Rigidity of 3-colorings of the discrete torus

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Abstract

We prove that a uniformly chosen proper 3-coloring of the d-dimensional discrete torus has a very rigid structure when the dimension d is sufficiently high. We show that with high probability the coloring takes just one color on almost all of either the even or the odd sub-torus. In particular, one color appears on nearly half of the torus sites. This model is the zero temperature case of the 3-state anti-ferromagnetic Potts model from statistical physics.

Our work extends previously obtained results for the hypercube, and for the discrete torus with specific boundary conditions. The main challenge in this extension is to overcome certain topological obstructions which appear when no boundary conditions are imposed on the model. These are addressed by developing discrete analogues of appropriate results from algebraic topology. This theory is developed in some generality and may be of use in the study of other models.

1 Introduction

We study proper 3-colorings of $\mathbb{T}_n^d$, the d-dimensional discrete torus $(\mathbb{Z}/n\mathbb{Z})^d$, whose side length n is even. Our main theorem is that in high dimensions, a uniformly chosen proper 3-coloring of $\mathbb{T}_n^d$ is nearly constant on one of the two bipartition classes of $\mathbb{T}_n^d$. Precisely, denote the partite classes of $\mathbb{T}_n^d$ by V^0 and V^1. A *proper* 3-*coloring* of $\mathbb{T}_n^d$ is a function $f\colon \mathbb{T}_n^d \to \{0,1,2\}$ satisfying $f(v) \neq f(w)$ whenever v and w are adjacent in $\mathbb{T}_n^d$. Denote by $\mathrm{CP}_{i,k}(f)$ the proportion of color k on V^i, that is,

$$\mathrm{CP}_{i,k}(f) := \frac{|\{v \in V^i \ : \ f(v) = k\}|}{|V^i|}.$$

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Theorem 1.1. *There exist $d_0, c > 0$ such that for every integer $d \geq d_0$ and every even n, a uniformly chosen proper 3-coloring $f\colon \mathbb{T}_n^d \to \{0,1,2\}$ satisfies*

$$\mathbb{E}\left(\min_{i\in\{0,1\}}(\mathrm{CP}_{i,k}(f))\right) \leq \exp\left(-\frac{cd}{\log^2 d}\right) \quad \textit{for all } k \in \{0,1,2\}.$$

Thus, the theorem asserts that typically in high dimensions, for each color there is a partite class on which the color hardly appears. Equivalently, one of the partite classes is dominated by a single color.

The next section describes the main idea of the proof. More precise definitions are given in Section 2.

1.1 Main idea of the proof

Our proof of Theorem 1.1 exploits a connection between proper 3-colorings and height functions, which we now describe. It is convenient to introduce the required notions on a general graph. Suppose G is a connected, bipartite graph with a fixed vertex $v_0 \in V(G)$. Let $\mathrm{Col}(G, v_0)$ be the set of all proper 3-colorings of G taking the value 0 at v_0. That is,

$$\mathrm{Col}(G, v_0) := \{f : V(G) \to \{0,1,2\} \,:\, f(v_0) = 0,\ f(v) \neq f(w) \text{ when } (v,w) \in E(G)\}. \tag{1}$$

An integer-valued function on $V(G)$ is called a *homomorphism height function* on G, or simply height function or HHF, if it differs by exactly one between adjacent vertices of G. Let $\mathrm{Hom}(G, v_0)$ be the set of all homomorphism height functions on G which take the value 0 at v_0. Precisely,

$$\mathrm{Hom}(G, v_0) := \{f : V(G) \to \mathbb{Z} \,:\, f(v_0) = 0,\ |f(v) - f(w)| = 1 \text{ when } (v,w) \in E(G)\}. \tag{2}$$

In this paper, we always take G to be either $\mathbb{T}_n^d$ or $\mathbb{Z}^d$ for some n and d. We consider both $\mathbb{T}_n^d$ and $\mathbb{Z}^d$ to come with a fixed coordinate system and denote by $\mathbf{0}$ the vector $(0, 0, \ldots, 0)$ in that system. For these graphs, we abbreviate $\mathrm{Col}(G, \mathbf{0})$ to $\mathrm{Col}(G)$ and $\mathrm{Hom}(G, \mathbf{0})$ to $\mathrm{Hom}(G)$.

The connection we need between proper colorings and height functions is summarized by the following two facts:

1. For any graph G, $v_0 \in V(G)$ and $h \in \mathrm{Hom}(G, v_0)$, the function $g : V(G) \to \{0,1,2\}$ defined by

   $$g(v) := h(v) \bmod 3$$

   belongs to $\mathrm{Col}(G, v_0)$.

2. When $G = \mathbb{Z}^d$, the above correspondence defines a *bijection* between $\mathrm{Hom}(\mathbb{Z}^d)$ and $\mathrm{Col}(\mathbb{Z}^d)$.

The first fact is straightforward. The second fact appears to be well-known to experts in the field. It is a consequence of the fact that the basic 4-cycles in $\mathbb{Z}^d$ span all other cycles.

Our goal in this work is to use the above correspondence to transfer known results on height functions, proved in [15], to results on colorings, thereby obtaining Theorem 1.1. Our task is, however, made complicated by the following obstruction. The above correspondence is not a bijection when the graph $G = \mathbb{T}_n^d$. In other words, there exist colorings in $\mathrm{Col}(\mathbb{T}_n^d)$ which are not the modulo 3 of any height function in $\mathrm{Hom}(\mathbb{T}_n^d)$. For instance, the coloring 012012 of $\mathcal{T}_6^1$ provides one such example. The source of this problem is of a *topological* nature, stemming from the fact that the torus has non-contractible cycles. This poses a major difficulty, preventing a direct use of the known results on height functions. The following theorem, whose proof occupies most of this paper, provides a way around this difficulty. It shows that the above correspondence is, nonetheless, close to being bijective when the dimension d is sufficiently high.

Theorem 1.2. *There exist d_0 and $c > 0$ such that for every integer $d \geq d_0$ and every even n, a uniformly chosen proper 3-coloring of $\mathbb{T}_n^d$ satisfies*

$$\mathbb{P}(f \text{ is not the modulo } 3 \text{ of some HHF on } \mathbb{T}_n^d) \leq \exp(-c_d n^{d-1}),$$

with $c_d = \frac{c}{d \log^2 d}$.

In the next section we explain how Theorem 1.1 follows from the above theorem and a result on height functions proved in [15]. In Section 1.3 we present some background. The rest of the paper is devoted to the proof of Theorem 1.2. Section 2 contains the first part of the proof and a proof overview. The proof is inspired by ideas from algebraic topology but the necessary tools are developed completely in the discrete setting. We believe that some of these tools could prove useful in other models as well, especially the trichotomy theorems of Section 3, Theorem 3.2 and Theorem 3.4, which deal with discrete counterparts of manifolds of codimension one. The connection between our work and algebraic topology is expounded upon in Section 2.4. Section 6 is dedicated to remarks and open problems.

1.2 Proof of Theorem 1.1

In this section we deduce Theorem 1.1 from Theorem 1.2 and a result of [15] on the fluctuations of typical homomorphism height functions on $\mathbb{T}_n^d$.

We start with the following lemma, which states the required result on the typical behavior of height functions.

Lemma 1.3. *There exist $c > 0$ and d_0 such that in all dimensions $d \geq d_0$, if h is uniformly sampled from $\mathrm{Hom}(\mathbb{T}_n^d)$ then*

$$\mathbb{P}(|h(u) - h(v)| \geq 3) \leq \exp\left(-\frac{cd}{\log^2 d}\right) \qquad \forall u, v \in \mathbb{T}_n^d.$$

Proof. Theorem 2.1 in [15] gives, in particular, that there exist $c > 0$ and d_0 such that in all dimensions $d \geq d_0$ and for every $u, v \in \mathbb{T}_n^d$, if h is uniformly sampled from $\mathrm{Hom}(\mathbb{T}_n^d, u)$, then

$$\mathbb{P}(|h(v)| \geq 3) \leq \exp\left(-\frac{cd}{\log^2 d}\right).$$

The lemma follows from this by using the fact that the mapping $T_u\colon \mathrm{Hom}(\mathbb{T}_n^d) \to \mathrm{Hom}(\mathbb{T}_n^d, u)$ defined by $T_u(h)(v) := h(v) - h(u)$ is a bijection. □

We are now ready to prove Theorem 1.1. First, observe that by symmetry, it suffices to prove the theorem for a uniformly chosen coloring in $\mathrm{Col}(\mathbb{T}_n^d)$, i.e., a coloring normalized at $\mathbf{0}$.

Let f be uniformly chosen from $\mathrm{Col}(\mathbb{T}_n^d)$. Recall that

$$\mathrm{CP}_{i,k}(f) = \frac{|\{v \in V^i \ : \ f(v) = k\}|}{|V^i|},$$

where V^0 and V^1 are the partite classes of $\mathbb{T}_n^d$. Fix $k \in \{0, 1, 2\}$ and let

$$X := \min_{i \in \{0,1\}} \mathrm{CP}_{i,k}\,.$$

We need to show that $\mathbb{E}(X) \leq \exp(-cd/\log^2 d)$ for some $c > 0$ and all sufficiently high d.

Fix d sufficiently high and $c > 0$ sufficiently small for the following arguments. Define the event

$$A := \{f \text{ is the modulo 3 of some HHF in } \mathrm{Hom}(\mathbb{T}_n^d)\}.$$

By symmetry again, Theorem 1.2 implies that

$$\mathbb{P}(A^c) \leq \exp\left(-\frac{c}{d\log^2 d} n^{d-1}\right).$$

Hence,

$$\mathbb{E}(X) = \mathbb{E}(X \mathbb{1}_A) + \mathbb{E}(X \mathbb{1}_{A^c}) \leq \mathbb{E}(X|A) + \exp\left(-\frac{c}{d\log^2 d} n^{d-1}\right). \tag{3}$$

Thus we focus on estimating $\mathbb{E}(X|A)$. Conditioning on A, there exists some $h \in \mathrm{Hom}(\mathbb{T}_n^d)$ for which $f \equiv h \pmod 3$. Moreover, since distinct functions in $\mathrm{Hom}(\mathbb{T}_n^d)$ give rise to distinct colorings in $\mathrm{Col}(\mathbb{T}_n^d)$ under the modulo 3 operation, it follows that, conditioned on A, h is uniformly distributed in $\mathrm{Hom}(\mathbb{T}_n^d)$.

Now note that if $u, v \in \mathbb{T}_n^d$ are vertices in different partite classes of $\mathbb{T}_n^d$ then $h(u)$ and $h(v)$ have different parity. Thus, for such vertices, we have the following containment of events,

$$\{f(u) = f(v)\} = \{h(u) \equiv h(v) \pmod 3\} \subseteq \{|h(u) - h(v)| \geq 3\}.$$

We conclude that X satisfies the following relation.

$$X^2 = \frac{1}{|V^0|^2} \min_{i \in \{0,1\}} |\{v \in V^i \ : \ f(v) = k\}|^2 \leq \frac{1}{|V^0|^2} |\{v \in V^0 \ : \ f(v) = k\}| \cdot |\{v \in V^1 \ : \ f(v) = k\}| \leq$$
$$\leq \frac{1}{|V^0|^2} \sum_{u \in V^0,\, v \in V^1} \mathbb{1}_{(f(u)=f(v))} \leq \frac{1}{|V^0|^2} \sum_{u \in V^0,\, v \in V^1} \mathbb{1}_{(|h(u)-h(v)| \geq 3)}.$$

Hence, we may use Lemma 1.3 to deduce that

$$\mathbb{E}(X|A) \leq \sqrt{\mathbb{E}(X^2|A)} \leq \frac{1}{|V^0|}\sqrt{\sum_{u\in V^0,\, v\in V^1} \mathbb{P}(|h(u)-h(v)|\geq 3)} \leq \exp\left(-\frac{cd}{\log^2 d}\right). \tag{4}$$

Together with (3), this establishes Theorem 1.1.

1.3 Background and related works

Our work is not the first to establish rigidity of proper 3-colorings in high dimensions. However, it is the first to do so when no boundary conditions are imposed. Previously, a result analogous to Theorem 1.1 in which the proper 3-coloring is sampled from the set of colorings with 'zero boundary conditions' was established in [15], and also by Galvin, Kahn, Randall and Sorkin in [6]. The restriction to such 'zero boundary conditions' makes the problem simpler from a topological point of view since it essentially removes the non-trivial cycles of $\mathbb{T}_n^d$, rendering the correspondence described in section 1.1 into a bijection of height functions and proper 3-colorings with these boundary conditions. The results of [15] and [6] imply Roman Kotecký's conjecture (see [12] for context and [6] for additional details), that the proper 3-coloring model admits at least 6 different Gibbs states in high dimensions.

Other earlier works include that of Galvin and Randall [7] who established bounds on the mixing time for Glauber dynamics in the proper 3-coloring model on $\mathbb{T}_n^d$. In addition, Kahn [10] and Galvin [4] established a version of Theorem 1.1 for the hypercube graph $\{0,1\}^d$.

In statistical physics terminology, the proper 3-coloring model is the same as the zero temperature case of the antiferromagnetic 3-state Potts model. It is expected that the analog of our result continues to hold for small, positive temperature, but this remains unproven. In two dimensions, the model is equivalent to the uniform six-vertex, or square ice, model (this was pointed out by Andrew Lenard, see [14]). It is expected that the analog of Theorem 1.1 fails in two dimensions, as the square ice model is conjectured to be in a disordered phase, in the sense that the model should have a unique Gibbs state when $d = 2$. However, it may well be that multiple Gibbs states exist already for any $d \geq 3$. Kotecký, Sokal and Swart [13] have shown that there are *planar* lattices for which the model does have multiple Gibbs states. Moreover, Huang et. al. [9] have shown that multiple Gibbs states exist for proper q-colorings with arbitrary large q on suitably chosen, q-dependent, planar lattices.

It is conjectured that the rigidity phenomenon described by Theorem 1.1 has an analog for proper colorings with more than 3 colors. Specifically, that for any $q \geq 4$ there exists a $d_0(q)$ such that a uniformly sampled proper q-coloring of $\mathbb{T}_n^d$, $d \geq d_0(q)$, has the following structure with high probability. The colors split into two sets of sizes $\lfloor q/2 \rfloor$ and $\lceil q/2 \rceil$, with the even sublattice colored predominantly by colors from one set and the odd sublattice colored predominantly by colors from the other set. While this conjecture remains open, several related results have appeared. Galvin and Tetali [8], following work of Kahn [11], gave approximate counts for the number of *graph*

homomorphisms from d-regular graphs to arbitrary finite graphs. Specializing to proper q-colorings of $\mathbb{T}_n^d$, their results support the above conjecture. Galvin and Engbers [3] established the analog of the above conjecture, and more general rigidity results for graph homomorphisms, in the limit when n is fixed and d tends to infinity. Similar rigidity results on expander and tree graphs are established in [17, 18, 19].

Of related interest is the hard-core model in $\mathbb{T}_n^d$. In this model, one samples an independent set I of $\mathbb{T}_n^d$ with probability proportional to $\lambda^{|I|}$. It is expected that there exists some $\lambda_c = \lambda_c(d)$ satisfying that, with high probability, if $\lambda > \lambda_c$ the sampled independent set resides predominantly in one of the two sublattices, whereas if $\lambda < \lambda_c$ no such structure appears. While the existence of λ_c is still open (and there are examples of graphs for which it does not exist, see [1]) one may still define $\lambda_c' = \lambda_c'(d)$ as the infimum over λ for which the model admits multiple Gibbs states. Dobrushin [2] proved that $\lambda_c' < \infty$ in every dimension $d \geq 2$, with an upper bound growing to infinity with d. Galvin and Kahn [5] significantly improved this result by showing that λ_c' tends to zero with d. The quantitative bound obtained in [5] was further improved in [16]. The main technical ingredient in both [5, 16], as well as the aforementioned [15, 6], is a careful analysis of the structure of certain special cutsets in $\mathbb{T}_n^d$, when the dimension d is sufficiently high. This is in contrast to this work, in which discrete analogs of topological considerations constitute the bulk of the argument.

2 Preliminaries and Overview

This section is divided into an introduction to the objects and notation of the paper, and to a reduction of Theorem 1.2 to a statement concerning quasi-periodic functions on the integer lattice. At the end of the section we give a glimpse into the ideas of the proof, and discuss the relation between our work and algebraic topology.

2.1 Preliminary definitions

Lattice and Torus. We write $\mathbb{Z}^d$ for the nearest-neighbor graph of the standard d-dimensional integer lattice, and $\mathbb{T}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ for the graph of the d-dimensional discrete torus with side length n. We assume n is an even integer greater or equal than 4, fixing it throughout the paper. We also assume both graphs come with a fixed coordinate system, letting $e_i \in \mathbb{Z}^d$ be the ith standard basis vector for $1 \leq i \leq d$. In both graphs, two vertices are adjacent if they differ by one in exactly one coordinate. As n is even, both graphs are bipartite. In both we thus refer to the vertices in the bipartition class of $\mathbf{0} = (0, \ldots, 0)$ as *even*, and to the rest of the vertices as *odd*. For a vector $v \in \mathbb{Z}^d$, and a set $U \in \mathbb{Z}^d$ we write $U + v$ to denote $\{u + v : \ u \in U\}$.

Distance and boundary. Let G be a connected graph. We write $u \sim v$ to denote that a pair of vertices $u, v \in V(G)$ are adjacent. For a set of vertices $U \subseteq V(G)$ we define the *boundary* of U to be

the set of edges

$$\partial U := \{e \in E(G) : e \cap U \neq \emptyset \text{ and } e \cap U^c \neq \emptyset\}.$$

We use $\mathrm{dist}(u, v)$ for the shortest-path distance between u and v, and extend this notion to non-empty sets $U, V \subseteq V(G)$, defining

$$\mathrm{dist}(U, V) := \min\{\mathrm{dist}(u, v) \ : \ u \in U, v \in V\}.$$

If one of the sets U, V is empty, we write $\mathrm{dist}(U, V) = \infty$. For a set of vertices U, we denote

$$U^+ := \{u \in V(G) : \mathrm{dist}(\{u\}, U) \leq 1\},$$
$$U^- := \{u \in V(G) : \mathrm{dist}(\{u\}, U^c) > 1\}.$$

Note that $U^- = ((U^c)^+)^c$. We also abbreviate $U^{++} := (U^+)^+$ and $U^{--} := (U^-)^-$. The following simple relations hold for any two sets $U, V \subseteq V(G)$:

$$U^+ \subseteq V \iff U \subseteq V \text{ and } \partial U \cap \partial V = \emptyset, \tag{5}$$

$$\mathrm{dist}(U^+, V) = \max(\mathrm{dist}(U, V) - 1, 0), \tag{6}$$

$$U \subseteq V \iff \forall\, W \subset V(G),\ \mathrm{dist}(U, W) \geq \mathrm{dist}(V, W). \tag{7}$$

For a set of vertices U, we define the *internal vertex boundary* of U to be

$$\partial_\bullet U := U \setminus U^-.$$

Similarly we define the *external vertex boundary* of U to be

$$\partial_\circ U := U^+ \setminus U.$$

In both $\mathbb{Z}^d$ and $\mathbb{T}_n^d$, we call a set of vertices U *odd* if all the vertices of $\partial_\bullet U$ have the same parity (in [15] a different convention is used, calling a set U odd if all vertices of $\partial_\bullet U$ are odd). The internal and external vertex boundaries of an odd set of vertices $U \subsetneq T_{10}^2$, as well as U^+ and U^-, are depicted in Figure 1.

Homomorphism height functions, 3-colorings and quasi-periodic functions. A proper 3-coloring of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ satisfying $f(v) \neq f(w)$ when $(v, w) \in E(G)$. An integer-valued function on $V(G)$ is called a *homomorphism height function* on G, or simply height function or HHF, if it differs by exactly one between adjacent vertices of G. We usually work with $\mathrm{Col}(G, v_0)$ and $\mathrm{Hom}(G, v_0)$, the sets of colorings and height functions normalized to take the value 0 at the vertex v_0, as defined in (1) and (2). When $G = \mathbb{T}_n^d$ or $\mathbb{Z}^d$ we abbreviate $\mathrm{Col}(G, \mathbf{0})$ to $\mathrm{Col}(G)$ and $\mathrm{Hom}(G, \mathbf{0})$ to $\mathrm{Hom}(G)$.

Let V be either $\mathbb{Z}$ or $\{0, 1, 2\}$. We say a function

$$f \colon \mathbb{Z}^d \to V \text{ is } \textit{periodic} \text{ if } f(v) = f(w) \text{ whenever } v - w = ne_i \text{ for some } i.$$

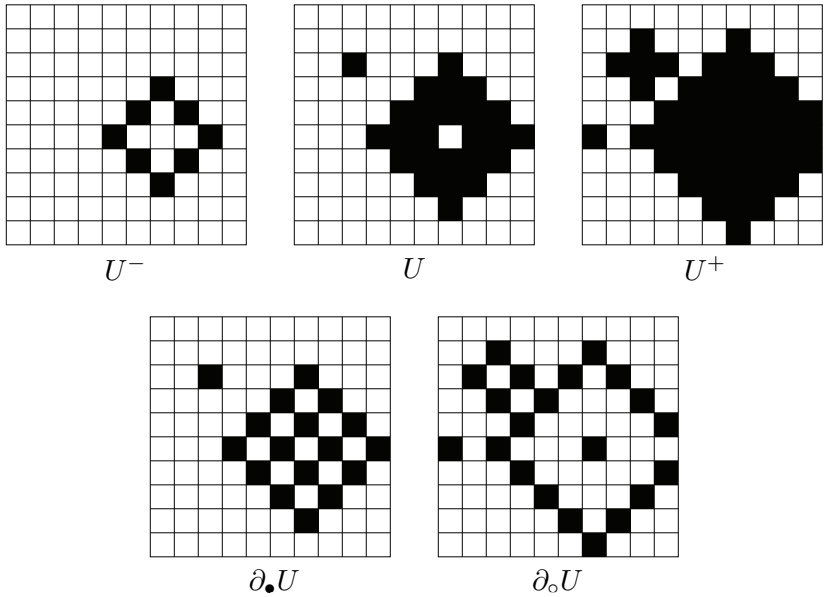

Figure 1: Boundary operations on some odd set U in T^2_{10}.

We denote by PC the set of all periodic proper 3-colorings in $\mathrm{Col}(\mathbb{Z}^d)$. Similarly, for $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$ we say that an HHF

$h\colon \mathbb{Z}^d \to \mathbb{Z}$ is *quasi-periodic* with slope m if $f(v) = f(w) + m_i$ whenever $v - w = ne_i$ for some i.

We write QP_m for the set of quasi-periodic functions with slope m in $\mathrm{Hom}(\mathbb{Z}^d)$. Note that for an HHF, being periodic is equivalent to being quasi-periodic with slope $\mathbf{0}$.

Observe that, in fact,

$$\mathrm{QP}_m = \emptyset \text{ if } m \notin 2\mathbb{Z}^d \text{ or if } |m_i| > n \text{ for some } i. \tag{8}$$

To see this, note that any $h \in \mathrm{Hom}(\mathbb{Z}^d)$ must take even values on even vertices, and satisfy $|h(v)| \leq \mathrm{dist}(v, 0)$, since h changes by one between adjacent vertices. Thus, we must have that $m_i = h(ne_i)$ is even and $|h(ne_i)| \leq n$ for all i. The quasi-periodic functions whose slope is not a multiple of 6 will not play a role in our work. Thus we define

$$\mathrm{QP} := \bigcup_{m \in 6\mathbb{Z}^d \cap [-n,n]^d} \mathrm{QP}_m\,. \tag{9}$$

Denote by $\pi\colon \mathbb{Z}^d \to \mathbb{T}^d_n$ the natural projection from the integer lattice to the torus, defined by

$$\pi((x_1, \ldots, x_d)) = (x_1 \bmod n, \ldots, x_d \bmod n)$$

(where we identify the coordinate system of the torus with $\{0, \ldots, n-1\}^d$). Observe that π extends naturally to a bijection between periodic proper 3-colorings (of $\mathbb{Z}^d$) and proper 3-colorings of $\mathbb{T}^d_n$,

as well as to a bijection between periodic HHFs (on $\mathbb{Z}^d$) and HHFs on $\mathbb{T}_n^d$. With a slight abuse of notation we also denote these extensions by π.

Relations between HHFs and 3-colorings. It is not difficult to see that the mapping Mod_3, which takes an HHF h to the function defined by

$$\mathrm{Mod}_3(h)(v) := h(v) \bmod 3,$$

maps every HHF to a proper 3-coloring. As mentioned in the introduction, it is a known fact that Mod_3 defines a bijection between $\mathrm{Hom}(\mathbb{Z}^d)$ and $\mathrm{Col}(\mathbb{Z}^d)$, that is between the set of HHFs on $\mathbb{Z}^d$ normalized at $\mathbf{0}$ and the set of proper 3-colorings of $\mathbb{Z}^d$ normalized at $\mathbf{0}$.

This bijection does not extend to $\mathbb{T}_n^d$, as there are colorings in $\mathrm{Col}(\mathbb{T}_n^d)$ which are not the image of any HHF through Mod_3. Nonetheless, $\mathrm{Col}(\mathbb{T}_n^d)$ is still in bijection with a subclass of quasi-periodic functions, as the following proposition states.

Proposition 2.1. *The mapping $\pi \circ \mathrm{Mod}_3 \colon \mathrm{QP} \to \mathrm{Col}(\mathbb{T}_n^d)$ is a bijection.*

Proof. We first show that the mapping is well-defined. Let $h \in \mathrm{QP}_m$ for some $m \in 6\mathbb{Z}^d$. By quasi-periodicity, $h(v) \equiv h(v + ne_i) \pmod 3$, for all $1 \le i \le d$ and $v \in \mathbb{Z}^d$. Consequently $\mathrm{Mod}_3(h) \in \mathrm{PC}$ and hence π may be applied to $\mathrm{Mod}_3(h)$ to produce an element of $\mathrm{Col}(\mathbb{T}_n^d)$.

Since Mod_3 is a bijection between $\mathrm{Hom}(\mathbb{Z}^d)$ and $\mathrm{Col}(\mathbb{Z}^d)$ and π is a bijection between PC and $\mathrm{Col}(\mathbb{T}_n^d)$, we deduce that $\pi \circ \mathrm{Mod}_3$ is one-to-one on QP. All that remains in order to show that this mapping is a bijection, is to prove that it is onto.

Let $f \in \mathrm{Col}(\mathbb{T}_n^d)$. Define $g := \pi^{-1}(f) \in \mathrm{PC}$ and an HHF h by $h := {\mathrm{Mod}_3}^{-1}(g)$. We need to show that $h \in \mathrm{QP}_m$ for some $m \in 6\mathbb{Z}^d \cap [-n, n]^d$. We first show that for any $v, w \in \mathbb{Z}^d$ and $1 \le i \le d$,

$$h(v + ne_i) - h(v) = h(w + ne_i) - h(w).$$

For this it suffices to show that for any $v \in \mathbb{Z}^d$ and $1 \le i, j \le d$,

$$h(v + ne_i) - h(v) = h(v + e_j + ne_i) - h(v + e_j). \tag{10}$$

Since $h(v + e_j) - h(v)$ and $h(v + e_j + ne_i) - h(v + ne_i)$ are both in $\{-1, 1\}$ by the definition of homomorphism height function, the equality (10) follows upon recalling that $g = \mathrm{Mod}_3(h)$ and noting that

$$g(v + e_j) - g(v) = g(v + e_j + ne_i) - g(v + ne_i),$$

since g is periodic. Thus $h \in \mathrm{QP}_m$ for some $m \in \mathbb{Z}^d$.

It remains to show that $m \in 6\mathbb{Z}^d \cap [-n, n]^d$. By (8) it suffices to show that $m \in 3\mathbb{Z}^d$. This follows from the fact that

$$m_i = h(ne_i) \equiv g(ne_i) \equiv g(0) \pmod 3. \qquad \square$$

Proposition 2.1 enables us to define the following partition of $\mathrm{Col}(\mathbb{T}_n^d)$,

$$\mathrm{Col}_m := (\pi \circ \mathrm{Mod}_3)(\mathrm{QP}_m). \tag{11}$$

It also implies the important fact that $\mathrm{Col}_{\mathbf{0}}$ and $\mathrm{Hom}(\mathbb{T}_n^d)$ are in bijection through $\pi \circ \mathrm{Mod}_3{}^{-1} \circ \pi^{-1}$. In other words,

$$\mathrm{Col}_{\mathbf{0}} = \{f \in \mathrm{Col}(\mathbb{T}_n^d) \,:\, f \text{ is the modulo 3 of some } h \in \mathrm{Hom}(\mathbb{T}_n^d)\}. \tag{12}$$

The relations between $\mathrm{Col}(\mathbb{T}_n^d), \mathrm{Hom}(\mathbb{T}_n^d), \mathrm{QP}$ and PC are summarized in Figure 2.

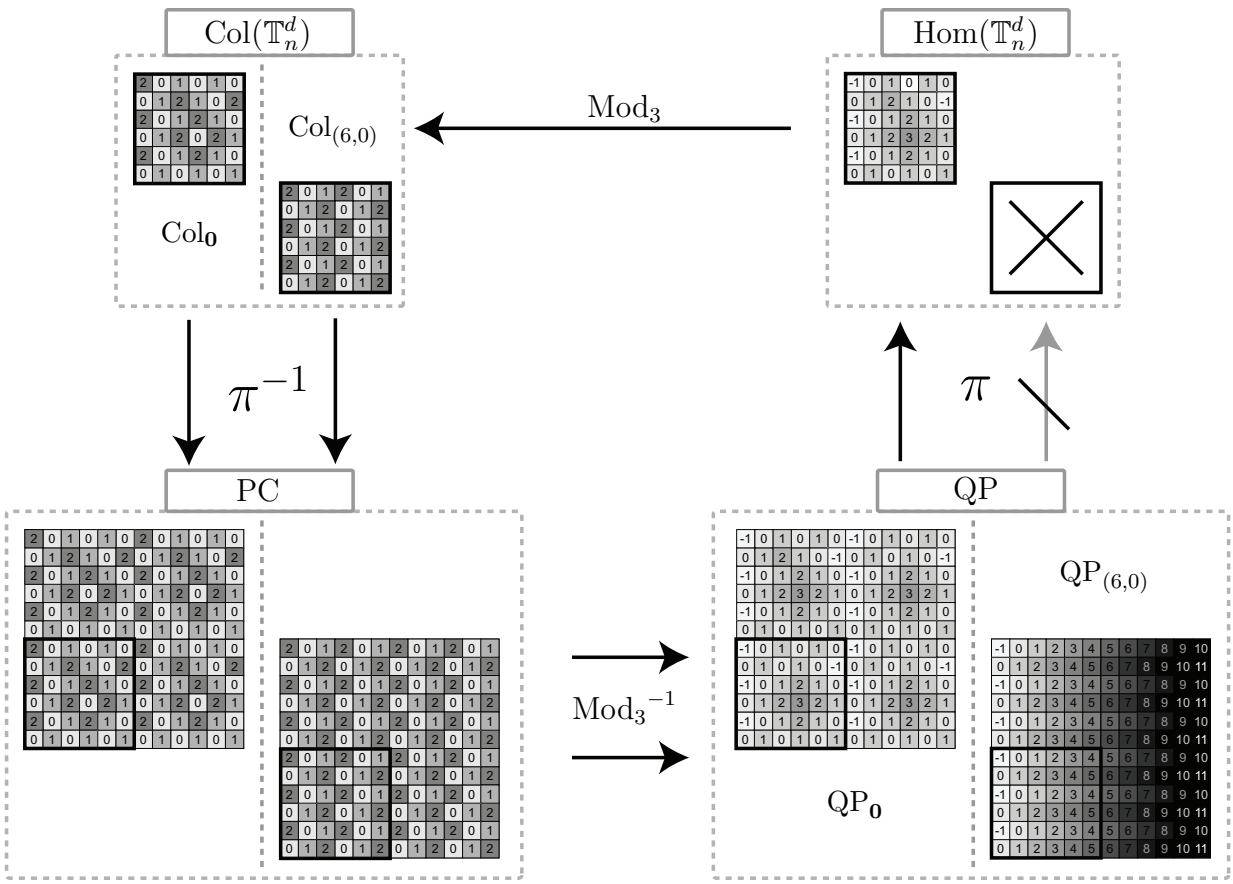

Figure 2: The relations between $\mathrm{Col}(\mathbb{T}_n^d)$ and $\mathrm{Hom}(\mathbb{T}_n^d)$ through periodic colorings and quasi-periodic HHFs on $\mathbb{Z}^d$. Notice that for PC and QP only a small region of the infinite lattice is illustrated. All functions are normalized at $\mathbf{0}$, at the lower left corner of the displayed region. The illustrations depicts the case $n = 6$, $d = 2$.

2.2 Most elements of QP are in $\mathrm{QP}_{\mathbf{0}}$

The following Theorem 2.2, which is the main technical statement of the paper, states that most elements of QP have slope 0.

Theorem 2.2. *There exist d_0 and $c > 0$ such that in all dimensions $d \geq d_0$, for every $m \in 6\mathbb{Z}^d \setminus \{\mathbf{0}\}$ we have*

$$\frac{|\,\mathrm{QP}_m\,|}{|\,\mathrm{QP}_{\mathbf{0}}\,|} \leq \exp(-c_d n^{d-1}), \tag{13}$$

with $c_d = \frac{c}{d \log^2 d}$.

Given (9), we observe that the above theorem is trivial in for $n \leq 4$, as in those cases $|\,\mathrm{QP}_m\,| = 0$. Naturally we focus our attention on the non-trivial cases.

Theorem 1.2 is an immediate consequence of (and is, in fact, equivalent to) the Theorem 2.2.

Proof of Theorem 1.2 from Theorem 2.2. By symmetry, it is enough to prove Theorem 1.2 for colorings normalized at $\mathbf{0}$. That is, to establish that for sufficiently large d, if f is uniformly sampled from $\mathrm{Col}(\mathbb{T}_n^d)$ then

$$\mathbb{P}\left(f \text{ is not the modulo 3 of some } h \in \mathrm{Hom}(\mathbb{T}_n^d)\right) \leq \exp\left(-\frac{c}{d \log^2 d} n^{d-1}\right). \tag{14}$$

Suppose then that f is uniformly sampled from $\mathrm{Col}(\mathbb{T}_n^d)$. By Proposition 2.1, (9), (11) and (12),

$$\mathbb{P}\left(f \text{ is not the modulo 3 of some } h \in \mathrm{Hom}(\mathbb{T}_n^d)\right) = \frac{\left|\bigcup_{m \in (6\mathbb{Z}^d \cap [-n,n]^d) \setminus \{\mathbf{0}\}} \mathrm{Col}_m\right|}{|\,\mathrm{Col}(\mathbb{T}_n^d)|} =$$

$$= \frac{\left|\bigcup_{m \in (6\mathbb{Z}^d \cap [-n,n]^d) \setminus \{\mathbf{0}\}} \mathrm{QP}_m\right|}{|\,\mathrm{QP}\,|} \leq (2n+1)^d \max_{m \in 6\mathbb{Z}^d \setminus \{\mathbf{0}\}} \frac{|\,\mathrm{QP}_m\,|}{|\,\mathrm{QP}_{\mathbf{0}}\,|}.$$

Thus (14) follows from Theorem 2.2. □

2.3 Proof overview

Most of the remainder of the paper is dedicated to proving Theorem 2.2. Our proof can be divided into two parts. First we construct a set of one-to-one mappings, $\Psi_m : \mathrm{QP}_m \to \mathrm{QP}_0$ for $m \in 6\mathbb{Z}^d \setminus \{\mathbf{0}\}$. We then apply results from [15] to show that the image of QP_m under Ψ_m is relatively small. Theorem 2.2 follows. In this section we present for the reader a rough sketch of the idea behind the construction of Ψ_m.

Let us first explain (a minor variant of) the construction of Ψ_m in dimension $d = 1$, where it is rather simple. Suppose that h is a 1-dimensional quasi-periodic HHF with slope $6 \cdot \ell > 0$ (the case that the slope is negative is treated analogously). One can look for the minimal $w \geq 0$ such that $h(w) = 2$ and for the maximal $u \leq 0$ such that $h(u) = -3\ell + 2$. Since h has slope 6ℓ it follows that $w - u < n$. Thus, we may partition $\mathbb{Z}$ to segments of the form $(u+in, w+in]$ and $(w+in, u+(i+1)n]$, $i \in \mathbb{Z}$. We may then define, for $v \in \mathbb{Z}$,

$$\Psi_{6\ell}(h)(v) = \begin{cases} h(v) - 6i\ell & u + in \leq v \leq w + in \text{ for some } i \in \mathbb{Z} \\ 4 - h(v) - 6i\ell & w + in \leq v \leq u + (i+1)n \text{ for some } i \in \mathbb{Z}. \end{cases}$$

An example is shown in Figure 3.

It is not difficult to check that $\Psi_{6\ell}(h)$ is still an HHF, noting that the action of $\Psi_{6\ell}$ can be seen as reversing the gradient of h between w and $u+n$ and each of their translations by multiples of n. Moreover, the resulting HHF will be periodic in the sense that $\Psi_{6\ell}(h)(v+n) = \Psi_{6\ell}(h)(v)$ for all $v \in \mathbb{Z}$. To see that $\Psi_{6\ell}$ is one-to-one, one may check that w is the minimal in $\mathbb{Z}_+$ satisfying $\Psi_{6\ell}(h)(w) = 2$ and u is the maximal in $\mathbb{Z}_-$ satisfying $\Psi_{6\ell}(h)(u) = -3\ell + 2$. Given ℓ, one can thereby recover u and w from $\Psi_{6\ell}(h)$ and use them to recover h.

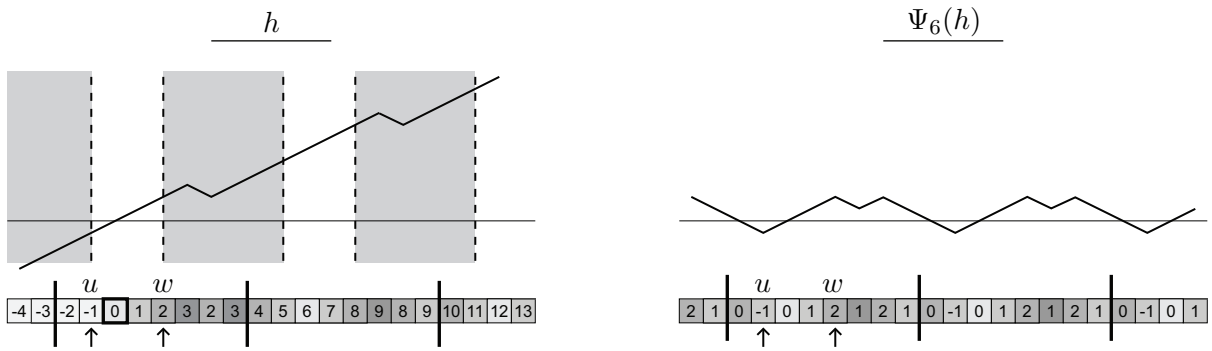

Figure 3: On the left - an example of a one-dimensional quasi periodic HHF with $n = 8$ and slope 6. The gray regions are the regions where Ψ_6 reverses the gradient of the function. On the right - the image of the same HHF through Ψ_6.

Generalizing this technique to higher dimensions is not immediate. The general idea is to use the given HHF h to carefully define two sets $U, W \subseteq \mathbb{Z}^d$ and a vector $\Delta \in n\mathbb{Z}^d$ suitable for our purposes. The set U is the analog of the interval $(-\infty, u]$ and the set W is the analog of the interval $(-\infty, w]$. Among the properties which these sets satisfy is the fact that if we define $U_i := U + i\Delta$ and $W_i := W + i\Delta$ then the sets $W_i \setminus U_i$ and $U_{i+1} \setminus W_i$ form a partition $\mathbb{Z}^d$. We then define Ψ_m, analogously to the above one-dimensional case, by reversing the gradient of h in the regions $U_{i+1} \setminus W_i$, see (44). The main difficulty is to find such sets W, U, and vector Δ, for which this operation yields an HHF, and, moreover, for which the operation is invertible, yields a *periodic* HHF, and such that the size of the range of Ψ_m will be small compared to $|\,\mathrm{QP}_{\mathbf{0}}\,|$.

The sets U, W which we define are closely related to the level sets of the function h in the sense that h is constant on $\partial_\bullet U, \partial_\bullet W, \partial_\circ U$ and $\partial_\circ W$. In addition, they satisfy special topological properties. The boundaries ∂U and ∂W, regarded as a collection of plaquettes in $\mathbb{R}^d$, are analogs of continuous hypersurfaces. Furthermore, the projection of these boundaries to the torus are analogs of hypersurfaces whose removal does not disconnect the torus.

The existence of sets U, W satisfying all the required properties is far from obvious. The intuition for it comes from algebraic topology, specifically de Rham cohomology theory, and some of the

connections are explained in the next section. However, our proof proceeds by developing the theory fully in the discrete setup. This is achieved in sections 3 and 4. This theory is then applied in Section 5 to define Ψ_m and prove that it satisfies the required properties.

To get a feeling of why the sets U and W exist, it may help to think first of continuous linear functions on $\mathbb{R}^d$. A multidimensional linear function is always simply a projection on its gradient vector. Such a linear function could be made periodic by periodically reversing its gradient between two hyperplanes which are perpendicular to the gradient vector. These hyperplanes are the analogs of ∂W and ∂U. This case is therefore very similar to the one-dimensional case. Algebraic topology tells us that every continuous function is a deformation of a linear function. Thus, a guiding intuition may be that for more general functions, the above hyperplanes are deformed into some hypersurfaces, and hence should still exist.

2.4 Relation with topology

The proof of Theorem 2.2 is motivated by ideas from algebraic topology. One element of the proof that might puzzle a reader who lacks topological background is our ability to find a domain, bounded by two hypersurfaces, such that reversing the gradient in translated copies of this domain suffices to make our HHF periodic. We dedicate this short section to highlight some of the analogies between concepts of the proof and their continuous topological counterparts and shed some light over this particular point.

We begin with a brief review of concepts from de Rham cohomology theory. A 0-form on a manifold is simply a smooth function. A 1-form is a differential form which can be integrated against paths. On Riemannian manifolds a 1-form can be identified with a vector field through the Riemannian metric. A 1-form is called *closed* if it satisfies that its integral over contractible loops is 0. The gradient of a 0-form is always a closed 1-form, and, locally, the converse is also true. Globally, however, on non-contractible manifolds such as the torus, there are many closed 1-forms which are not the gradient of any 0-form. The group of closed 1-forms modulo the gradients of the 0-forms is called the first de Rham cohomology group of the manifold.

In the context of our work, 0-forms correspond to HHFs on the torus. Closed 1-forms correspond to proper 3-colorings of the torus, in the sense that, locally, they describe the discrete gradient of an HHF. In the continuous torus every closed 1-form is locally the gradient of a 0-form. Similarly, in the discrete torus, every 3-coloring is locally the gradient of an HHF. However, the local information does not always add up to form the global structure of an HHF.

Algebraic topology tells us that the first de Rham cohomology measures this global obstruction, in the sense that a 1-form corresponds to the zero class of the cohomology group if and only if it is globally the gradient of a 0-form. The first de Rham cohomology of the d-dimensional torus is $\mathbb{R}^d$. The class of a particular 1-form can be identified by the integral of the form over a loop in each of the standard basis directions. In the terminology of this paper, this vector of integrals is called the

slope of the form. Another way to represent the slope of a 1-form is to look at what is called the universal cover of our space. In the case of the torus we look at quasi-periodic functions over $\mathbb{R}^d$. Taking this point of view, the slope is the vector of differences between the quasi-periodic function at standard basis points and at 0.

Poincaré duality identifies H^1, the first cohomology group of the torus, with H_{n-1}, the $(n-1)$th homology group of the torus, which corresponds, if the slope consists of integers, to a class of hypersurfaces of codimension 1. The duality further tells us that for every nice enough 1-form in a class of H^1, there exist hypersurfaces in the dual class in H_{n-1}, orthogonal to the gradient of the form and with the following property. Cutting the torus along such a hypersurface leaves the torus connected, but nullifies the cohomology class, i.e., on the cut torus the 1-form becomes the gradient of a 0-form.

Much of the above description carries over to the discrete case. Here too, we match proper 3-colorings with quasi-periodic HHFs, and classify them according to their slope. We find "level sets", corresponding to the above hypersurfaces, along which one may cut the torus, that is, remove the corresponding edges, to make the coloring the gradient of an HHF. We consider two such level sets with a specific height difference. Deleting the edges of these level sets splits the torus into two connected components such that on each component, the coloring is the gradient of an HHF. Since the height of the HHF is constant along each boundary of the cut torus (as we have cut along level sets), we may reverse the gradient of the coloring on one of the connected components of the cut torus to obtain a coloring which is globally the gradient of an HHF (here, our specific choice of the height difference of the level sets enters). This illustrates the operation of Ψ_m. In practice, we transfer most of the topological part of the proof to statements involving HHFs on $\mathbb{Z}^d$, the universal cover of the torus. This gives us more direct access to the level sets.

The main difficulties in our task are to define the level sets in the discrete setup and to do so in such a way that would allow their recovery after applying the gradient-reversal operation. As mentioned above, the topological arguments are applicable to nice functions, with nice level sets. In the discrete setting the level sets are made out of plaquettes that can have complicated intersections, of various dimensions. Proving that discrete level sets still possess a nice structure requires the theory developed in sections 3 and 4.

It remains unclear whether it is possible to avoid any combinatorial argument in our proof, and use only topology. One can hope to achieve this either by defining a clever discrete variant of the de Rham cohomology, or by mapping the discrete problem to an analogous question in $\mathbb{R}^d$ with the hope of tackling it there. This, however, is a path we did not pursue.

3 Closed Hypersurfaces in $\mathbb{Z}^d$

In this section we introduce a class of subsets of $\mathbb{Z}^d$ and discuss the topological properties of its members. The definitions and results are inspired by continuous topological analogs in $\mathbb{R}^d$ but are

given directly in the discrete setting without requiring knowledge of the continuous notions (but see Section 2.4 for more on the connection). We make no mention of neither colorings nor height functions here and thus the section may be read using only the definitions regarding set operations in Section 2. The tools developed here are applied to the study of colorings and height functions in the following section, but we believe that they are also of independent interest and may be of use for other purposes.

The ultimate conclusion of the discussion here, Theorem 3.4 below, is a certain trichotomy for systems of translates in $\mathbb{Z}^d$. This trichotomy is later applied to level sets of quasi-periodic HHFs.

We remind the reader that in the beginning of section 2 we fixed an even integer n for the remainder of the paper. This integer plays the role of the side length of the torus $\mathbb{T}_n^d$ in later sections. In this section n will also play a role, though the torus $\mathbb{T}_n^d$ will not be explicitly mentioned. However, unlike the rest of the paper, the results and proofs presented in this section remain valid regardless of whether n is even or odd.

The structure of the section is as follows. In Section 3.1 we present the fundamental properties of the sets that we investigate and state our two main results, in the form of certain trichotomies. Section 3.2 describes corollaries of the main results, which will be of use in our application. The proofs of the main results are given in Sections 3.3 and 3.4.

3.1 Topology of $\mathbb{Z}^d$

We begin by defining three properties of sets in $\mathbb{Z}^d$: *co-connectedness*, *boundary disjointness*, and *translation respecting*. These are repeatedly used throughout the paper.

Co-connectedness. A set $U \subseteq \mathbb{Z}^d$ is called *co-connected* if $U \neq \emptyset$, $U \neq \mathbb{Z}^d$ and U and U^c are connected.

A useful property of co-connected sets is that their boundaries are, in a sense, connected. Namely,

Proposition 3.1. *If A is a co-connected set in $\mathbb{Z}^d$ then $\partial_\bullet A \cup \partial_\circ A$, $A^{++} \setminus A$ and $A \setminus A^{--}$ are all connected sets.*

We delay the proof of this proposition to Section 3.3, as it requires the tools developed there.

In order to get a more intuitive grasp of the theorems and definitions of this section the reader might find it useful to regard $\mathbb{Z}^d$ as a lattice of d-dimensional cubes where the edges between adjacent vertices represent plaquettes of codimension 1. Taking this continuous view, co-connected sets are analogous to continuous sets whose boundary is a connected, oriented, closed hypersurface. A set and its complement should be thought of as defining opposite orientations on the same surface.

Boundary disjointness. Two sets $U_1, U_2 \subseteq \mathbb{Z}^d$ are called *boundary disjoint* if

1. $\partial U_1 \cap \partial U_2 = \emptyset$,

2. there is no 4-cycle in $\mathbb{Z}^d$ whose vertices, in order, are $(v_{00}, v_{01}, v_{11}, v_{10})$ such that $v_{00} \in U_1^c \cap U_2^c$, $v_{01} \in U_1^c \cap U_2$, $v_{11} \in U_1 \cap U_2$ and $v_{10} \in U_1 \cap U_2^c$.

Here and below, by a cycle in $\mathbb{Z}^d$ we mean a finite set $\{(u_1, v_1), \ldots, (u_k, v_k)\}$ of distinct edges of $\mathbb{Z}^d$ satisfying that $u_{i+1} = v_i$, $1 \leq i \leq k-1$, and $u_1 = v_k$. A 4-*cycle* is a cycle with $k = 4$, and by its vertices, in order, we mean (u_1, u_2, u_3, u_4).

Continuing the analogy with hypersurfaces, two sets are boundary disjoint if their boundaries neither overlap nor intersect transversally.

When both U_1 and U_2 are odd, as will always be the case from Section 4 and on, the second condition for boundary disjointness is trivially fulfilled, yielding the simpler relation:

$$\text{odd } U_1, U_2 \text{ are boundary disjoint iff } \partial U_1 \cap \partial U_2 = \emptyset. \tag{15}$$

Observe that by definition, boundary disjointness is preserved under taking complements, i.e., if U_1, U_2 are boundary disjoint sets, then U_1^c and U_2 are also boundary disjoint.

The containment relations between two co-connected boundary disjoint sets are restricted by the following theorem.

Theorem 3.2. (Pair trichotomy) *If $U_1, U_2 \subseteq \mathbb{Z}^d$ are co-connected and boundary disjoint sets, then exactly one of the following alternatives hold:*

- $U_1 \cap U_2 = \emptyset$,
- $U_1^c \cap U_2^c = \emptyset$,
- $U_1 \subsetneq U_2$ *or* $U_2 \subsetneq U_1$.

The proof of this theorem is postponed to Section 3.3.

The following proposition relates containment of boundary disjoint sets and their distance from a third set.

Proposition 3.3. *If $U_1, U_2 \subseteq \mathbb{Z}^d$ are non-empty, boundary disjoint sets satisfying $U_1 \subset U_2$ then for every non-empty set V satisfying $V \cap U_2 = \emptyset$ we have* $\mathrm{dist}(U_1, V) > \mathrm{dist}(U_2, V)$.

Proof. Using boundary disjointness and (5), we have $U_1^+ \subseteq U_2$. By (6) and (7) we thus have $\mathrm{dist}(U_1, V) > \mathrm{dist}(U_2, V)$ as required. □

Translation respecting sets. For a set $U \subseteq \mathbb{Z}^d$, we define $T_U = T_U^n$, the set of *translates* of U by multiples of n in each of the coordinate directions, as

$$T_U := \{U + x : \ x \in n\mathbb{Z}^d\}.$$

Recalling that $U + v := \{u + v \ : \ u \in U\}$. We note that it may well be the case that different translations of U yield the same set.

A set $U \subseteq \mathbb{Z}^d$ is called *translation respecting* if U is co-connected and every *distinct* $U_1, U_2 \in T_U$ are boundary disjoint. Observe that by definition, if U is translation respecting, then so is U^c.

Continuing the analogy with hypersurfaces, a set is translation respecting if the image of its boundary through π is a connected, closed hypersurface on the torus.

The main result of this section, is that the trichotomy of Theorem 3.2 extends to translation respecting sets in the following strong sense.

Theorem 3.4. (Translation trichotomy) *If $U \subseteq \mathbb{Z}^d$ is translation respecting and $|T_U| > 1$, then exactly one of the following alternatives holds:*

- [Type 1] *If $U_1, U_2 \in T_U$ and $U_1 \neq U_2$ then $U_1 \cap U_2 = \emptyset$.*
- [Type -1] *If $U_1, U_2 \in T_U$ and $U_1 \neq U_2$ then $U_1^c \cap U_2^c = \emptyset$.*
- [Type 0] *If $U_1, U_2 \in T_U$ then $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.*

Moreover, if U satisfies the Type 0 alternative of the theorem, then there exists an order-preserving bijection *$o\colon T_U \to \mathbb{Z}$. Here, order preserving means that $o(U_1) < o(U_2)$ if and only if $U_1 \subsetneq U_2$. Furthermore, there exists a $\Delta \in n\mathbb{Z}^d$ such that $o^{-1}(i+1) = o^{-1}(i) + \Delta$ for all $i \in \mathbb{Z}$. We call any such Δ a* minimal translation *of U.*

The proof of this theorem is postponed to Section 3.4. Observe that in dimension $d > 2$ the requirement that $|T_U| > 1$ is not equivalent to $U \notin \{\mathbb{Z}^d, \emptyset\}$ (e.g., the set U of vertices in $\mathbb{Z}^d$ having at most one coordinate which is not a multiple of n is a translation respecting set which satisfies $|T_U| = 1$).

Theorem 3.4 allows us to assign a type to every translation respecting set U satisfying $|T_U| > 1$. For $i \in \{-1, 0, 1\}$, we write $\mathrm{Type}(U) = i$ if U satisfies the Type i alternative of the theorem. The case $|T_U| = 1$ does not play a role in our application. However, for completeness, we say in this case, with a slight abuse of notation, that both $\mathrm{Type}(U) = 1$ and $\mathrm{Type}(U) = -1$ hold. An illustration of sets of the various types is given in Figure 4.

3.2 Corollaries of the trichotomy

In this section we state several useful corollaries of Theorem 3.4. The next proposition discusses how the type of translation respecting sets is affected by taking complements.

Proposition 3.5. *If U is translation respecting of type i then:*

- *U^c is translation respecting of type $-i$.*
- *If U is of type 0 with minimal translation Δ, then $-\Delta$ is a minimal translation of U^c.*

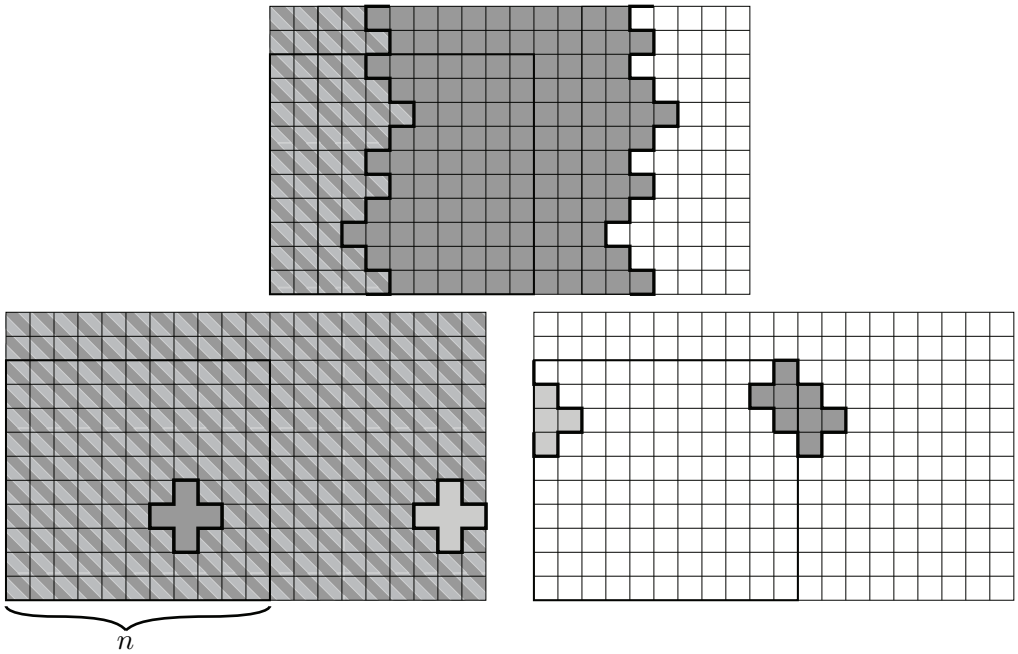

Figure 4: Examples of translation respecting sets of the three types. In each image a portion of the plane is depicted, on which a set U and its translation $U+ne_1$ are emphasized in light gray and in dark gray respectively. Vertices contained in both sets are striped. In each image a different alternative of Theorem 3.4 holds: At the top type 0, at the bottom-left type -1 and at the bottom-right type 1.

The proof of this proposition is straightforward from Theorem 3.4.

The following proposition investigates the possible containment relations between translation respecting sets.

Proposition 3.6. *Let U, V be two translation respecting sets satisfying that $|T_U|, |T_V| > 1$ and $U \subseteq V$. Then* $\mathrm{Type}(U) \geq \mathrm{Type}(V)$.

Proof. Checking the possible cases we see that it suffices to prove that if $\mathrm{Type}(U) = -1$ then also $\mathrm{Type}(V) = -1$, and that if $\mathrm{Type}(V) = 1$ then also $\mathrm{Type}(U) = 1$.

Suppose $\mathrm{Type}(U) = -1$. Let $\Delta \in n\mathbb{Z}^d$ be such that $U + \Delta \neq U$, using that $|T_U| > 1$. Since $\mathrm{Type}(U) = -1$, $U^c \subseteq U + \Delta$. Thus,

$$V^c \subseteq U^c \subseteq U + \Delta \subseteq V + \Delta.$$

It follows that $V^c \cap (V + \Delta)^c = \emptyset$ and hence $\mathrm{Type}(V) = -1$.

The case that $\mathrm{Type}(V) = 1$ follows similarly. □

Translation respecting sets of type 0. These have a unique structure, as the following proposition indicates.

Proposition 3.7. *If U is translation respecting of type 0 then:*

- $\bigcup_{V \in T_U} V = \mathbb{Z}^d$.
- *There exists $1 \leq i \leq d$ such that for every $v \in \mathbb{Z}^d$, $\{v + ke_i \ : \ k \in \mathbb{Z}\}$ intersects both U and U^c.*

Proof. Let $v \in \mathbb{Z}^d$ and let Δ be a minimal translation of U. Observe that by definition, $U \subsetneq U + \Delta$, and $U, U + \Delta$ are co-connected and boundary disjoint. Applying Proposition 3.3 we get $\mathrm{dist}(U + \Delta, \{v\}) \leq \max(\mathrm{dist}(U, \{v\}) - 1, 0)$. Iterating, we obtain that there exists some k such that $v \in U + k\Delta$. We deduce the first item of the proposition.

Towards proving the second item, observe that there exists some $1 \leq i \leq d$ such that $U + ne_i \neq U$ (otherwise we would have $U + \Delta = U$, contradicting the fact that U is of type 0). By the last part of Theorem 3.4, there exists some $\ell \in \mathbb{Z} \setminus \{0\}$ such that $U + ne_i = U + \ell\Delta$. Notice that both U and U^c are translation respecting of type 0 with $-\Delta$ being a minimal translation for U^c (by Proposition 3.5). Thus, the first item of the proposition and the last part of Theorem 3.4 show that there exist $k_1, k_2 \in \mathbb{Z}$ such that

$$v \in (U + k_2\ell\Delta) \cap (U^c + k_1\ell\Delta).$$

Equivalently $v - k_2 ne_i \in U$ while $v - k_1 ne_i \notin U$, as required. □

3.3 Proof of the pair trichotomy

In this section we prove Proposition 3.1 and Theorem 3.2 using the approach of Timár in [20]. To do so, we make use of the well-known fact that 4-cycles span the cycles of $\mathbb{Z}^d$, i.e., every cycle σ in $\mathbb{Z}^d$ can be written as

$$\sigma = \sum_{c \in \mathcal{C}} c, \tag{16}$$

where $\mathcal{C}$ is a set of 4-cycles, and we interpret the sum as meaning that an edge is in σ if it appears in an odd number of cycles in $\mathcal{C}$.

To aid our proof we introduce the following family of graphs.

Definition 3.8. *Given $U \subseteq \mathbb{Z}^d$, a set of vertices, we define a graph G_U as follows. The vertices of G_U are the vertices of $\mathbb{Z}^d$. Two vertices u, v are adjacent in G_U if there exist $e_u, e_v \in \partial U$ and a 4-cycle c, such that $v \in e_v, u \in e_u$, and $e_v, e_u \in c$.*

The following lemma connects this definition with co-connected sets.

Lemma 3.9. *If $U \subset \mathbb{Z}^d$ is a co-connected set of vertices, then $\partial_\bullet U$ is connected in G_U.*

Proof. The proof is heavily based on ideas developed in [20]. It suffices to show that for any non-trivial partition S_1, S_2 of $\partial_\bullet U$ there exists an edge of G_U connecting S_1 and S_2. Here, a non-trivial

partition means that $S_1, S_2 \neq \emptyset$, $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = \partial_\bullet U$. Let S_1, S_2 be such a partition. We set

$$E_1 := \{e \in \partial U \ : \ e \cap S_1 \neq \emptyset\},$$
$$E_2 := \{e \in \partial U \ : \ e \cap S_2 \neq \emptyset\}.$$

By the connectedness of U and U^c in $\mathbb{Z}^d$, there exists some cycle σ in $\mathbb{Z}^d$ which contains exactly one edge of E_1 and one edge of E_2 (in fact, we can even pick those boundary edges arbitrarily). As 4-cycles span the cycles of $\mathbb{Z}^d$, we write σ as a sum of such cycles

$$\sigma = \sum_{c \in \mathcal{C}} c, \tag{17}$$

as in (16). We notice that as σ contains an odd number of E_1 edges (in fact, just one), there must also be a 4-cycle $c_0 \in \mathcal{C}$ containing an odd number of E_1 edges. However as every cycle contains an even number of edges from the boundary $\partial U = E_1 \uplus E_2$, c_0 must contain an edge of E_2 as well. Thus S_1 and S_2 are connected by an edge of G_U, concluding the proof. □

Lemma 3.9 allows us to prove Proposition 3.1 and Theorem 3.2. In this proof we will make use of [20, Theorem 4]. For convenience, we state a special case of this theorem in the context of our work.

Theorem (Timár)**.** *For any co-connected $A \subsetneq \mathbb{Z}^d$, the set*

$$\{y \in A^c \ : \ y \textit{ differs from some point in } A \textit{ by } \pm 1 \textit{ in each of exactly one or two coordinates}\}$$

is connected in $\mathbb{Z}^d$.

To see that this is a special case of [20, Theorem 4], take $G = \mathbb{Z}^d$, and let G^+ be G with an edge between every two vertices who differ by ± 1 on each of exactly one or two coordinates. Also, take $C = A$, and let x be some arbitrary point in A^c.

Proof of Proposition 3.1. Let A be a co-connected set in $\mathbb{Z}^d$. The first part of the proposition is an immediate result of Lemma 3.9, as connectivity of $\partial_\bullet A \cup \partial_\circ A$ in $\mathbb{Z}^d$ is weaker than connectivity of $\partial_\bullet A$ in G_A. The proof of the second part uses the above stated version of [20, Theorem 4]. By the theorem,

$$B := \{y \in A^c \ : \ y \text{ differs from some point in } A \text{ by } \pm 1 \text{ in each of exactly one or two coordinates}\}$$

is connected in $\mathbb{Z}^d$. In addition B satisfies that $B \subset A^{++} \setminus A$ and that every vertex in $A^{++} \setminus A$ has a neighbor in B. We therefore have that $A^{++} \setminus A$ is connected in $\mathbb{Z}^d$ as required. To get the third part of the proposition, we recall that if A is co-connected, then so is A^c, and that $A \setminus A^{--} = (A^c)^{++} \setminus A^c$. We can therefore derive the third part of the proposition by applying the second part to A^c. □

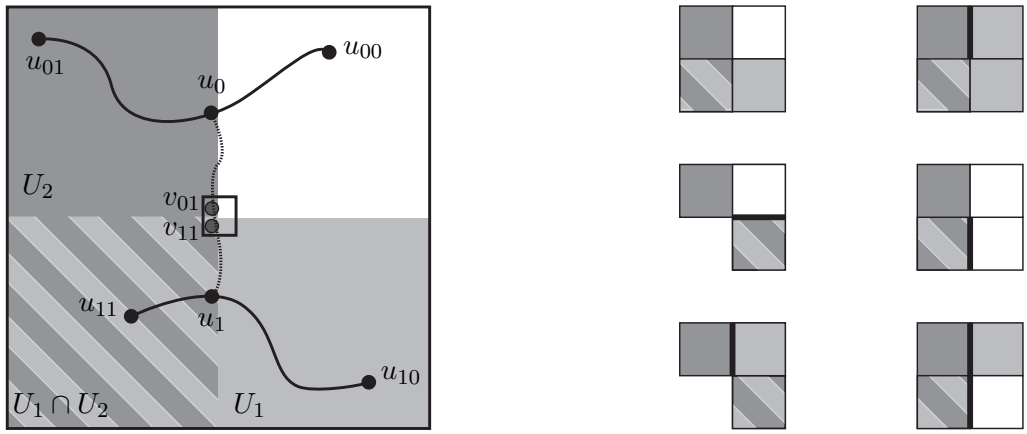

Figure 5: Illustration accompanying the proof of Theorem 3.2. On the left the roles of u_{00}, u_{10}, u_{11} and u_{01} are illustrated, as well as these of u_0, u_1, v_{01} and v_{11}. On the right, all the possible configurations of the 4-cycles c, up to rotation and reflection, are illustrated. When the boundary disjointness is ruled out due to the existence of an edge violating $\partial U_1 \cap \partial U_2 = \emptyset$, this edge is marked. When no edge is marked, the alternative is ruled out due to the existence of a "forbidden cycle" (as in the definition of boundary disjointness).

Proof of Theorem 3.2. We accompany the proof with Figure 5. Assume to the contrary all the alternatives in the theorem do not hold. We can therefore pick $u_{11} \in U_1 \cap U_2$, $u_{10} \in U_1 \cap U_2^c$, $u_{01} \in U_1^c \cap U_2$ and $u_{00} \in U_1^c \cap U_2^c$. As U_1 is connected, there exists a path inside U_1 between u_{10} and u_{11}. This path must contain a vertex $u_1 \in U_1 \cap \partial_\bullet U_2$. Similarly there exists a path outside U_1 between u_{00} and u_{01} which contains a vertex $u_0 \in U_1^c \cap \partial_\bullet U_2$.

By Lemma 3.9, $\partial_\bullet U_2$ is connected in G_{U_2}. In particular, if we partition $\partial_\bullet U_2$ into $U_1 \cap \partial_\bullet U_2$ and $U_1^c \cap \partial_\bullet U_2$, we must have an edge in G_{U_2} crossing this partition. In other words, there exists a 4-cycle c which contains two edges $e_0, e_1 \in \partial U_2$, and two vertices $v_{01} \in e_0$ and $v_{11} \in e_1$ such that $v_{01} \in U_1^c \cap \partial_\bullet U_2$ and $v_{11} \in U_1 \cap \partial_\bullet U_2$. A careful case study of all the possible configurations of such a cycle (see Figure 5) yields that its existence must contradict the boundary disjointness for U_1 and U_2. The theorem follows. □

3.4 Proof of the translation trichotomy

This section is dedicated to the proof of Theorem 3.4.

We begin by showing the trichotomy itself. The pair intersection trichotomy, Theorem 3.2, guarantees that every two sets $U_1, U_2 \in T_U$ satisfy one of the three alternatives of the theorem. Thus it is sufficient to show that for any three distinct sets $U_1, U_2, U_3 \in T_U$, the same alternative holds for both pairs U_1, U_2 and U_1, U_3. In particular, the theorem is immediate if $|T_U| = 2$. Fix distinct

$U_1, U_2, U_3 \in T_U$. We shall rule out three cases.

1. Alternatives 0 and 1 cannot coexist. Let $\delta, \Delta \in n\mathbb{Z}^d$ be such that $U_2 = U_1 + \delta$ and $U_3 = U_1 + \Delta$. Assume, WLOG, that $U_1 \cap U_3 = \emptyset$ and $U_1 \subsetneq U_2$. As U_1 and U_2 are boundary disjoint, by Proposition 3.3 we get that $\mathrm{dist}(U_1, U_3) > \mathrm{dist}(U_2, U_3)$. We note that, $U_1 + \Delta \subseteq U_1 + \Delta + \delta$, as $U_1 \subseteq U_1 + \delta$. We deduce, using (7), that $\mathrm{dist}(U_1 + \delta, U_1 + \Delta) \geq \mathrm{dist}(U_1 + \delta, U_1 + \Delta + \delta)$. Putting all of this together, we get:

$$\mathrm{dist}(U_1, U_1 + \Delta) > \mathrm{dist}(U_1 + \delta, U_1 + \Delta) \geq \mathrm{dist}(U_1 + \delta, U_1 + \Delta + \delta) = \mathrm{dist}(U_1, U_1 + \Delta),$$

   which is a contradiction.

2. Alternatives 0 and -1 cannot coexist. The argument follows similarly to the previous part by passing from U_1, U_2, U_3 to U_1^c, U_2^c, U_3^c.

3. Alternatives 1 and -1 cannot coexist. To see this, assume, WLOG, that $U_1 \cap U_2 = \emptyset$ and $U_1^c \cap U_3^c = \emptyset$. It follows that $U_1 \cup U_3 = \mathbb{Z}^d$ and hence $U_2 \subseteq U_3$. A contradiction follows since alternatives 0 and 1 cannot coexist.

Next, we show the second part of the theorem, i.e., that if $\mathrm{Type}(U) = 0$, then there exists a translation $\Delta \in n\mathbb{Z}^d$ and an order-preserving bijection $o\colon T_U \to \mathbb{Z}$, such that $o^{-1}(i+1) = o^{-1}(i) + \Delta$ for all $i \in \mathbb{Z}$. Assume $\mathrm{Type}(U) = 0$. Define $o(U) := 0$ and for any $V \in T_U$ let

$$o(V) := \begin{cases} \left|\{W \in T_U \ : \ U \subsetneq W \subseteq V\}\right| & U \subseteq V \\ -\left|\{W \in T_U \ : \ V \subseteq W \subsetneq U\}\right| & V \subseteq U \end{cases}.$$

To see that this is well defined, let us explain why $\{W \in T_U \ : \ U \subsetneq W \subseteq V\}$ is finite. A similar argument will show that $\{W \in T_U \ : \ V \subseteq W \subsetneq U\}$ is finite. Since T_U is ordered by inclusion, applying Proposition 3.3 to the complements of two distinct sets $\{W \in T_U \ : \ U \subsetneq W \subseteq V\}$, taking the V of the proposition to be our U, shows that each set W in $\{W \in T_U \ : \ V \subseteq W \subsetneq U\}$ is uniquely characterized by $\mathrm{dist}(W^c, U)$. Since $\mathrm{dist}(W^c, U) \leq \mathrm{dist}(V^c, U)$ we conclude that $\{W \in T_U \ : \ U \subsetneq W \subseteq V\}$ is finite, as we wanted to show.

To show that o is one-to-one, suppose $V_1, V_2 \in T_U$ satisfy $o(V_1) = o(V_2)$. Assume WLOG that $o(V_1) \geq 0$ and $V_1 \subseteq V_2$. This implies that

$$\{W \in T_U \ : \ U \subsetneq W \subseteq V_1\} \subseteq \{W \in T_U \ : \ U \subsetneq W \subseteq V_2\}.$$

However, as $o(V_1) = o(V_2)$, we get

$$\{W \in T_U \ : \ U \subsetneq W \subseteq V_1\} = \{W \in T_U \ : \ U \subsetneq W \subseteq V_2\}$$

and, in particular, $V_2 \subseteq V_1$. Thus $V_1 = V_2$.

Finally, we show that there is a $\Delta \in n\mathbb{Z}^d$ such that $o^{-1}(i+1) = o^{-1}(i) + \Delta$ for all $i \in \mathbb{Z}$. We begin by observing that $o^{-1}(1)$ is nonempty. To see this recall that $|T_U| > 1$ and therefore $U \subsetneq U + z$ for some $z \in n\mathbb{Z}^d$. This implies that $o(U+z) \geq 1$ and therefore there must exist some $\Delta \in n\mathbb{Z}^d$ such that $o(U+\Delta) = 1$. Equivalently, there is no $W \in T_U$ for which $U \subsetneq W \subsetneq U + \Delta$. Since this situation is preserved under translations it follows that $o^{-1}(i) = U + i\Delta$ for all $i \in \mathbb{Z}$. □

4 Level Sets of HHFs

In this section we establish the theoretical basis for dealing with quasi-periodic HHFs. Much of the intuition behind the theorems of this section stems from algebraic topology, viewing quasi-periodic HHFs as a discrete analogue of co-cycles on the torus, and periodic HHFs as a discrete analogue of co-boundaries. Nonetheless, we avoid making any direct reference to topology, and restrict ourselves to purely combinatorial proofs.

We begin by introducing the notions of *sublevel sets* and *sublevel components*. Roughly, these are discrete analogues of continuous sublevel sets, and of regions bounded by a single connected component of a level set.

Formally, let G be either $\mathbb{Z}^d$ or $\mathbb{T}_n^d$. Let $k \in \mathbb{Z}$, $h \in \mathrm{Hom}(G)$ and let $u, v \in V(G)$ satisfy

$$h(u) \leq k < h(v). \tag{18}$$

We define the *k-sublevel set* of u,

$$\mathrm{LC}_h^{k+}(u) \text{ is the connected component of } u \text{ in } G \setminus \{w \in V(G) \ : \ h(w) = k+1\}.$$

While the sublevel set is itself connected, by definition, its complement may be disconnected. We wish to isolate a single connected component of the complement and do this by enlarging the sublevel set. Precisely, we define the *k-sublevel component from u to v*,

$$\mathrm{LC}_h^{k+}(u,v) \text{ is the complement of the connected component of } v \text{ in } G \setminus \mathrm{LC}_h^{k+}(u).$$

Figure 6 illustrates a sublevel component and a sublevel set in $\mathbb{Z}^d$. In our applications sublevel sets are mostly used as a part of the definition of sublevel components, without a significant role of their own. To simplify our notation we write $\mathrm{LC}_h^{+}(u)$ for $\mathrm{LC}_h^{h(u)+}(u)$ and $\mathrm{LC}_h^{+}(u,v)$ for $\mathrm{LC}_h^{h(u)+}(u,v)$.

4.1 Basic properties of level components

Let G be either $\mathbb{Z}^d$ or $\mathbb{T}_n^d$. Let $h \in \mathrm{Hom}(G)$ and suppose $u, v \in G$ satisfy (18). Let

$$U := \mathrm{LC}_h^{k+}(u,v).$$

The next proposition collects several basic properties of sublevel components of h.

0	-1	0	1	2	1	2	1	0	-1	0	1
1	0	1	2	1	0	1	2	1	0	1	2
2	1	2	1	0	1	2	1	2	1	2	1
3	2	1	2	1	2	1	2	3	2	1	2
2	1	2	1	0	1	2	1	2	1	2	1
1	2	1	2	1	0	1	0	1	2	1	2
2	1	2	1	0	u_{-1}	0	1	2	1	2	1
1	0	1	2	1	0	1	2	1	0	1	2
0	-1	0	1	2	1	2	1	0	-1	0	1
1	0	1	2	1	0	1	2	1	0	1	2
2	1	2	1	0	1	2	1	2	1	2	1
3	2	1	2	1	2	1	2	3	2	1	2

0	-1	0	1	2	1	2	1	0	-1	0	1
1	0	1	2	1	0	1	2	1	0	1	2
2	1	2	1	0	1	2	1	2	1	2	1
3	2	1	2	1	2	1	2	3	2	1	2
2	1	2	1	0	1	2	1	2	1	2	1
1	2	1	2	1	0	1	0	1	2	1	2
2	1	2	1	0	u_{-1}	0	1	2	1	2	1
1	0	1	2	1	0	1	v_2	1	0	1	2
0	-1	0	1	2	1	2	1	0	-1	0	1
1	0	1	2	1	0	1	2	1	0	1	2
2	1	2	1	0	1	2	1	2	1	2	1
3	2	1	2	1	2	1	2	3	2	1	2

Figure 6: An illustration of sublevel components for a certain periodic $h \in \mathrm{Hom}(\mathbb{Z}^d)$, with respect to the two vertices $u, v \in \mathbb{Z}^d$. On the left: a portion of $\mathrm{LC}_h^{1+}(u)$ is highlighted. on the right: a portion of $\mathrm{LC}_h^{1+}(u,v)$. Observe that $\mathrm{LC}_h^{1+}(u,v)$ is co-connected while $\mathrm{LC}_h^{1+}(u)$ is not.

Proposition 4.1. *The sublevel component U satisfies:*

1. *$u \in U$ and $v \notin U$.*
2. *$h(x) = k$ for all $x \in \partial_\bullet U$, and $h(x) = k+1$ for all $x \in \partial_\circ U$. In particular, U is odd.*
3. *U is co-connected.*
4. *$\partial_\bullet U \subseteq \mathrm{LC}_h^{k+}(u) \subseteq U$.*

All of these properties are straightforward from the definition and we omit their proof.

In view of the second item of the proposition, we write, with a slight abuse of notation, $h(\partial_\bullet U)$ and $h(\partial_\circ U)$ for the common height of all vertices in $\partial_\bullet U$ and $\partial_\circ U$, respectively.

In the next corollary, we give a criterion for a set to have certain containment relations with a sublevel component.

Corollary 4.2. *The sublevel component U satisfies:*

- *If V is a connected set satisfying $v \in V$, $u \notin V$ and $h(w) > k$ for all $w \in \partial_\bullet V$, then $V \subseteq U^c$.*
- *If V is a connected set satisfying $V \cap U \neq \emptyset$, $\partial_\circ U \subseteq V^c$, then $V \subseteq U$.*

Proof. To get the first item, observe that every path between a vertex of height smaller than $k+1$ and a vertex of height greater or equal to $k+1$ must contain a vertex of height $k+1$. Therefore, $h(x) \leq k$ for all $x \in \mathrm{LC}_h^{k+}(u)$ and in particular $\partial_\bullet V \subset \mathrm{LC}_h^{k+}(u)^c$. Every connected set containing a vertex in V and a vertex outside V must contain a vertex in $\partial_\bullet V$. Since $\partial_\bullet V \subset \mathrm{LC}_h^{k+}(u)^c$ and

$u \in \mathrm{LC}_h^{k+}(u) \cap V^c$, we deduce that $V \subset \mathrm{LC}_h^{k+}(u)^c$. Together with the fact that V is a connected set containing v, the first item follows.

The second item is straightforward and we omit its proof. □

4.2 Level components on $\mathbb{Z}^d$

Until the end of Section 4 we discuss the structure of the set of level components of a single HHF on $\mathbb{Z}^d$. Throughout the rest of Section 4, we denote by h an arbitrary function in $\mathrm{Hom}(\mathbb{Z}^d)$. In the beginning of Sections 4.3 we shall impose additional restrictions on h. Note that dependence on h will often be implicit in our notation.

Boundary disjointness. The following proposition relates sublevel components to the theory developed in Section 3.

Proposition 4.3. *Distinct sublevel components of a function $h \in \mathrm{Hom}(\mathbb{Z}^d)$ are boundary disjoint.*

Proof. Consider $U := \mathrm{LC}_h^{k+}(u,v)$ and $V := \mathrm{LC}_h^{\ell+}(x,y)$, where $k, \ell \in \mathbb{Z}$ and $u, v, x, y \in \mathbb{Z}^d$ satisfy $h(u) \le k < h(v)$ and $h(x) \le \ell < h(y)$. Observe that if $k \neq \ell$, the proposition holds trivially, by the second item of Proposition 4.1 and (15). We thus assume $k = \ell$. Suppose U and V are not boundary disjoint and let us show that this implies them being equal. From the second item of Proposition 4.1, and using (15), we get that there exists $e = (w_1, w_2) \in \partial U \cap \partial V$, such that $w_1 \in \partial_\bullet U \cap \partial_\bullet V$. By the fourth item of Proposition 4.1 we have $w_1 \in \mathrm{LC}_h^{k+}(u) \cap \mathrm{LC}_h^{k+}(x)$ and thus $\mathrm{LC}_h^{k+}(u) = \mathrm{LC}_h^{k+}(x)$, by the definition of sublevel sets. Since w_2 is in the connected component of both v and y in $\mathbb{Z}^d \setminus \mathrm{LC}_h^{k+}(u)$, then these connected components are equal and we get $\mathrm{LC}_h^{k+}(u,v) = \mathrm{LC}_h^{k+}(x,y)$, as required. □

From Proposition 4.3 we derive the following corollary.

Corollary 4.4. *Every edge $(u,v) \in \mathbb{Z}^d$ is contained in the boundary of a unique sublevel component.*

Proof. Assume WLOG that $h(v) = h(u) + 1$. By definition, $(u,v) \in \partial\, \mathrm{LC}_h^{+}(u,v)$. By Proposition 4.3 no other sublevel component has (u,v) in its edge boundary. □

The next proposition shows that in $\mathbb{Z}^d$, the fact that A is a sublevel component of h depends only on a certain neighborhood of the boundary of A.

Proposition 4.5. *Let $h_1, h_2 \in \mathrm{Hom}(\mathbb{Z}^d)$ be two HHFs. Let A be a sublevel component of h_1 and let $u \in \partial_\bullet A$. Suppose there exists $S \supseteq \partial_\bullet A \cup \partial_\circ A$ satisfying that $h_1(w) = h_2(w)$ for all $w \in S$ and that $\mathrm{LC}_{h_1}^{+}(u) \cap S$ is a connected set. Then A is also a level component of h_2.*

Proof. By our assumption $h_1(w) = h_2(w)$ for all $w \in S$, and by definition $h_1(w) \le h(u)$ for all $w \in \mathrm{LC}_{h_1}^{+}(u)$. We get that $h_2(w) \le h(u)$ for all $w \in \mathrm{LC}_{h_1}^{+}(u) \cap S$. Putting this together with our assumptions that $u \in \partial_\bullet A \subseteq S$, and that $\mathrm{LC}_{h_1}^{+}(u) \cap S$ is connected, we get that

$$\mathrm{LC}_{h_1}^{+}(u) \cap S \subseteq \mathrm{LC}_{h_2}^{+}(u), \tag{19}$$

by the definition of sublevel sets.

Next, let $v \in \partial_\circ A$ be such that $u \sim v$. Observe that by Corollary 4.4, we have $A = \mathrm{LC}^+_{h_1}(u,v)$. Let $U := \mathrm{LC}^+_{h_2}(u,v)$. We shall show that $A = U$, establishing the proposition. By the fourth item of Proposition 4.1 we have that $\partial_\bullet A \subseteq \mathrm{LC}^+_{h_1}(u)$ so that, using (19) and our assumption that $\partial_\bullet A \subseteq S$, we get that $\partial_\bullet A \subseteq \mathrm{LC}^+_{h_2}(u)$. Thus, using the fourth item of Proposition 4.1 again yields that

$$\partial_\bullet A \subseteq U. \tag{20}$$

By our assumptions and Proposition 4.1, A^c is connected and satisfies $v \in A^c$, $u \notin A^c$ and $h_2(\partial_\circ A) = h_1(\partial_\circ A) = h(u) + 1$. Thus, the first item of Corollary 4.2 implies that $A^c \subseteq U^c$. Thus, using (20) and the fact that U^c is connected by Proposition 4.1, shows that $A^c = U^c$. Hence $U = A$ as we wanted to show. □

Expressing height differences via level components. Here we develop a formula expressing the difference between the height assigned to a pair of vertices u and v in terms of sublevel components.

Let $u, v \in \mathbb{Z}^d$. We define the set of sublevel components *separating* u from v by

$$\mathcal{L}_{(u,v)} := \{A \ : \ \exists u', v', k \ \ \textbf{s.t.}\ h(u') \le k < h(v') \text{ and } A = LC^{k+}_h(u', v') \text{ satisfies } u \in A,\ v \notin A\}. \tag{21}$$

Proposition 4.6. *Let $u, v \in \mathbb{Z}^d$. $\mathcal{L}_{(u,v)}$ is finite and ordered by inclusion. Furthermore, the following formula holds:*

$$h(v) - h(u) = \left|\mathcal{L}_{(u,v)}\right| - \left|\mathcal{L}_{(v,u)}\right|.$$

Proof. Let U, V be distinct elements of $\mathcal{L}_{(u,v)}$. We begin by showing that $\mathcal{L}_{(u,v)}$ is ordered by inclusion. By Proposition 4.1, U and V are co-connected and by Proposition 4.3 they are boundary disjoint. Thus, U and V satisfy the conditions of Theorem 3.2. By the definition of $\mathcal{L}_{(u,v)}$, we have $u \in U \cap V$ and $v \in U^c \cap V^c$. We deduce that either $U \subseteq V$ or $V \subseteq U$. As containment relations are transitive we deduce that $\mathcal{L}_{(u,v)}$ is ordered by inclusion.

To prove the remaining claims we use induction on the distance between u and v. Indeed, the case $u = v$ is trivial. Assume the proposition holds for every pair of vertices exactly at distance ρ and suppose u, v satisfy $\mathrm{dist}(u,v) = \rho + 1$. Next, let w be a vertex satisfying $w \sim u$ and $\mathrm{dist}(w,v) = \rho$. By our assumption

$$h(v) - h(w) = \left|\mathcal{L}_{(w,v)}\right| - \left|\mathcal{L}_{(v,w)}\right|,$$

and thus

$$h(v) - h(u) = \left|\mathcal{L}_{(w,v)}\right| - \left|\mathcal{L}_{(v,w)}\right| + h(w) - h(u). \tag{22}$$

Suppose that $h(w) = h(u) + 1$. Thus $U = \mathrm{LC}^+_h(u,w)$ is well defined. By Corollary 4.4, it is the only sublevel component containing u and not containing w, and there is no sublevel component which contains w and does not contain u. If $v \in U$, we get that $\mathcal{L}_{(u,v)} = \mathcal{L}_{(w,v)}$ and that $\mathcal{L}_{(v,w)} = \mathcal{L}_{(v,u)} \uplus \{U\}$. If $v \notin U$, we get that $\mathcal{L}_{(u,v)} = \mathcal{L}_{(w,v)} \uplus \{U\}$ and that $\mathcal{L}_{(v,u)} = \mathcal{L}_{(v,w)}$. In either case, by (22),

$$h(v) - h(u) = \left|\mathcal{L}_{(u,v)}\right| - \left|\mathcal{L}_{(v,u)}\right|.$$

The case $h(u) = h(w) + 1$ follows similar lines. $\square$

4.3 Level components of quasi-periodic HHFs

In this section we require h to satisfy $h \in \mathrm{QP}_m$, for some $m \in \mathbb{Z}^d$. We show that sublevel components of such functions have a special structure.

The first property we observe is that the set of sublevel components of h is itself periodic.

Proposition 4.7. *Let $k \in \mathbb{Z}$ and $u, v \in \mathbb{Z}^d$ be such that $h(u) \leq k < h(v)$. For any $x \in n\mathbb{Z}^d$ we have $\mathrm{LC}_h^{(k+\delta_x)+}(u + x, v + x) = \mathrm{LC}_h^{k+}(u, v) + x$ where $\delta_x := h(x) - h(\mathbf{0})$.*

The proposition follows directly from the definition of sublevel component and quasi-periodic function and we omit its proof. Combining this with the third item of Proposition 4.1, Proposition 4.3, and recalling the definition of translation respecting sets from Section 3.1, we get the following corollary.

Corollary 4.8. *Every sublevel component of h is translation respecting.*

Corollary 4.8 tells us that sublevel components of quasi-periodic HHFs may be assigned a type, as in Section 3.1.

The next proposition establishes a duality between $\mathcal{L}_{(u,v)}$ and $\mathcal{L}_{(v,u)}$ when $u - v \in n\mathbb{Z}^d$.

Proposition 4.9. *Let $u, z \in \mathbb{Z}^d$ with $z \neq \mathbf{0}$. If $A \in \mathcal{L}_{(u,u+nz)}$ has $\mathrm{Type}(A) \neq 0$ then*

$$A + \mathrm{Type}(A) \cdot nz \in \mathcal{L}_{(u+nz,u)}.$$

Proof. Suppose $A \in \mathcal{L}_{(u,u+nz)}$ satisfies $\mathrm{Type}(A) \neq 0$, i.e., $\mathrm{Type}(A) \in \{-1, 1\}$. Recall that by definition, $u \in A$ and $u + nz \notin A$. Since A is a sublevel component then, by Proposition 4.7, $A \pm nz$ are also sublevel components. Both are distinct from A since $u + nz \in A + nz$ and $u \notin A - nz$. If $\mathrm{Type}(A) = 1$, then by the trichotomy of Theorem 3.4, $u \in A$ implies that $u \notin A + nz$. Similarly if $\mathrm{Type}(A) = -1$, then by the same trichotomy $u + nz \notin A$ implies $u + nz \in A - nz$. In either case the proposition holds. $\square$

An important corollary of the above proposition is the following:

Corollary 4.10. *If $h \in \mathrm{QP}_m$ for m satisfying $m_1 > 0$, then there exists a sublevel component of type 0 which contains $\mathbf{0}$ and does not contain ne_1.*

Proof. Suppose to the contrary that every sublevel component in $\mathcal{L}_{(\mathbf{0},ne_1)}$ is either of type 1 or of type -1. By Proposition 4.9 we get that $|\mathcal{L}_{(\mathbf{0},ne_1)}| \leq |\mathcal{L}_{(ne_1,\mathbf{0})}|$. By Proposition 4.6 this implies $h(ne_1) \leq h(\mathbf{0})$, in contradiction to our premise. Here, we have also used the fact that the type of a level component is preserved under translation, thus distinct level components $A \in \mathcal{L}_{(\mathbf{0},ne_1)}$ are mapped to distinct level components in $\mathcal{L}_{(ne_1,\mathbf{0})}$ by the mapping $A \mapsto A + \mathrm{Type}(A) \cdot ne_1$. $\square$

4.4 Superlevel components and level components of type 0

In the construction of our embedding (in Section 5) we make use of superlevel components. These are counterparts of sublevel components, in which the role of the sublevel set is replaced by a superlevel set. While these could be defined in an analogous way to that of sublevel components, as given at the beginning of Section 4, we rather define them through a duality.

Definition 4.11. *For any $u, v \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$ satisfying $h(v) < k \leq h(u)$, we define*

$$\mathrm{LC}_h^{k-}(u,v) := \mathrm{LC}_{-h}^{(-k)+}(u,v).$$

This definition allows us to apply propositions dealing with sublevel components to superlevel components. For instance, combining the definition with Corollary 4.8 and Theorem 3.4 we can assign a type to every superlevel component. In addition, by Proposition 4.1, a superlevel component $U = \mathrm{LC}_h^{k-}(u,v)$ satisfies $h(x) = k$ for all $x \in \partial_\bullet U$, and $h(x) = k-1$ for all $x \in \partial_\circ U$. However, to avoid confusion, we remark that the complement of a superlevel component is not necessarily a sublevel component.

The next lemma shows that certain sublevel and superlevel components which are "sandwiched" between two type 0 sublevel components must also be of type 0.

Lemma 4.12. *Let $U \subsetneq W$ be a pair of type 0 sublevel components, such that $h(\partial_\circ U) < h(\partial_\circ W)$ and let $u \in \partial_\bullet U$, $w \in \partial_\bullet W$ and $k \in \mathbb{Z}$. Then:*

- *If $h(u) \leq k < h(w)$ then $V_+ := \mathrm{LC}_h^{k+}(u,w)$ is a sublevel component of type 0, satisfying $U \subseteq V_+ \subsetneq W$.*
- *If $h(u) < k \leq h(w)$ then $V_- := (\mathrm{LC}_h^{k-}(w,u))^c$ satisfies that $(V_-)^c$ is a superlevel component of type 0 and $U \subseteq V_- \subsetneq W$.*

Proof. We start by proving the first item and let V_+ be as in the proposition. We first show that

$$U = \mathrm{LC}_h^{+}(u,w). \tag{23}$$

By our assumptions, $U = \mathrm{LC}_h^{h(u)+}(u',v')$ for some u', v'. By the fourth item of Proposition 4.1 we have $\mathrm{LC}_h^{+}(u) = \mathrm{LC}_h^{h(u)+}(u')$. Next, $w \notin U$ since $U \subsetneq W$ and U and W are boundary disjoint by Proposition 4.3. Hence (23) follows.

Now observe that by applying (23), Proposition 4.1 and the first item of Corollary 4.2 to U and $(V_+)^c$, we get that $(V_+)^c \subseteq U^c$, i.e., $U \subseteq V_+$. Similarly, by Proposition 3.1,

$$\partial_\bullet W \cup \partial_\circ W \text{ is a connected set containing } w\text{, whose vertices are of height greater than } k, \tag{24}$$

and hence $u \notin \partial_\bullet W \cup \partial_\circ W$. Thus, applying (24), the first item of Corollary 4.2, we deduce that $(\partial_\bullet W \cup \partial_\circ W) \subseteq V_+^c$. We can now use the second item of Corollary 4.2 to deduce that $V_+ \subseteq W$.

Consequently, $U \subseteq V_+ \subsetneq W$, where we have used also that $w \in W \setminus V_+$. It remains to show that V_+ is of type 0. All that we need in order to draw this conclusion from Proposition 3.6 is to show that $|T_W|, |T_{V_+}|, |T_U| > 1$. To see this first observe that since $\mathrm{Type}(U) = \mathrm{Type}(W) = 0$, we have by definition $|T_W|, |T_U| > 1$. By Proposition 3.7 there exists some $\Delta \in n\mathbb{Z}^d$ satisfying $(U+\Delta)\cap(V_+)^c \neq \emptyset$ while $U + \Delta \subseteq V_+ + \Delta$. We deduce that $|T_{V_+}| > 1$, so that V_+ is of type 0.

The second item is proved similarly. Let V_- be as in the proposition. By the definition of superlevel component and Proposition 4.1, we have that $(V_-)^c$ is connected, $u \notin (V_-)^c$, $w \in (V_-)^c$ and $h(\partial_\circ V_-) > h(u)$. Applying (23) and the first item of Corollary 4.2 to $(V_-)^c$ we deduce that $(V_-)^c \subseteq U^c$, i.e., $U \subseteq V_-$.

Applying (24), the definition of a superlevel component, and the fourth item of Proposition 4.1 we get that $\partial_\bullet W \cup \partial_\circ W \subseteq (V_-)^c$, as it is contained in the corresponding superlevel set. We deduce that V_- is a connected set satisfying $u \in V_-$ and $\partial_\circ W \subseteq (V_-)^c$. Therefore by the second item of Corollary 4.2, we have $V_- \subseteq W$. Consequently, $U \subseteq V_- \subsetneq W$, where we have used also that $w \in W \setminus V_-$. It remains to show that V_- is of type 0. All that we need in order to draw this conclusion from Proposition 3.6 is to show that $|T_{V_-}| > 1$. This is done in exactly the same way as in the proof of the first part of the lemma. □

We conclude this section with a criterion for applying Proposition 4.5.

Proposition 4.13. *Let $h_1, h_2 \in \mathrm{Hom}(\mathbb{Z}^d)$ be two HHFs and let A be a sublevel component of h_1. Suppose that*

$$h_1(w) = h_2(w) \text{ for all } w \in A^+ \setminus B^-, \tag{25}$$

for some $B \subsetneq A$ which is either a sublevel component of h_1 or the complement of a superlevel component of h_1. Then A is also a sublevel component of h_2.

Proof. Let $u \in \partial_\bullet A$. Let $v \in \partial_\circ A$ be such that $u \sim v$. By Corollary 4.4,

$$A = \mathrm{LC}^+_{h_1}(u, v). \tag{26}$$

Let us show that $u \notin B$. Suppose to the contrary that $u \in B$. Hence $u \in \partial_\bullet B$ by our assumption that $B \subsetneq A$. Then, by Proposition 4.1 and the definition of superlevel component, B^c is a connected set satisfying $v \in B^c$ and satisfying $h_1(\partial_\circ B) = h_1(u) + 1 > h_1(\partial_\bullet A)$. Thus, by the first item of Corollary 4.2, we have that $B^c \subseteq A^c$. However, this contradicts the fact that $B \subsetneq A$.

We continue by considering separately two cases. First, assume that

$$\text{either } h_1(\partial_\bullet B) > h_1(u) \text{ or } h_1(\partial_\circ B) > h_1(u). \tag{27}$$

Since $u \notin B$, the definition of $\mathrm{LC}^+_{h_1}(u)$ and the assumption (27) imply that $\mathrm{LC}^+_{h_1}(u) \cap B = \emptyset$. Now, Proposition 4.1 and (26) imply that $\mathrm{LC}^+_{h_1}(u) \subseteq A$. Thus, by (25), $h_1(w) = h_2(w)$ for all $w \in (\mathrm{LC}^+_{h_1}(u))^+$. Hence the definition of sublevel set yields that $\mathrm{LC}^+_{h_1}(u) = \mathrm{LC}^+_{h_2}(u)$, which, in turn, implies that $\mathrm{LC}^+_{h_1}(u, v) = \mathrm{LC}^+_{h_2}(u, v)$. Thus, recalling (26), A is also a sublevel component of h_2.

Second, let us assume that (27) does not hold. That is, that

$$h_1(\partial_\bullet B) \leq h_1(u) \text{ and } h_1(\partial_\circ B) \leq h_1(u). \tag{28}$$

Denote $S := A^+ \setminus B^-$. Recalling (25) and observing that

$$A^+ \setminus A^- = \partial_\bullet A \cap \partial_\circ A \subseteq S,$$

all that we need to show in order to apply Proposition 4.5 and derive the proposition, is that

$$\mathrm{LC}^+_{h_1}(u) \cap S \text{ is connected.} \tag{29}$$

Observe that, by Proposition 4.1, $\mathrm{LC}^+_{h_1}(u) \subseteq \mathrm{LC}^+_{h_1}(u,v) = A$ we have

$$\mathrm{LC}^+_{h_1}(u) \cap S = \mathrm{LC}^+_{h_1}(u) \setminus B^-.$$

Let $H_0 \uplus H_1$ be a non-trivial partition of $\mathrm{LC}^+_{h_1}(u) \setminus B^-$. Assume for the sake of obtaining a contradiction that there is no edge in $\mathbb{Z}^d$ connecting H_0 and H_1 (that is an edge between a vertex in H_0 and a vertex in H_1). Since $H_0 \uplus H_1 \uplus (\mathrm{LC}^+_{h_1}(u) \cap B^-) = \mathrm{LC}^+_{h_1}(u)$, and $\mathrm{LC}^+_{h_1}(u)$ is a connected set, there must be an edge of $\mathbb{Z}^d$ connecting H_0 and $\mathrm{LC}^+_{h_1}(u) \cap B^-$, and an edge of $\mathbb{Z}^d$ connecting H_1 and $\mathrm{LC}^+_{h_1}(u) \cap B^-$. The existence of these edges implies that

$$\begin{aligned} &(B^+ \setminus B^-) \cap H_0 \neq \emptyset \quad \text{and} \\ &(B^+ \setminus B^-) \cap H_1 \neq \emptyset. \end{aligned} \tag{30}$$

In particular,

$$(B^+ \setminus B^-) \cap (\mathrm{LC}^+_{h_1}(u) \setminus B^-) \neq \emptyset. \tag{31}$$

By Proposition 3.1, we have that

$$B^+ \setminus B^- \text{ is a connected set.} \tag{32}$$

Observe that $\mathrm{LC}^+_{h_1}(u)$ is a connected component of $\{w : h_1(w) \leq h_1(u)\}$, and, by (28), $B^+ \setminus B^- \subseteq \{w : h_1(w) \leq h_1(u)\}$. Thus, using (31) and (32) we may deduce that

$$(B^+ \setminus B^-) \subseteq \mathrm{LC}^+_{h_1}(u) \setminus B^- = H_0 \cup H_1. \tag{33}$$

Putting together (33) and (30) we get that $H_0 \uplus H_1$ induces a non-trivial partition on $B^+ \setminus B^-$ that is not crossed by any edge. Since this contradicts (32), we deduce that (29) holds. □

5 Proof of the Embedding Theorem

In this section we use the theory developed in the previous sections to prove Theorem 2.2. In Section 5.1 we present a one-to-one mapping from QP_m, the set of quasi-periodic HHFs with slope m, to $\mathrm{QP}_{\mathbf{0}}$, the set of periodic HHFs. In Section 5.2 we prove Theorem 2.2 using a probabilistic bound taken from [15] and an auxiliary lemma. This lemma, which relates the boundaries of level components in $\mathrm{QP}_{\mathbf{0}}$ with the boundaries of level components of HHFs on $\mathrm{Hom}(\mathbb{T}^d_n)$, is then proved in Section 5.3.

5.1 Mapping quasi-periodic to periodic functions

Throughout this section we fix some $m \in 6\mathbb{Z}^d$ such that

$$m_1 > 0 \text{ and } \mathrm{QP}_m \neq \emptyset.$$

We also fix $h \in \mathrm{QP}_m$. With the structural results of Sections 3 and 4 in our toolkit, we are ready to construct Ψ_m, our one-to-one mapping from QP_m into $\mathrm{QP}_{\mathbf{0}}$. We start by defining three sets, U_0, W_0 and V_0. The definition relies on the fact that by Corollary 4.8, sublevel and superlevel components of h are translation respecting and can therefore be assigned a type by Theorem 3.4. The first and the third sets will be used to construct Ψ_m. The second set will be used in Section 5.2 to show that the image of Ψ_m is small. Proposition 5.1 below shows that the three sets are well-defined.

In the following definition, and throughout the entire section, we say that a set $S \subset \mathbb{Z}^d$ is the *minimal set* with a given property, if S is contained in every other set with that property.

- $W_0 = W_0(h)$ is the minimal type 0 *sublevel* component satisfying

$$\mathbf{0} \in W_0 \text{ and } ne_1 \notin W_0. \tag{34}$$

  We let Δ be a minimal translation of W_0 as in Theorem 3.4. We choose Δ in some prescribed manner, e.g., as the minimal translation which is first in lexicographic order among the minimal translations with smallest ℓ_1 norm. Write

$$\delta := h(\Delta).$$

- $V_0 = V_0(h)$ is the minimal type 0 *sublevel* component satisfying

$$h(\partial_\bullet V_0) = h(\partial_\bullet W_0) - 1,\ W_0 - \Delta \subseteq V_0 \subseteq W_0,\ \mathbf{0} \notin V_0 \text{ and } -ne_1 \in V_0. \tag{35}$$

- $U_0 = U_0(h, \delta)$ is defined by the property that its complement U_0^c is the minimal type 0 *superlevel* component such that

$$h(\partial_\bullet U_0) = h(\partial_\circ W_0) - \delta/2,\ W_0 - \Delta \subseteq U_0 \subseteq W_0,\ \mathbf{0} \notin U_0 \text{ and } -ne_1 \in U_0. \tag{36}$$

U_0, V_0 and W_0 of a certain $h \in \mathrm{QP}_{(6,0)}$ are illustrated in Figure 7.

Proposition 5.1. *W_0, V_0 and U_0 are well-defined, and satisfy*

$$W_0 - \Delta \subsetneq U_0 \subsetneq V_0 \subsetneq W_0. \tag{37}$$

Proof. For brevity we write U, V and W, for U_0, V_0 and W_0 respectively. We begin by showing that W is well defined. Write $\mathcal{W}$ for the set of type 0 *sublevel* components which contain $\mathbf{0}$ and do not contain ne_1. Recalling (21) we observe that $\mathcal{W} \subseteq \mathcal{L}_{(0,ne_1)}$. Thus, by Proposition 4.6, $\mathcal{W}$ is ordered

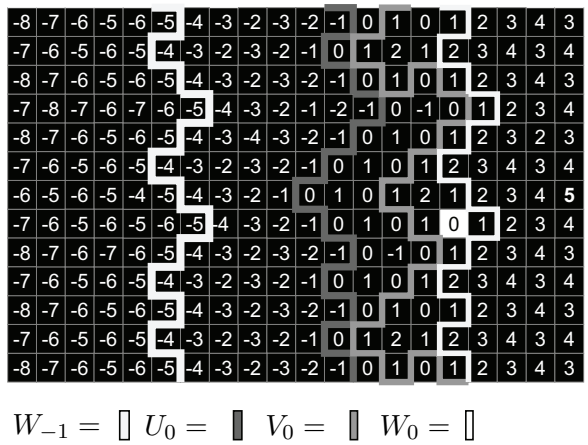

Figure 7: An illustration of the boundaries of U_0, V_0, W_0 and $W_{-1} = W_0 - \Delta$ for $\Delta = ne_1$ and $\delta = 6$. The sets themselves are in all cases to the left of the boundary. $\mathbf{0}$ is marked in white.

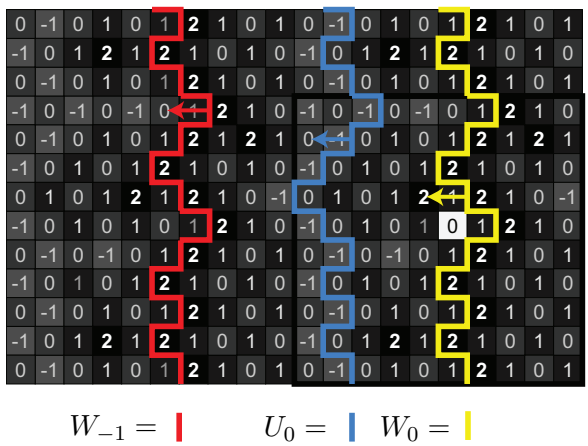

Figure 8: The image through Ψ of the HHF illustrated in figure 7. The boundaries of U_0, W_0 and W_{-1} are highlighted to allow the reader to follow the behavior of Ψ in different regions. $\mathbf{0}$ is marked in white.

by inclusion and finite. By Corollary 4.10, $\mathcal{W} \neq \emptyset$, and thus W, the minimal element of $\mathcal{W}$, is well defined.

Next, towards showing that V is well defined, we write $\mathcal{V}$ for the set of type 0 *sublevel* components V' satisfying $h(\partial_\bullet V') = h(\partial_\bullet W) - 1$, $W - \Delta \subsetneq V' \subsetneq W$, $\mathbf{0} \notin V'$ and $-ne_1 \in V'$. We observe that $\mathcal{V} \subseteq \mathcal{L}_{(-ne_1,0)}$, and thus by Proposition 4.6, $\mathcal{V}$ is ordered by inclusion and finite. To derive the existence of V, all that remains is to show that $\mathcal{V} \neq \emptyset$.

To see that $\mathcal{V} \neq \emptyset$, we make some observations about Δ and δ. Since $h \in \mathrm{QP}_m$, $m \in 6\mathbb{Z}^d$ and

$\Delta \in n\mathbb{Z}^d$, it follows that

$$\delta \equiv 0 \pmod 6. \tag{38}$$

Since W is of type 0, $\mathbf{0} \in W$ and $ne_1 \notin W$ we get that $W \subsetneq W + ne_1$ and therefore, by Theorem 3.4,

$$W + ne_1 = W + k\Delta \text{ for some positive } k. \tag{39}$$

We deduce, using Proposition 4.7, that $h(\partial_\bullet W + ne_1) = h(\partial_\bullet W) + h(ne_1) = h(\partial_\bullet W) + m_1$, and therefore that $m_1 = k\delta$. In particular, since $m_1 > 0$, we see that

$$\delta \geq 6. \tag{40}$$

By subtracting ne_1 and $k\Delta$ from both sides of (39) we have that $W - ne_1 = W - k\Delta$. Thus, recalling that $0 \in W$ and $W - k\Delta \subseteq W - \Delta$, we obtain that

$$-ne_1 \in W - \Delta. \tag{41}$$

By Proposition 4.7 and (40) we get that $W - \Delta$ is a sublevel component satisfying $h(\partial_\circ(W - \Delta)) = h(\partial_\circ W) - \delta \leq h(\partial_\circ W) - 6$. Thus, the first item of Lemma 4.12 guarantees the existence of a type 0 sublevel component V' satisfying $h(\partial_\bullet V') = h(\partial_\bullet W) - 1$, $W - \Delta \subsetneq V' \subsetneq W$. Since $-ne_1 \in W - \Delta$ by (41) we get that $-ne_1 \in V'$. By the minimality of W, we get that $\mathbf{0} \notin V'$ implying that $V' \in \mathcal{V}$ so that $\mathcal{V} \neq \emptyset$.

To show that U is well defined, we write $\mathcal{U}$ for the set containing all U' such that $(U')^c$ is a type 0 *superlevel* component such that $h(\partial_\bullet U') = h(\partial_\circ W) - \delta/2$, $W - \Delta \subseteq U' \subseteq W$, $\mathbf{0} \notin U'$ and $-ne_1 \in U'$. Recalling Definition 4.11 of superlevel sets we use Proposition 4.6 to deduce that the set of superlevel sets containing $\mathbf{0}$ and not containing $-ne_1$ is finite and ordered by inclusion, and therefore $\mathcal{U}$ is also finite and ordered by inclusion. All that remains in order to deduce the existence of U is to show that $\mathcal{U} \neq \emptyset$.

This time we apply (40) and the second item of Lemma 4.12, to h, V and $W - \Delta$, to show the existence of U' satisfying that $(U')^c$ is a superlevel component of type 0, $W - \Delta \subseteq U' \subsetneq V$ and $h(\partial_\bullet U') = h(\partial_\circ W) - \delta/2$. Since $\mathbf{0} \notin V$ by definition and $-ne_1 \in W - \Delta$ by (41) we get that $\mathbf{0} \notin U'$ and $-ne_1 \in U'$. Thus $U' \in \mathcal{U}$, $\mathcal{U} \neq \emptyset$ so that U is well defined. The definition of $\mathcal{U}$ and the fact that $U \subseteq U'$ imply that $W - \Delta \subseteq U \subsetneq V$. In fact, by (40) we have $h(\partial_\bullet(U)) > h(\partial_\bullet(W - \Delta))$ so that

$$W - \Delta \subsetneq U \subsetneq V.$$

This relation and the definition of $\mathcal{V}$ imply (37). □

For $i \in \mathbb{Z}$, we write

$$U_i := U_0 + i\Delta,\ V_i := V_0 + i\Delta \text{ and } W_i := W_0 + i\Delta. \tag{42}$$

Proposition 5.2. *For every $z \in n\mathbb{Z}^d$ and $i \in \mathbb{Z}$ the following are equivalent:*

- $U_0 + z = U_i$,
- $V_0 + z = V_i$,
- $W_0 + z = W_i$.

Proof. We begin by showing that U_0, V_0 and W_0 all have Δ as a minimal translation. For W_0, this is the case by the definition of Δ. We now show this for V_0. The proof for U_0 is similar. Let Δ_V be a minimal translation of V_0. Since

$$V_0 - \Delta \subsetneq W_0 - \Delta \subsetneq V_0 \subsetneq W_0$$

by (37), we have $V_0 - k\Delta_V = V_0 - \Delta$ for some integer $k \geq 1$. By Proposition 3.3, we have

$$\operatorname{dist}(V_0 - \Delta_V, W^c) > \operatorname{dist}(V_0, W_0^c) = \operatorname{dist}(V_0 - \Delta_V, W_0^c - \Delta_V).$$

We deduce that $W_0 - \Delta_V \subsetneq W_0$, and thus $W_0 - \Delta_V \subseteq W_0 - \Delta$ (by the minimality of Δ). Suppose to the contrary that $W_0 - \Delta_V \subsetneq W_0 - \Delta$. Since Δ is a minimal translation of W_0, we get that

$$V_0 - \Delta_V \subsetneq W_0 - \Delta_V \subseteq W_0 - 2\Delta \subsetneq V_0 - \Delta,$$

contradicting the minimality of Δ_V. We conclude that $W_0 - \Delta = W_0 - \Delta_V$.

Fix $z \in n\mathbb{Z}^d$. Since U_0, V_0 and W_0 are of type 0 with Δ as a minimal translation, there exist i, j, k for which $U_0 + z = U_i$, $V_0 + z = V_j$, $W_0 + z = W_k$. Translating (37) by z, we have

$$W_{k-1} \subsetneq U_i \subsetneq V_j \subsetneq W_k. \tag{43}$$

However, (37) and (42) imply that

$$W_{-1} \subsetneq U_0 \subsetneq V_0 \subsetneq W_0 \subsetneq U_1 \subsetneq V_1.$$

Hence we conclude from (43) and the fact that $(U_i), (V_i)$ and (W_i) are ordered by inclusion that

$$k - 1 < i \leq j \leq k$$

and therefore that $i = j = k$. □

We define the mapping $\Psi_m \colon \mathrm{QP}_m \to \mathrm{QP}_{\mathbf{0}}$ by

$$\Psi_m(h)(v) := \begin{cases} h(v - i\Delta) = h(v) - i\delta, & v \in W_i \setminus U_i \text{ for some } i \in \mathbb{Z} \\ 2h(\partial_\circ W_0) - h(v - i\Delta) = 2h(\partial_\circ W_0) - h(v) + i\delta, & v \in U_{i+1} \setminus W_i \text{ for some } i \in \mathbb{Z} \end{cases}. \tag{44}$$

The remainder of the section is dedicated to showing that Ψ_m is well defined and has the required properties.

By Theorem 3.4, for every $i \in \mathbb{Z}$ we have $W_i \subsetneq W_{i+1}$. Thus, applying Proposition 3.7 to W_0, we have that every $v \in \mathbb{Z}^d$ belongs to exactly one set of the form $W_{i+1} \setminus W_i$. Hence $\Psi_m(h)(v)$ is defined for every $v \in \mathbb{Z}^d$. The image through Ψ of the HHF illustrated in Figure 7 is depicted in Figure 8.

By definition, $\Psi_m(h)$ is Δ-periodic , i.e., it satisfies $\Psi_m(h)(v) = \Psi_m(h)(v + \Delta)$ for every $v \in \mathbb{Z}^d$. Thus to understand $\Psi_m(h)$ it suffices to understand its values on $v \in W_0 \setminus W_{-1}$. As a first step to this end we point out that on the region $W_0 \setminus U_0$, Ψ_m is the identity while on the region $U_0 \setminus W_{-1}$ it is a reflection with respect to height $h(\partial_\circ W_0) - \delta/2 = h(\partial_\bullet U_0)$.

Proposition 5.3. *Ψ_m is a one-to-one mapping from QP_m to $\mathrm{QP}_{\mathbf{0}}$.*

Proof. Write $t := \Psi_m(h)$. We need to show is that t is periodic in ne_i for every $1 \le i \le d$, that it is a height function, and that Ψ_m is one-to-one.

t is Periodic. First we show that for every $\Delta' \in n\mathbb{Z}^d$, $a \in \mathbb{Z}$ such that $W_0 + \Delta' = W_a$, we have

$$h(v) = h(v + \Delta' - a\Delta) \text{ for all } v \in \mathbb{Z}^d. \tag{45}$$

By quasi-periodicity, for all $v \in \mathbb{Z}^d$, we have $h(v + \Delta' - a\Delta) = h(v) + (h(\Delta' - a\Delta) - h(\mathbf{0}))$. Hence it suffices to prove (45) for a single $v \in \mathbb{Z}^d$. Next, note that since $W_0 = W_a - \Delta' = W_0 + a\Delta - \Delta'$ we have that if $v \in \partial_\bullet W_0$, then $v + \Delta' - a\Delta$ is also a member of $\partial_\bullet W_0$, implying that $h(v) = h(v + \Delta' - a\Delta)$. This establishes (45).

Now, let $1 \le j \le d$, and suppose that $o_{W_0}(W_0 + ne_j) = a \in \mathbb{Z}$ where o_{W_0} is the order function of W_0 given by Theorem 3.4. Observe that $W_0 + ne_j = W_a$. Note that if $v \in W_i \setminus U_i$ then, by Proposition 5.2, $v + ne_j \in W_{i+a} \setminus U_{i+a}$. Thus, using (45), if $v \in W_i \setminus U_i$ then

$$t(v) = h(v - i\Delta) = h(v + ne_j - (i + a)\Delta) = t(v + ne_j).$$

Similarly, if $v \in U_{i+1} \setminus W_i$ then, using Proposition 5.2, we have

$$t(v) = 2h(\partial_\circ W_0) - h(v - i\Delta) = 2h(\partial_\circ W_0) - h(v + ne_j - (i + a)\Delta) = t(v + ne_j).$$

t is an HHF. We claim that $t \in \mathrm{Hom}(\mathbb{Z}^d)$, i.e., that the values which t assigns to adjacent vertices differ by exactly 1. Let u, v be adjacent vertices in $\mathbb{Z}^d$. We need to show that

$$|t(u) - t(v)| = 1. \tag{46}$$

Since t is Δ-periodic, for every vertex $w \in \mathbb{Z}^d$ there exists $j \in \mathbb{Z}$ such that $w + j\Delta \in U_1 \setminus U_0$ and $t(w) = t(w + j\Delta)$. We may therefore assume WLOG $u \in U_1 \setminus U_0$, and $v \in U_1$. We consider three cases separately.

First, if both $u, v \in U_1 \setminus W_0$ or both $u, v \in W_0 \setminus U_0$ then (46) follows directly from the definition of Ψ_m.

Second, note that

$$t(\partial_\circ W_0) - t(\partial_\bullet W_0) = 2h(\partial_\circ W_0) - h(\partial_\circ W_0) - h(\partial_\bullet W_0) = h(\partial_\circ W_0) - h(\partial_\bullet W_0) = 1.$$

Hence (46) holds if either $u \in \partial_\circ W_0$ and $v \in \partial_\bullet W_0$ or vice versa.

Third,

$$t(\partial_\circ U_0) - t(\partial_\bullet U_0) = h(\partial_\circ U_0) - (2h(\partial_\circ W_0) - h(\partial_\bullet U_0) - \delta),$$

and plugging the relation $h(\partial_\circ W_0) = h(\partial_\bullet U_0) + \delta/2$ from (36) yields

$$t(\partial_\circ U_0) - t(\partial_\bullet U_0) = h(\partial_\circ U_0) - h(\partial_\bullet U_0) = 1.$$

Thus (46) holds if $u \in \partial_\circ U_0$ and $v \in \partial_\bullet U_0$.

Ψ_m is one-to-one. To show that Ψ_m is one-to-one, we explain how to construct an inverse for it. Suppose that we are able to recover U_0, W_0, Δ and δ from t and m. Then we may define $U_i = U_0 + i\Delta$, $W_i = W_0 + i\Delta$ and the mapping

$$\Psi_m^{-1}(t)(v) := \begin{cases} t(v) + i\delta, & v \in W_i \setminus U_i \text{ for some } i \in \mathbb{Z} \\ 2t(\partial_\circ W_0) - t(v) + i\delta, & v \in U_{i+1} \setminus W_i \text{ for some } i \in \mathbb{Z} \end{cases}.$$

It is simple to check that this Ψ_m^{-1} is indeed an inverse to Ψ_m. It is therefore sufficient to show that U_0, W_0, Δ and δ may be recovered from t and m.

We begin by recovering W_0. To do this we follow the lines of the proof of proposition 5.1. Write $\mathcal{W}_t$ for the set of type 0 *sublevel* components of t which contain $\mathbf{0}$ and do not contain ne_1. Again we recall (21) and observe that $\mathcal{W}_t \subseteq \mathcal{L}_{(0,ne_1)}$, where $\mathcal{L}$ is defined with respect to t. Thus, by Proposition 4.6, $\mathcal{W}_t$ is ordered by inclusion and finite. We now argue that $\mathcal{W}_t$ is a non-empty set whose minimal element is W_0.

The definition (44) of Ψ_m and the relation $h(\partial_\circ W_0) = h(\partial_\bullet U_0) + \delta/2$ from (36) imply that

$$t(x) = h(x) \quad \text{for } x \in W_0^+ \setminus U_0^-. \tag{47}$$

We can therefore apply Proposition 4.13 with $h_1 = h$, $h_2 = t$, $A = W_0$ and $B = U_0$ to get that

$$W_0 \in \mathcal{W}_t. \tag{48}$$

Applying the same proposition with $A = V_0$ yields that

$$V_0 \text{ is a sublevel component of } t. \tag{49}$$

Let us write W_t for the minimal element of $\mathcal{W}_t$. Since $W_0 \in \mathcal{W}_t$ we conclude that

$$W_t \subseteq W_0. \tag{50}$$

To obtain the opposite inclusion we now show that W_t is also a sublevel component of h. Observe that since W_t is of type 0, and since $\mathbf{0} \in W_t$ and $ne_1 \notin W_t$ we have by Theorem 3.4 that $W_t - ne_1 \subsetneq W_t$. We deduce that $-ne_1 \in W_t \cap V_0$. In addition, our definitions imply that $ne_1 \in (W_t)^c \cap (V_0)^c$ and $\mathbf{0} \in (W_t \setminus V_0)$. By Theorem 3.2, using that distinct sublevel components of t are boundary disjoint by Proposition 4.3, we deduce that $V_0 \subsetneq W_t$. Applying Proposition 4.13 with $h_1 = t$, $h_2 = h$, $A = W_t$ and $B = V_0$, using (47) and (50) to check the condition (25), we get that W_t is a sublevel component of h. Together with (50), the minimality of W_0 now implies that $W_t = W_0$, allowing the recovery of W_0 from t. After recovering W_0, we can recover Δ and δ using the fact that Δ is a minimal translation of W_0 chosen in a prescribed manner and the fact that $\delta \cdot o_{W_0}(W_0 + ne_1) = m_1$, where o_{W_0} is the order function on translations of W_0, given by Theorem 3.4.

All that remains is to recover U_0. Following again the lines of the proof of Proposition 5.1, we write $\mathcal{U}_t$ for the set containing all U' such that $(U')^c$ is a type 0 *superlevel* component of t and $t(\partial_\bullet U') = t(\partial_\circ W_0) - \delta/2$, $W_0 - \Delta \subseteq U' \subseteq W_0$, $\mathbf{0} \notin U'$ and $-ne_1 \in U'$. Recalling Definition 4.11 of superlevel sets we again use Proposition 4.6 to deduce that the set of superlevel components containing $\mathbf{0}$ and not containing $-ne_1$ is finite and ordered by inclusion, implying that $\mathcal{U}_t$ is also finite and ordered by inclusion. We now use (47) and Proposition 4.13, with $h_1 = -h$, $h_2 = -t$, $A = (U_0)^c$ and $B = (W_0)^c$, to get that U_0^c is a superlevel component of t (again, using Definition 4.11 of superlevel components). It follows from (47) that $U_0 \in \mathcal{U}_t$. Write U_t for the maximal element of $\mathcal{U}_t$, that is, the complement of the minimal element amongst complements of elements in $\mathcal{U}_t$. Since $U_0 \in \mathcal{U}_t$ we conclude that

$$U_t^c \subseteq U_0^c. \tag{51}$$

Recall that, by the definition of $\mathcal{U}_t$, we have $(W_0)^c \subsetneq (U_t)^c$ and that, by (48), W_0 is also a sublevel component of t. Applying Proposition 4.13 to $h_1 = -t$, $h_2 = -h$, $A = (U_t)^c$ and $B = (W_0)^c$, using (47) and (51) to check the condition (25), we get that U_t^c is also a superlevel component of h. We also have $h(\partial_\bullet U_t) = h(\partial_\circ W_0) - \delta/2$ by (47). Thus, together with (51), the minimality of U_0^c now implies that $U_0 = U_t$. As W_0, U_0, Δ and δ can be recovered from t and m, we deduce that Ψ_m is one-to-one. □

5.2 Proof of Theorem 2.2

In this section we prove Theorem 2.2 using a bound on the probability for a uniformly chosen HHF on the torus to have a level component with long boundary. Here, for the first time, we use level components on $\mathbb{T}_n^d$ (defined in Section 4). To clarify our proof we will always denote HHFs in $\mathrm{Hom}(\mathbb{T}_n^d)$ by r, HHFs in $\mathrm{QP}_{\mathbf{0}}$ by t and HHFs in QP_m, for arbitrary m, by h.

Recall that for $u \in \mathbb{T}_n^d$ we denoted by $\mathrm{Hom}(\mathbb{T}_n^d, u)$ the set of all homomorphism height functions on $\mathbb{T}_n^d$ which are zero at u. We use the following theorem of [15] to derive the estimates of Theorem 2.2.

Theorem 5.4 ([15, special case of Theorem 2.8])**.** *There exist $c > 0$ and d_0 such that in all dimensions $d \geq d_0$, for all even n, all $u, v \in \mathbb{T}_n^d$ and all $L \geq 1$, if h is uniformly sampled from* $\mathrm{Hom}(\mathbb{T}_n^d, u)$

then

$$\mathbb{P}\left(|\partial \mathrm{LC}_h^{0+}(u,v)| \geq L\right) \leq d \exp\left(-\frac{cL}{d\log^2 d}\right),$$

where we mean that $\mathrm{LC}_h^{0+}(u,v) = \emptyset$ *if* $h(v) \leq 0$.

We adapt Theorem 5.4 to our setting through the following corollary.

Corollary 5.5. *There exist* $c > 0$ *and* d_0 *such that in all dimensions* $d \geq d_0$*, for all even* n *and all* $L \geq 1$*, denoting*

$$A := \left\{ r \in \mathrm{Hom}(\mathbb{T}_n^d) \ : \ \textit{there exists a sublevel component } A \textit{ such that } |\partial A| \geq L \right\},$$

the following holds,

$$\frac{|A|}{|\mathrm{Hom}(\mathbb{T}_n^d)|} \leq 2d^2 n^d \exp\left(-\frac{cL}{d\log^2 d}\right).$$

Proof. Fix $L \geq 1$ and let $B := \{r \in \mathrm{Hom}(\mathbb{T}_n^d) \ : \ \exists v \in \mathbb{T}_n^d, v \sim \mathbf{0}, \text{ s.t. } |\partial \mathrm{LC}_r^{+}(\mathbf{0}, v)| \geq L\}$. By Theorem 5.4 with $u = \mathbf{0}$, and using a union bound on all $v \sim \mathbf{0}$, we have

$$|B| \leq 2d \cdot d \exp\left(-\frac{cL}{d\log^2 d}\right) \left|\mathrm{Hom}(\mathbb{T}_n^d)\right| \qquad \text{for all } d \text{ greater then some fixed } d_0.$$

Now, for every $w \in \mathbb{T}_n^d$ define the mapping $\eta_w : \mathrm{Hom}(\mathbb{T}_n^d) \to \mathrm{Hom}(\mathbb{T}_n^d)$ by

$$\eta_w(r)(v) := r(v+w) - r(w).$$

It is not difficult to check that this mapping is well defined and is a bijection. Moreover, for every $r \in A$ there exists a $w \in \mathbb{T}_n^d$ such that $\eta_w(r) \in B$. The corollary follows. □

In order to apply Corollary 5.5, we must show that HHFs in the image of Ψ_m, when projected to the torus, contain a sublevel component with a long boundary. We proceed in two steps. First, we claim that the projection of the boundary of the set V_0 from Proposition 5.1 is contained in the boundary of a level component of the projection of $\Psi_m(h)$. Then we claim that this boundary is long. This strategy is expressed in the following two lemmata. Recall that π was defined in Section 2.1 to be the natural projection from $\mathbb{Z}^d$ to $\mathbb{T}_n^d$. Here we use also the natural extension of π to edges of $\mathbb{Z}^d$.

Lemma 5.6. *Let* $h \in \mathrm{QP}_m$ *for* $m \in 6\mathbb{Z}^d$ *satisfying* $m_1 > \mathbf{0}$*. Let* $r = \pi \circ \Psi_m(h)$ *and* V_0 *be as in Proposition 5.1. There exists a sublevel component* R *of* r *such that* $\pi(\partial V_0) \subseteq \partial R$.

We delay the proof of this lemma to Section 5.3.

Lemma 5.7. *Let* $t \in \mathrm{QP}_{\mathbf{0}}$*. Let* $u, v \in \mathbb{Z}^d$ *and* $k \in \mathbb{Z}$ *satisfy* $t(u) \leq k < t(v)$*. Suppose* $V := \mathrm{LC}_t^{k+}(u,v) \subseteq \mathbb{Z}^d$ *is a sublevel component of type* 0*. Then*

$$\max_{1 \leq i \leq d} |\{(w_0, w_1) \in \pi(\partial V) \ : \ w_0 - w_1 = e_i\}| \geq n^{d-1}.$$

Proof. By Proposition 3.7, there exists $1 \le i \le d$ such that for every $x \in \mathbb{Z}^d$ there exists $\ell \in \mathbb{Z}$ such that

$$(x + \ell e_i, x + (\ell+1)e_i) \in \partial V. \tag{52}$$

We deduce that $\pi(x + \ell e_i, x + (\ell+1)e_i) \in \pi(\partial V)$. Using (52) for all x in

$$\{z \in \mathbb{Z}^d \ : \ z_i = 0 \ \text{ and } \ \forall j \neq i, \ 0 \le z_j < n\}$$

yields that $|\{(x, x+e_i) \in \pi(\partial V)\}| \ge n^{d-1}$, as required. □

At last we are ready to prove the theorem.

Proof of Theorem 2.2. Let $m \in 6\mathbb{Z}^d \setminus \{\mathbf{0}\}$. Using the appropriate rotation we may assume without loss of generality that $m_1 > 0$. Fixing $h \in \mathrm{QP}_m$ and applying Lemma 5.6 and Lemma 5.7, we obtain the existence of a sublevel component R of $\pi \circ \Psi_m(h)$ such that $|\partial R| \ge n^{d-1}$. Thus

$$\pi(\Psi_m(\mathrm{QP}_m)) \subset \{r \in \mathrm{Hom}(\mathbb{T}_n^d) \ : \ \text{there exists a sublevel component } A \text{ of } r \text{ such that } |\partial A| \ge n^{d-1} \ \}.$$

Recall that π is a bijection from $\mathrm{QP}_{\mathbf{0}}$ to $\mathrm{Hom}(\mathbb{T}_n^d)$. Thus, applying Corollary 5.5, we get that for large enough d,

$$|\Psi_m(\mathrm{QP}_m)| \le 2d^2 n^d \exp\left(-\frac{cn^{d-1}}{d\log^2 d}\right) |\,\mathrm{Hom}(\mathbb{T}_n^d)| \le \exp\left(-\frac{c'n^{d-1}}{d\log^2 d}\right) |\,\mathrm{QP}_{\mathbf{0}}\,|$$

for some $c, c' > 0$. Thus, since Ψ_m is one-to-one, the theorem follows. □

5.3 Projecting type 0 level components to the torus

In this section we prove Lemma 5.6 connecting level components on $\mathrm{QP}_{\mathbf{0}}$ with those on $\mathrm{Hom}(\mathbb{T}_n^d)$. While the relation between sublevel *components* of HHFs on the integer lattice and those of HHFs on the torus is non-trivial, the relation between sublevel *sets* of the two spaces is much simpler. In particular,

$$\pi(\mathrm{LC}^+_{\pi^{-1}(r)}(u)) = \mathrm{LC}^+_r(\pi(u)) \text{ for all } r \in \mathrm{Hom}(\mathbb{T}_n^d) \text{ and } u \in \mathbb{Z}^d. \tag{53}$$

This can be easily verified from the definition of sublevel sets.

Next, we prove a proposition relating the boundaries of level components on $\mathbb{Z}^d$ to level components on $\mathbb{T}_n^d$. We then show that this proposition applies to the set V_0 from Proposition 5.1, and use this fact to prove Lemma 5.6. We remind the reader that A^+ and A^{++} were introduced in Section 2.1.

Proposition 5.8. *Let $r \in \mathrm{Hom}(\mathbb{T}_n^d)$ and $t = \pi^{-1}(r) \in \mathrm{QP}_{\mathbf{0}}$. Suppose $V := \mathrm{LC}^+_t(u,v)$ for adjacent vertices $u, v \in \mathbb{Z}^d$ satisfying $t(v) = t(u) + 1$. If*

$$\pi(V^{++} \setminus V) \cap \pi(\mathrm{LC}^+_t(u)) = \emptyset \tag{54}$$

then

$$\pi(\partial V) \subseteq \partial\,\mathrm{LC}^+_r(\pi(u), \pi(v)). \tag{55}$$

Proof. Let $R := \mathrm{LC}_r^+(\pi(u), \pi(v))$. We first note that (55) follows from the following two claims,

$$\pi(\partial_\bullet V) \subseteq R, \tag{56}$$

$$\pi(\partial_\circ V) \subseteq R^c. \tag{57}$$

We begin by showing (56). Indeed, we have:

$$\pi(\partial_\bullet V) \subseteq \pi(\mathrm{LC}_t^+(u)) = \mathrm{LC}_r^+(\pi(u)) \subseteq R,$$

where the equality follows from (53), and the two containment relations follow from Proposition 4.1.

Next we show (57). By Proposition 3.1, using the fact that V is co-connected by Proposition 4.1, we get that $\pi(V^{++} \setminus V)$ is a connected set which contains v (recall that $u \sim v$). By (53) and (54), $\pi(V^{++} \setminus V)$ is disjoint from $\mathrm{LC}_r^+(\pi(u))$. By the definition of sublevel component this implies that $\pi(V^{++} \setminus V) \subseteq R^c$. Since $\pi(\partial_\circ V) \subseteq \pi(V^{++} \setminus V)$, we deduce (57). □

At last, we prove Lemma 5.6. Let $h \in \mathrm{QP}_m$ for $m \in 6\mathbb{Z}^d$ satisfying $m_1 > \mathbf{0}$. Let $U = U_0$, $V = V_0$, $W = W_0$ and Δ be as in Proposition 5.1. Let also $t := \Psi_m(h)$ and $r := \pi(t)$. Our goal is to show that V satisfies the conditions of Proposition 5.8, from which Lemma 5.6 will follow.

Write $\mathcal{T}$ for the set of type 0 sublevel components T' satisfying $h(\partial_\bullet T') = h(\partial_\circ W) - \delta + 1$ and $W - \Delta \subsetneq T' \subsetneq V$. Recall that $W - \Delta \subsetneq V$ by (37), $h(\partial_\bullet(W - \Delta)) = h(\partial_\bullet W) - \delta$, $h(\partial_\bullet V) = h(\partial_\bullet W) - 1$ by (35) and that $\delta \geq 6$ by (40). Hence, by Lemma 4.12, we conclude that $\mathcal{T}$ is non-empty. Write T for the minimal element of $\mathcal{T}$.

Let us show that $T \subseteq U$. By Lemma 4.12 applied to $T \subsetneq V$, using that $h(\partial_\bullet T) = h(\partial_\circ W) - \delta + 1$ and $h(\partial_\bullet V) = h(\partial_\bullet W) - 1$, there exists a U' satisfying that $(U')^c$ is a type 0 *superlevel* component such that $h(\partial_\bullet U') = h(\partial_\circ W) - \delta/2$ and $T \subseteq U' \subsetneq V$. Next, observe that $\mathbf{0} \notin U'$, since $0 \notin V$ by (35), and that $-ne_1 \in U'$, since $-ne_1 \in W - \Delta \subsetneq T$ by (41). Thus, (37) and the definition of U (in particular, the fact that U^c is minimal), imply that $U' \subseteq U$. We conclude that

$$W - \Delta \subsetneq T \subseteq U. \tag{58}$$

Next, the definition (44) of Ψ_m, (58) and the definition of T imply that

$$t(\partial_\bullet T) = 2h(\partial_\circ W) - h(\partial_\bullet T) - \delta = h(\partial_\circ W) - 1.$$

Now, since $U \subsetneq V \subsetneq W$ by (37), the definition of Ψ_m implies that

$$h(\partial_\circ V) = t(\partial_\circ V).$$

Thus, by (35),

$$t(\partial_\bullet T) = t(\partial_\circ V). \tag{59}$$

We now check that V satisfies the conditions of Proposition 5.8. Recall that by (49), V is a sublevel component of t. Let $u \in \partial_\bullet V$, $v \in \partial_\circ V$ be two adjacent vertices. By Corollary 4.4 we have $V = \mathrm{LC}_t^+(u, v)$. Observe that the condition (54) is equivalent to

$$((V^{++} \setminus V) + z) \cap \mathrm{LC}_t^+(u) = \emptyset \quad \text{for all } z \in n\mathbb{Z}^d.$$

Since $(V^{++} \setminus V) + z = (V^{++} + z) \setminus (V + z)$ and since V is of type 0 having, by Proposition 5.2, Δ as a minimal translation, this is equivalent to

$$((V^{++} + k\Delta) \setminus (V + k\Delta)) \cap \mathrm{LC}_t^+(u) = \emptyset \quad \text{for all } k \in \mathbb{Z}. \tag{60}$$

We note that $T \subsetneq V$ by the definition of T. It follows from (59) that the set $S := V \setminus T$ satisfies $t(s) = t(\partial_\circ V)$ for all $s \in \partial_\circ S$. This implies that $\mathrm{LC}_t^+(u) \subseteq S$. Thus, to check condition (60) it suffices to show that

$$((V^{++} + k\Delta) \setminus (V + k\Delta)) \cap S = \emptyset \quad \text{for all } k \in \mathbb{Z},$$

which, since $S = V \setminus T$, is itself implied by

$$\begin{aligned} &(V^{++} + k\Delta) \subseteq T \quad \text{for all } k \leq -1, \\ &(V + k\Delta) \supseteq V \qquad \text{for all } k \geq 0. \end{aligned} \tag{61}$$

Since Δ is a minimal translation for V, the second part of (61) follows trivially and it suffices to check the first part for $k = -1$. Finally, the condition that $(V^{++} - \Delta) \subseteq T$ follows from the fact that $V - \Delta \subsetneq W - \Delta \subsetneq T$, a consequence of (37) and the definition of T. Thus the condition of Proposition 5.8 is satisfied. Lemma 5.6 follows from (55). □

6 Remarks and Open Problems

In this section we discuss a few open problems and make a remark.

1. (Tori with odd side length) In this work we consider a uniformly sampled proper 3-coloring of a high-dimensional discrete torus with even side length. Our main result is that for such a coloring, with high probability, one of the two bipartition classes is dominated by a single color. How will this result change if we take the side length of the torus to be odd? Since tori with odd side length are no longer bipartite, some change must occur. We expect that in this situation, a typical coloring will exhibit two 'pure phases', regions in which one of the bipartition classes is dominated by a single color, separated by a single, roughly straight, interface.

2. (Positive temperature) In physical terminology, the proper 3-coloring model is the zero-temperature case of the antiferromagnetic 3-state Potts model. The positive temperature version of this model is defined as follows. A 3-coloring f, not necessarily proper, of the underlying graph is

sampled with probability proportional to $\exp(-\beta H(f))$, where $\beta > 0$ is a parameter proportional to the inverse temperature and $H(f)$ is the number of edges (u, v) for which $f(u) = f(v)$. We expect that the analog of Theorem 1.1 continues to hold when the temperature is small, but positive (that is, when β is sufficiently large). Proving this is complicated by the fact that non-proper 3-colorings are no longer related to height functions.

3. (Larger amount of colors) As explained in Section 1.3, it is expected that Theorem 1.1 has a natural extension to proper colorings of the torus with more than 3 colors. Specifically, that for each q there is some $d_0(q)$ such that if $d \geq d_0(q)$ then a typical proper q-coloring of $\mathbb{T}_n^d$ has the property that the q colors split into two sets of sizes $\lfloor q/2 \rfloor$ and $\lceil q/2 \rceil$ with each bipartition class dominated by colors from one of the two sets. Proving this is wide open even for the case $q = 4$. A result of Vigoda [21] implies that $d_0(q) \geq \frac{3}{11}q$. In [3, Conjecture 5.3] it is conjectured that $d_0(q) = q/2$, at least in the sense that certain "long range influences" exist if and only if $d \geq q/2$. However, any result showing that $d_0(q) < \infty$ will constitute a major advance.

We end with the following remark. Our work extends certain results from [15]. The results in [15] were proven in greater generality than simply for the torus $\mathbb{T}_n^d$. There, also tori with non-equal side lengths were considered, of the form $\mathbb{T}_{n_1}^1 \times \mathbb{T}_{n_2}^1 \times \cdots \times \mathbb{T}_{n_d}^1$. These include, in particular, "two-dimensional" tori of the form $\mathbb{T}_n^2 \times \mathbb{T}_2^d$ for d a fixed large constant. In our work, for simplicity, we considered only the case of the torus $\mathbb{T}_n^d$. However, it seems that our results can be adopted with no difficulty to the more general tori for which results were obtained in [15].

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