{y_0+1, y_1, y_2, y_3\}$ control time y. Otherwise, $\Delta\mu_0(y_0+1) < qh$ so, by Lemma 14, there is $y_0 < s_0 < y_0 + 2h$ with $g_0(s_0) = 0$. We consider two cases, corresponding to the two reasons in (16) to have $g_0(s_0) = 0$. First, in the case

$$\mu_0(s_0) - \mu_0(y_0) \ge 2qh^2 \ge qh(s_0 - y_0), \tag{21}$$

use (20) for (y_0, y_1, y_2, y_3) to obtain that

$$\begin{aligned}\mu_0(s_0) + \sum_{j\in[3]} \mu_j(y_j) &= \mu_0(s_0) - \mu_0(y_0) + \sum_{i\in\mathcal{I}} \mu_i(y_i)\\ &\ge qh(s_0 - y_0) + \varphi(0) - qhy_0 + \sum_{i\in\mathcal{I}} qhy_i\\ &\ge \varphi(0) - qh\min\{s_0, y_1, y_2, y_3\} + qhs_0 + \sum_{j\in[3]} qhy_j\end{aligned}$$

and thus (s_0, y_1, y_2, y_3) control time y.

If (21) does not hold, then $\varphi_0(s_0) < 4qh^5$ by (16). For $j \in [3]$, set $s_j = y_j$ if $y_j \geq s_0$; otherwise, apply Lemma 15 with (y_j, s_0) to deduce the existence of $s_0 \leq s_j < s_0 + 2h$ with $g_j(s_j) = 0$ that satisfies

$$\mu_j(s_j) - \mu_j(y_j) \geq qh(s_j - y_j).$$

The conditions for Lemma 15 are satisfied since $g_j(y_j) = 0$ and for all $s_0 - 2h \leq y_j \leq \tau \leq s_0 + 2h$ we may apply Proposition 18 for $s = \min\{s_0, \tau\}$ and $\max\{s_0, \tau\}$ to obtain

$$\varphi_j(\tau) > \varphi_j(\tau) + \varphi_0(s_0) - 4qh^5 \geq 8qh^5 - 4qh^5 = 4qh^5.$$

Note that $s \leq s_0 < y_0 + 2h < t$ so $\mathcal{H}(s)$ indeed holds.

Next, use (20) for (y_0, y_1, y_2, y_3) to obtain that

$$\begin{aligned}\mu_0(y_0) + \sum_{j\in[3]} \mu_j(s_j) &= \sum_{j\in\mathcal{I}} \mu_j(y_j) + \sum_{j\in[3]} (\mu_j(s_j) - \mu_j(y_j)) \\ &\geq \varphi(0) + \sum_{j\in[3]} qhy_j + \sum_{j\in[3]} qh(s_j - y_j) = \varphi(0) + \sum_{j\in[3]} qhs_j.\end{aligned}$$

Now $\mu_0(y_0) \leq \mu_0(s_0)$ since $y_0 < s_0$, so

$$\sum_{i\in\mathcal{I}} \mu_i(s_i) \geq \varphi(0) + \sum_{j\in[3]} qhs_j = \varphi(0) - qhs_{\min} + \sum_{i\in\mathcal{I}} qhs_i,$$

as $s_0 = s_{\min}$. This establishes that time $y + 1$ is controlled by (s_0, s_1, s_2, s_3), which concludes the proof. □

4.4 Proofs of auxiliary lemmata

Proof of Lemma 13. Recall that $\Phi_i^k(t) = L_i^k(t) \cap B_i^k(t)$. First we show that there exist $k, \ell \in [h]$, $|k - \ell| = 1$ such that

$$\begin{array}{lll} r_i^k(t) = r_i^k(t-1) + 1 & r_i^k(t-1) = r_i^\ell(t-1) + 1 & r_i^\ell(t) = r_i^\ell(t-1) + 1 \\ a_i^k(t) = 1 & & a_i^\ell(t) = 0 \end{array}$$

To see this, recall the definition of $r_i^k(t)$ given in (3) and let $\vec{v}_i := \vec{\rho}_i(t) - \big(\vec{\rho}_i(t-1) + \vec{\alpha}_i(t)\big)$. Observe that $\vec{v}_i$ is not the all-zero vector by the assumption $\|\vec{v}_i\| = \Delta r_i(t) - a_i(t) > 0$. We select $\ell \in [h]$ among the non-zero coordinates of $\vec{v}_i$ and an adjacent $k = \ell \pm 1$ for which $a_i^k(t) = 1$. By the Lipschitz property, such k and ℓ exist.

Observe that $a_i^k(t) = 1, a_i^\ell(t) = 0$, so by (17),

$$\begin{aligned}\varphi_i^k(t-1) &> \Big(4h\big(1 - g_i(t-1)\big) + 4qh^3 g_i(t-1)\Big)\Big(r_i^k(t-1) - r_i^{\min}(t-1)\Big);\\ \varphi_i^\ell(t-1) &\leq \Big(4h\big(1 - g_i(t-1)\big) + 4qh^3 g_i(t-1)\Big)\Big(r_i^\ell(t-1) - r_i^{\min}(t-1)\Big).\end{aligned}$$

Since $r_i^k(t-1) = r_i^\ell(t-1) + 1$ we get

$$\varphi_i^k(t-1) - \varphi_i^\ell(t-1) > 4h\big(1 - g_i(t-1)\big) + 4qh^3 g_i(t-1) \geq 4h.$$

Together with $\varphi_i^\ell(t) + \Delta f_i^\ell(t) \geq \varphi_i^k(t-1) - 2$, obtained by Proposition 12, we have

$$\Delta\mu_i^\ell(t) = \Delta\varphi_i^\ell(t) + \Delta f_i^\ell(t) + \Delta p_i^\ell(t) > 4h - 4,$$

where $\Delta p_i^\ell(t) \geq -2$ by (14). Now

$$\Delta\mu_i(t) \geq \Delta\mu_i^\ell(t) > 4h - 4 \geq qh,$$

using $h \geq 2$, and the proposition follows. □

Proof of Lemma 14. Apply Proposition 6 together with $\Delta\mu_i(t+1) < qh$ to obtain $a_i(t+1) < h$, which ensures the existence of $\ell \in [h]$ for which $a_i^\ell(t+1) = 0$. This implies

$$\varphi_i^\ell(t) \le 4h\Big(r_i^\ell(t) - r_i^{\min}(t)\Big) \le 4h(h-1), \tag{22}$$

by (17) and our assumption $g_i(t) = 0$. Let $t' = \min\{\tau > t \ : \ g_i(\tau) = 0\}$. We need to show $t' \le t + 2h$. Note that

$$\Delta\varphi_i^\ell(\tau) \le \Delta\mu_i^\ell(\tau) - \Delta p_i^\ell(\tau) \le \Delta\mu_i(\tau) + 2 \le 2qh^2 + 2 \tag{23}$$

for all $t+1 \le \tau \le t'$ by (14) and (16). We consider three cases separately.

First, we consider the case in which there exists $t+2 \le \tau \le t+2h-1$ for which $a_i^\ell(\tau) = 1$, denoting by T the minimal such time. If $t' < T-1$ we are done; otherwise, by (22) and (23) we have

$$\begin{aligned}\varphi_i^\ell(T-2) = \varphi_i^\ell(t) + \sum_{\tau=t+1}^{T-2} \Delta\varphi_i^\ell(\tau) &\le 4h(h-1) + (2qh^2+2)(T-t-2) \\ &\le 4h(h-1) + (2qh^2+2)(2h-3) < 4qh^3 - 2qh^2 - 2,\end{aligned}$$

while $\varphi_i^\ell(T-1) > 4qh^3$ (by (17)). Hence $\Delta\varphi_i^\ell(T-1) > 2qh^2 + 2$. Using again the fact that $\Delta p_i^\ell(T-1) \ge -2$, we obtain, by (16), that $t' = T-1$.

Second, we consider the case in which there is a $k \in [h]$ such that $a_i^k(\tau) = 1$ for every $t+1 \le \tau \le t+2h-1$. This implies

$$r_i^k(t+2h-1) - r_i^k(t) = 2h-1.$$

By the Lipschitz property of $\vec{\rho}_i(t)$ and $\vec{\rho}_i(t+2h-1)$ we have

$$r_i^\ell(t) - (h-1) \le r_i^k(t) \quad \text{and} \quad r_i^k(t+2h-1) - (h-1) \le r_i^\ell(t+2h-1),$$

so

$$r_i^\ell(t+2h-1) - r_i^\ell(t) \ge r_i^k(t+2h-1) - r_i^k(t) - 2(h-1) \ge 1.$$

Let $T = \min\{\tau \ge t \ : \ \Delta r_i^\ell(\tau) = 1\}$ and note that $a_i^\ell(T) = 0 < \Delta r_i^\ell(T)$. Now $\Delta r_i(T) > a_i(T)$ and by Lemma 13 we obtain $t' = T$.

Lastly, in the remaining case, for each $k \in [h]$ there exists $t+1 \le \tau^k \le t+2h-1$ satisfying $a_i^k(\tau^k) = 0$. In this case, by (17) we have $\varphi_i^k(\tau^k) < 4qh^3(h-1)$. If $t' < \tau^{\max}$ we are done; otherwise, by (23) we have $\Delta\varphi_i^k(\tau) \le 2qh^2 + 2$ for $\tau^k < \tau \le \tau^{\max}$. Thus

$$\begin{aligned}\varphi_i(\tau^{\max}) &= \sum_{k\in[h]} \varphi_i^k(\tau^k) + \Big(\varphi_i^k(\tau^{\max}) - \varphi_i^k(\tau^k)\Big) \\ &\le \sum_{k\in[h]} 4qh^3(h-1) + (2qh^2+2)(\tau^{\max} - \tau^k) \\ &\le \sum_{k\in[h]} 4qh^3(h-1) + (2qh^2+2)(2h-2) \\ &= 4h(h-1)(h+1)(qh^2+1) < 4qh^5,\end{aligned}$$

which gives $t' \le \tau^{\max} + 1$. The proposition follows. □

Proof of Lemma 15. For all $s \le \tau \le t + 2h$ we have

$$g_i(t) := \begin{cases} 1 & \Delta\mu_i(\tau) < qh(1 - g_i(\tau-1)) + 2qh^2 g_i(\tau-1) \\ 0 & \text{otherwise.}\end{cases} \tag{24}$$

by our assumption $\varphi_i(\tau) > 4q^5$ and the definition of g_i in (16). Let $\tau_0 = s$ and inductively set $\tau_n = \min\{\tau > \tau_{n-1} \,:\, g_i(\tau) = 0\}$ for $n \geq 1$. We show that $t' = \tau_m$ satisfies the required property, where $m = \min\{n \in \mathbb{N} \,:\, \tau_n > t\}$.

First note that $\tau_n \leq \tau_{n-1} + 2h$ for all $n \in [m]$, and in particular $t' \leq \tau_{m-1} + 2h \leq t + 2h$. Indeed, if $\tau_n > \tau_{n-1}+1$, then $g_i(\tau_{n-1}+1) = 1$, and, by (24), $\Delta\mu_i(\tau_{n-1}+1) < qh$. Thus we may apply Lemma 14 with τ_{n-1} to obtain $\tau_n \leq \tau_{n-1} + 2h$. Next, $g_i(\tau_n) = 0$ for all $n \in [m]$ so

$$\Delta\mu_i(\tau_n) \geq qh(1 - g_i(\tau_n - 1)) + 2qh^2 g_i(\tau_n - 1)$$

by (24). Now either $\tau_n - 1 = \tau_{n-1}$ so that

$$\Delta\mu_i(\tau_n) \geq qh = qh(\tau_n - \tau_{n-1}),$$

or $g_i(\tau_n - 1) = 1$, in which case

$$\Delta\mu_i(\tau_n) \geq 2qh^2 \geq qh(\tau_n - \tau_{n-1}).$$

In either case, by (10) this implies

$$\mu_i(\tau_n) - \mu_i(\tau_{n-1}) \geq qh(\tau_n - \tau_{n-1}).$$

Summing this for $n \in [m]$ we obtain

$$\mu_i(t') - \mu_i(s) = \mu_i(\tau_m) - \mu_i(\tau_0) \geq qh(\tau_m - \tau_0) = qh(t' - s),$$

and the lemma follows. □

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