

# Quantitative Relation Between Noise Sensitivity and Influences

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March 9, 2010

## Abstract

A Boolean function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  is said to be noise sensitive if inserting a small random error in its argument makes the value of the function almost unpredictable. Benjamini, Kalai and Schramm [BKS99] showed that if the sum of squares of influences in  $f$  is close to zero then  $f$  must be noise sensitive. We show a quantitative version of this result which does not depend on  $n$ , and prove that it is tight for certain parameters. Our results hold also for a general product measure  $\mu_p$  on the discrete cube, as long as  $\log 1/p \ll \log n$ . We note that in [BKS99], a quantitative relation between the sum of squares of the influences and the noise sensitivity was also shown, but only when the sum of squares is bounded by  $n^{-c}$  for a constant  $c$ .

Our results require a generalization of a lemma of Talagrand on the Fourier coefficients of monotone Boolean functions. In order to achieve it, we present a considerably shorter proof of Talagrand's lemma, which easily generalizes in various directions, including non-monotone functions.

## 1 Introduction

The noise sensitivity of a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  is a measure of how likely its value is to change, when evaluated on a slightly perturbed input. Noise sensitivity became an important concept in various areas of research in recent years, with applications in percolation theory, complexity theory, and learning theory (see e.g. [BKS99], [Hás01], [BJT99]). We work with a dual notion of noise sensitivity, namely noise stability, defined as follows.

**Definition 1.** For  $x \in \{0, 1\}^n$ , the  $\epsilon$ -noise perturbation of  $x$ , denoted by  $N_\epsilon(x)$ , is a distribution obtained from  $x$  by independently keeping each coordinate of  $x$  unchanged with probability  $1 - \epsilon$ , and replacing it by a random value with probability  $\epsilon$ . For this purpose we assume that a product distribution  $\mu$  on the discrete cube  $\{0, 1\}^n$  is defined, however we leave it implicit in the notation.

The noise stability of  $f$  is defined by

$$\mathcal{S}_\epsilon(f) \stackrel{\text{def}}{=} \text{COV}_{x \sim \mu, y \sim N_\epsilon(x)}[f(x), f(y)].$$

Roughly saying, a function is noise-sensitive if its noise stability is close to zero.

Another concept that was intensively studied in recent decades is that of the influences of coordinates on a function.

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**Definition 2.** Let  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  be a Boolean function, and let  $i \in \{1, \dots, n\}$ . The influence of the  $i$ 'th coordinate on  $f$  is defined as

$$I_i(f) \stackrel{\text{def}}{=} \Pr_{x \sim \mu}[f(x) \neq f(x \oplus e_i)],$$

where  $x \oplus e_i$  is the vector obtained from  $x$  by flipping the  $i$ 'th coordinate.

Influences were studied in economics for decades, and first found their way into computer science in [BOL90], in the context of cryptography. The study of influences has numerous applications in mathematical physics, economics, and various areas of computer science, such as cryptography, hardness of approximation, and computational lower-bounds (see e.g. [LMN93], [DS05], [Mos09], or the survey [KS06]).

**Relations between influences and noise sensitivity.** The noise sensitivity of a function and its influences both measure how likely it is to change its value when the input is slightly perturbed. It makes sense to study the relations between these concepts. Perhaps counterintuitively, it turns out that functions with very low influences must be very sensitive to noise. This phenomenon was first shown in a paper by Benjamini, Kalai, and Schramm [BKS99]. They proved the following theorem:

**Theorem 3** ([BKS99]). Let  $\{f_m : \{0, 1\}^{n_m} \rightarrow \{0, 1\}\}_{m=1,2,\dots}$  be a sequence of Boolean functions, such that

$$\sum_{i=1}^{n_m} I_i(f)^2 \xrightarrow{m \rightarrow \infty} 0.$$

Then for any  $\epsilon > 0$ ,  $\mathcal{S}_\epsilon(f_m) \xrightarrow{m \rightarrow \infty} 0$ .

The BKS theorem was proved in [BKS99] only with respect to the uniform measure on the Boolean cube, and the case of highly biased product measures was left open. However, for some applications, such as in the study of threshold phenomena, one is often interested in biased measures. Moreover, the BKS theorem is qualitative, and does not show a concrete relation between the influences of a function and its noise stability (a quantitative relation was shown in [BKS99], but only for the case where for a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , the sum of squares of the influences is inverse polynomially small in  $n$ ).

## 1.1 Our results

In this paper we show a quantitative version of the BKS theorem. With respect to the uniform measure, we prove that

**Theorem 4.** There exists a constant  $C > .234$  such that the following holds. Let  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , and denote

$$\mathcal{W}(f) = \frac{1}{4} \cdot \sum_{i=1}^n I_i(f)^2.$$

Then

$$\mathcal{S}_\epsilon(f) \leq 20 \cdot \mathcal{W}(f)^{C \cdot \epsilon}.$$

The main technical tool used in the proof of Theorem 4, and also in the original qualitative result of [BKS99], is a generalization of a lemma by Talagrand [Tal96]. Talagrand’s result considers the Fourier-Walsh expansion of a monotone Boolean function, and bounds its weight on second-level Fourier coefficients in terms of its weight on first level coefficients. This lemma is of independent interest, and was used by Talagrand to estimate the correlation between monotone families [Tal96], and the size of the boundary of subsets of the discrete cube [Tal97]. The generalization gives a similar bound on the weight on  $d$ -level coefficients.

While in the paper of [BKS99] only a qualitative estimate was given in the generalization of Talagrand’s lemma, the main technical tool in our proof is a quantitative version of it. To obtain it, we simplify the proof of Talagrand’s lemma in a way which makes its generalization quite simple and straightforward. Our result for the uniform measure is the following:

**Lemma 5.** *For all  $d \geq 2$ , and for every function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  such that*

$$\mathcal{W}(f) \leq \exp(-2(d-1)) \tag{1}$$

(where  $\mathcal{W}(f)$  is as in Theorem 4), we have

$$\sum_{|S|=d} \hat{f}(S)^2 \leq \frac{5e}{d} \cdot \left(\frac{2e}{d-1}\right)^{d-1} \cdot \mathcal{W}(f) \cdot (\log(d/\mathcal{W}(f)))^{d-1}. \tag{2}$$

Lemma 5, as well as Theorem 4, hold also for functions into the segment  $[-1, 1]$ , if one appropriately extends the definition of the influence. Specifically, one should define  $I_i(f) = \|f(x^{i \leftarrow 0}) - f(x^{i \leftarrow 1})\|_1$  where  $x^{i \leftarrow a}$  is the vector obtained from  $x$  by inserting  $a$  in the  $i$ ’th coordinate. This holds also for the biased case, discussed below. For simplicity, we assume that  $f$  is Boolean in the proofs.

We note that Talagrand also proves a “decoupled” version of his lemma. While we do not need a decoupled version for the proof of Theorem 4, we prove one for the sake of completeness in the Appendix.

**Biased measure.** In the study of threshold phenomena, and for other applications, often one is interested in biased measures rather than the uniform measure over the discrete cube. Once the proof of Talagrand’s lemma is simplified, it becomes easier to apply it also for biased measures. Below are our analogous results with respect to the  $p$ -biased measure on the discrete cube. The coefficients below are with respect to the “ $p$ -biased” Fourier-Walsh expansion (see Section 2).

**Lemma 6.** *Let  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , and denote*

$$\mathcal{W}(f) = p(1-p) \cdot \sum_{i=1}^n I_i(f)^2.$$

For all  $d \geq 2$ , if

$$\mathcal{W}(f) \leq \exp(-2(d-1)), \tag{3}$$

then we have

$$\sum_{|S|=d} \hat{f}(S)^2 \leq \frac{5e}{d} \cdot \left(\frac{2B(p) \cdot e}{d-1}\right)^{d-1} \cdot \mathcal{W}(f) \cdot \left(\log\left(\frac{d}{\mathcal{W}(f)}\right)\right)^{d-1}, \tag{4}$$

where  $B(p)$  is the hypercontractivity constant defined in Section 2.

We note that a bound slightly weaker than in Lemma 6 can be obtained from Lemma 5 by a general reduction technique, as was observed in [Kel10]. However we prove Lemma 6 directly, and Lemma 5 follows as an immediate corollary.

Using Lemma 6 we can prove an analogue of Theorem 4 for the case of biased measure.

**Theorem 7.** *For all  $d \geq 2$ , and for every function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  the following holds. Denoting  $\mathcal{W}(f) = p(1-p) \cdot \sum_{i=1}^n I_i(f)^2$ , we have*

$$\mathcal{S}_\epsilon(f) \leq (6e + 1)\mathcal{W}(f)^{\alpha(\epsilon) \cdot \epsilon}, \tag{5}$$

where

$$\alpha(\epsilon) = \frac{1}{\epsilon + \log(2B(p)e) + 3 \log \log(2B(p)e)},$$

and  $B(p)$  is the hypercontractivity constant defined in Section 2.

We note that for small  $p$ ,  $B(p) \approx \frac{1}{p \log(1/p)}$ , and thus Theorem 7 is useful only when  $\log(1/p)$  is asymptotically smaller than  $\log(n)$ . Indeed, when  $p$  is inverse polynomially small in  $n$  the BKS theorem does not hold even qualitatively – there exist functions which have asymptotically small influences but are noise stable. The graph property of containing a triangle with respect to the critical probability  $p$  is an example of such a function.

**Tightness.** Our main result (Theorem 7) is tight for small  $p$  up to a constant factor in the exponent of  $\mathcal{W}(f)$ , which tends to 1 for small  $\epsilon$ , and for  $p = 1/2$  it is tight up to a constant factor in the exponent. In Section 5 we prove this, and also discuss the tightness of Lemma 6, showing that it is essentially tight.

**Organization.** This paper is organized as follows: in Section 2 we recall the definitions of the biased Fourier-Walsh expansion, hypercontractivity estimates, and some related large deviation bounds. In Section 3 we present the proof of Lemma 6 (which immediately implies Lemma 5 as well). In Section 4 we show that Lemma 6 implies Theorem 7 (and also Theorem 4). In Section 5 we discuss the tightness of our results. Finally, in the Appendix we prove a decoupled version of Lemma 6.

## 2 Preliminaries

### 2.1 Biased Fourier-Walsh Expansion of Functions on the Discrete Cube

Throughout the paper we consider the discrete cube  $\Omega = \{0, 1\}^n$ , endowed with a probability product measure  $\mu = \mu_p \otimes \dots \otimes \mu_p$ , i.e.,

$$\mu(x) = \mu((x_1, \dots, x_n)) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}.$$

Elements of  $\Omega$  are represented either by binary vectors of length  $n$ , or by subsets of  $\{1, 2, \dots, n\}$ . Denote the set of all real-valued functions on the discrete cube by  $Y$ . The inner product of functions  $f, g \in Y$  is defined as usual as

$$\langle f, g \rangle = \mathbb{E}[fg] = \sum_{x \in \{0, 1\}^n} \mu(x) f(x) g(x).$$

This inner product induces a norm on  $Y$ :

$$\|f\|_2 = \sqrt{\langle f, f \rangle} = \sqrt{\mathbb{E}[f^2]}.$$

**Walsh Products.** Consider the functions  $\{\omega_i\}_{i=1}^n$ , defined as:

$$\omega_i(x_1, \dots, x_n) = \begin{cases} \sqrt{\frac{1-p}{p}}, & x_i = 1 \\ -\sqrt{\frac{p}{1-p}}, & x_i = 0. \end{cases}$$

As was observed in [Tal94], these functions constitute an orthonormal system in  $Y$  (with respect to the measure  $\mu$ ). Moreover, this system can be completed to an orthonormal basis in  $Y$  by defining

$$\omega_T = \prod_{i \in T} \omega_i$$

for all  $T \subset \{1, \dots, n\}$ . The functions  $\omega_T$  are called (biased) Walsh products.

**Fourier-Walsh expansion.** Every function  $f \in Y$  can be represented by its Fourier-Walsh expansion with respect to the system  $\{\omega_T\}_{T \subset \{1, \dots, n\}}$ :

$$f = \sum_{T \subset \{1, \dots, n\}} \langle f, \omega_T \rangle \omega_T.$$

The coefficients in this expansion are denoted

$$\hat{f}(T) = \langle f, \omega_T \rangle.$$

A coefficient  $\hat{f}(T)$  is called *k-th level coefficient* if  $|T| = k$ . By the Parseval identity, for all  $f \in Y$  we have

$$\sum_{T \subset \{1, \dots, n\}} \hat{f}(T)^2 = \|f\|_2^2.$$

**Relation between Fourier-Walsh expansion, noise stability, and influences** The noise stability of a Boolean function can be expressed in a convenient way in terms of the Fourier-Walsh expansion of the function.

**Claim 8.** For any function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  and for any  $\epsilon > 0$ , we have

$$\mathcal{S}_\epsilon(f) = \sum_{S \neq \emptyset} (1 - \epsilon)^{|S|} \hat{f}(S)^2.$$

The assertion is obtained by direct computation in the case where  $f$  is a linear character, and it follows for general characters by multiplicativity of expectation for independent random variables. It then follows for the general case by linearity of expectation.

The influences are also related to the Fourier-Walsh expansion. It can be easily shown that  $p(1-p) \cdot \sum_{i=1}^n I_i(f) = \sum_S |S| \hat{f}(S)^2$ . Moreover, the influences are specifically related to the first-level Fourier coefficients. Indeed, denoting by  $x_{-i} \in \{0, 1\}^{[n] \setminus \{i\}}$  the vector obtained from  $x$  by omitting the  $i$ 'th coordinate, we have (for any Boolean function  $f$  and for any  $1 \leq i \leq n$ ):

$$|\hat{f}(\{i\})| = |\mathbb{E}_x[\omega_i(x)f(x)]| \leq \mathbb{E}_{x_{-i} \in \{0, 1\}^{[n] \setminus \{i\}}} [|\mathbb{E}_{x_i \in \{0, 1\}}[\omega_i(x)f(x)]|] = \sqrt{p(1-p)} I_i(f),$$

and thus,

$$\sum_{i=1}^n \hat{f}(\{i\})^2 \leq \mathcal{W}(f). \tag{6}$$

These expressions of noise stability and influences in terms of the Fourier-Walsh expansion play an important role in our proof.

## 2.2 Sharp Bound on Large Deviations Using the Hypercontractive Inequality

A crucial component in the proof of Lemma 5 is a bound on the large deviations of low-degree multivariate polynomials. Formally, for any  $d \geq 1$ , we would like to bound the probability  $\Pr[|f| \geq t]$ , for every function  $f$  whose Fourier degree is at most  $d$ . In the uniform measure case, such bound was obtained in [BKS99] using the Bonami-Beckner hypercontractive inequality [Bon70, Bec75]. In the biased case, one should use a biased version of the Bonami-Beckner inequality instead, and the strength of the obtained bound depends on the *hypercontractivity constant*, which depends on the bias. The optimal value of the hypercontractivity constant for biased measures was obtained by Oleszkiewicz in 2003 [Ole03]. For ease of presentation, we cite a large deviation bound, presented in [DFKO07]<sup>1</sup>, that relies on a slightly weaker estimate of the hypercontractivity constant.

**Definition 9.** For all  $0 < p < 1$ , let

$$B(p) = \frac{\frac{1-p}{p} - \frac{p}{1-p}}{2 \ln \frac{1-p}{p}} = \frac{(1-p) - p}{2p(1-p)(\ln(1-p) - \ln p)}. \quad (7)$$

**Theorem 10** (Lemma 2.2 in [DFKO07]). Let  $f : \{0, 1\}^n \rightarrow \mathbb{R}$  have Fourier degree at most  $d$ , and assume that  $\|f\|_2 = 1$ . Then for any  $t \geq (2B(p)e)^{d/2}$ ,

$$\Pr[|f| \geq t] \leq \exp\left(-\frac{d}{2B(p)e} t^{2/d}\right). \quad (8)$$

The next lemma, which easily follows from Fubini's theorem, allows using large deviation bounds to evaluate certain expectations. The integral that we get when we later apply it, using the bounds in Theorem 10, is considered in Lemma 12.

**Lemma 11.** Let  $\Omega$  be a probability space, and let  $f, g : \Omega \rightarrow \mathbb{R}$  be functions, where  $g$  is non-negative. For any real number  $t$ , let  $L(t) \subseteq \Omega$  be defined by  $L(t) = \{x : g(x) > t\}$ , and let  $\mathbf{1}_{L(t)}$  be the indicator of the set  $L(t)$ . Then we have

$$\mathbb{E}_{x \in \Omega}[f(x)g(x)] = \int_{t=0}^{\infty} (\mathbb{E}_{x \in \Omega}[f(x) \cdot \mathbf{1}_{L(t)}(x)]) dt$$

**Proof:**

$$\begin{aligned} \mathbb{E}_{x \in \Omega}[f(x)g(x)] &= \mathbb{E}_{x \in \Omega} \left[ f(x) \cdot \int_{t=0}^{g(x)} \mathbf{1} dt \right] = \mathbb{E}_{x \in \Omega} \left[ \int_{t=0}^{\infty} f(x) \cdot \mathbf{1}_{L(t)}(x) dt \right] \\ &= \int_{t=0}^{\infty} (\mathbb{E}_{x \in \Omega}[f(x) \cdot \mathbf{1}_{L(t)}(x)]) dt, \end{aligned}$$

where the last equality follows from Fubini's theorem.  $\square$

**Lemma 12.** Let  $d \geq 2$  be a positive integer, and let  $t_0$  be such that  $t_0 > (4B(p) \cdot e)^{(d-1)/2}$ . Then

$$\int_{t=t_0}^{\infty} t^2 \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t^{2/(d-1)}\right) dt \leq 5B(p) \cdot e \cdot t_0^{3-\frac{2}{d-1}} \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)}\right). \quad (9)$$

<sup>1</sup>In fact there is a small typo in the formula for  $B(p)$  in [DFKO07], which is fixed here.

**Proof:** To bound the l.h.s. of (9) we first apply a change of variables, setting

$$s = \frac{(d-1)}{2B(p) \cdot e} \cdot t^{2/(d-1)}, \quad (10)$$

and obtaining

$$\begin{aligned} \int_{t=t_0}^{\infty} t^2 \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t^{2/(d-1)}\right) dt \\ = \left(\frac{2B(p) \cdot e}{d-1}\right)^{3(d-1)/2} \cdot \frac{d-1}{2} \cdot \int_{s=s_0}^{\infty} s^{(3d-5)/2} \cdot \exp(-s) ds \end{aligned} \quad (11)$$

where  $s_0 = \frac{(d-1)}{2B(p) \cdot e} \cdot (t_0)^{2/(d-1)}$ . If we denote the integrand on the r.h.s. of (11) by  $\varphi(s)$ , one notes that for  $s \geq s_0$ ,  $\varphi(s)$  is decreasing and  $\varphi(s+1)/\varphi(s) \leq \exp(-1/4)$  – this follows from the condition on  $t_0$  and from (10). It therefore follows that the integral on the r.h.s. of (11) is bounded by  $(s_0)^{(3m-5)/2} \cdot \exp(-s_0) \cdot \frac{1}{1-\exp(-1/4)} \leq 5(s_0)^{(3m-5)/2} \cdot \exp(-s_0)$ . Substituting into (10) gives the lemma.  $\square$

### 3 Proof of Lemma 6

**Notation 1.** Throughout the proof, we use a “normalized” variant of the influences:

$$I'_i(f) = \sqrt{p(1-p)} I_i(f).$$

This notation is only technical, and is intended to avoid carrying the factor  $\sqrt{p(1-p)}$  along the proof. Note that  $\mathcal{W}(f) = \sum_{i=1}^n I'_i(f)^2$ .

#### 3.1 Two key observations

The key to the proof of Lemma 6 is based on two observations, as was the proof in [Tal97].

**First observation.** We write the space  $\{0, 1\}^n$  as a product of two probability spaces  $\{0, 1\}^I$  and  $\{0, 1\}^J$ . We consider for every  $j \in J$ , the part of the Fourier-Walsh expansion of  $f$ , which consists of Walsh products whose sole representative in  $J$  is  $j$ .

We now note that it is sufficient to prove that for every partition  $\{I, J\}$  of  $\{1, \dots, n\}$ ,

$$\sum_{\substack{T \subset I \\ |T|=d-1}} \sum_{j \in J} \hat{f}(\{T, j\})^2 \leq 5 \cdot \left(\frac{2B(p) \cdot e}{d-1}\right)^{d-1} \cdot \left(\sum_{j \in J} I'_j(f)^2\right) \cdot \left(\log\left(\frac{1}{\sum_{j \in J} I'_j(f)^2}\right)\right)^{d-1}. \quad (12)$$

The assertion of Lemma 6 will follow from (12) by taking expectation over the partitions  $\{I, J\}$ , such that every coordinate is independently put into  $J$  with probability  $1/d$ . We give the exact details at the end of the proof.

**Second observation.** The second observation is that we can write the left-hand-side of (12) as the inner-product of  $f$  with a function of the form  $\sum f'_j \omega_j$ , where the functions  $\{f'_j\}$  are all of low degree, and depend only on coordinates from  $I$ . The low degree of the  $f'_j$ 's will allow us to use Theorem 10 to bound them.

For a given partition  $\{I, J\}$  of  $\{1, \dots, n\}$  and an index  $j \in J$ , let

$$f'_j = \sum_{T \subset I, |T|=m-1} \hat{f}(T, j) \omega_T.$$

Note that  $f'_j$  indeed depends only on coordinates from  $I$ . We have

$$\langle f'_j \cdot \omega_j, f \rangle = \left\langle \sum_{T \subset I, |T|=d-1} \hat{f}(T, j) \omega_{T \cup \{j\}}, f \right\rangle = \sum_{T \subset I, |T|=d-1} \hat{f}(\{T, j\})^2, \quad (13)$$

and summing over  $j$  we have

$$\left\langle \sum_j f'_j \cdot \omega_j, f \right\rangle = \sum_{T \subset I, |T|=d-1} \sum_{j \in J} \hat{f}(\{T, j\})^2 = \text{l.h.s. of (12)}. \quad (14)$$

It will be convenient for us to normalize  $f'_j$ , hence we take  $f_j = f'_j / \|f'_j\|_2$ . It follows from (13) that for every  $j \in J$ ,

$$\langle f_j \cdot \omega_j, f \rangle = \left( \sum_{T \subset I, |T|=d-1} \hat{f}(\{T, j\})^2 \right)^{1/2}. \quad (15)$$

### 3.2 Proof of (12).

Using (15) and the fact that  $f_j$  only depends on coordinates from  $I$ , we have for every  $j \in J$  that

$$\sum_{T \subset I, |T|=d-1} \hat{f}(T, j)^2 = (\langle f_j \cdot \omega_j, f \rangle)^2 = (\langle f_j, \omega_j \cdot f \rangle)^2 \quad (16)$$

$$\begin{aligned} &= \left( \mathbb{E}_{x \in \{0,1\}^I} \left[ f_j(x) \cdot \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right] \right)^2 \\ &\leq \left( \mathbb{E}_{x \in \{0,1\}^I} \left[ |f_j(x)| \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] \right)^2. \end{aligned} \quad (17)$$

We now use Lemma 11 to bound (17) by considering the two multiplicands in the expectations as functions over  $\{0, 1\}^I$ , and obtain

$$(17) \leq \left( \int_{t=0}^{\infty} \mathbb{E}_{x \in \{0,1\}^I} \left[ \mathbf{1}_{\{|f_j(x)| > t\}} \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] dt \right)^2.$$

Using the inequality  $(a+b)^2 \leq 2a^2 + 2b^2$  we thus have that for any parameter  $t_0$ , (17) is bounded above by

$$2 \left( \int_{t=0}^{t_0} \mathbb{E}_{x \in \{0,1\}^I} \left[ \mathbf{1}_{\{|f_j(x)| > t\}} \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] dt \right)^2 \quad (18)$$

$$+ 2 \left( \int_{t=t_0}^{\infty} \mathbb{E}_{x \in \{0,1\}^I} \left[ \mathbf{1}_{\{|f_j(x)| > t\}} \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] dt \right)^2. \quad (19)$$

We will bound separately each of these summands.

**Bounding (18).** For  $z \in \{0, 1\}^{[n]}$ , we denote by  $z_{-j} \in \{0, 1\}^{[n] \setminus \{j\}}$  the vector obtained from  $z$  by omitting the  $j$ 'th coordinate. Since an indicator function is bounded by 1, we have

$$\begin{aligned}
(18) &\leq 2 \left( \int_{t=0}^{t_0} \mathbb{E}_{x \in \{0,1\}^I} \left[ \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] \right)^2 \\
&\leq 2t_0^2 \cdot \left( \mathbb{E}_{x \in \{0,1\}^I} \left[ \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] \right)^2 \\
&\leq 2t_0^2 \cdot \left( \mathbb{E}_{x \in \{0,1\}^I} \mathbb{E}_{y' \in \{0,1\}^{J \setminus \{j\}}} \left[ \left| \mathbb{E}_{y_j \in \{0,1\}} [\omega_j(x, y', y_j) f(x, y', y_j)] \right| \right] \right)^2 \\
&= 2t_0^2 \cdot \left( \mathbb{E}_{z_{-j} \in \{0,1\}^{[n] \setminus \{j\}}} \left[ \left| \mathbb{E}_{z_j \in \{0,1\}} [\omega_j(z) f(z)] \right| \right] \right)^2 = 2t_0^2 \cdot I'_j(f)^2.
\end{aligned}$$

**Bounding (19).** In the computation below, we explain some transitions below the corresponding line. We note that the two last implications apply if  $t_0 > (4B(p)e)^{(d-1)/2}$ , which is indeed the case for the  $t_0$  that is chosen later.

$$\begin{aligned}
(19) &= 2 \left( \int_{t=t_0}^{\infty} \frac{1}{t} \cdot t \cdot \mathbb{E}_{x \in \{0,1\}^I} \left[ \mathbf{1}_{\{|f_j(x)| > t\}} \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] dt \right)^2 \\
&\leq 2 \left( \int_{t=t_0}^{\infty} \frac{1}{t^2} dt \right) \cdot \int_{t=t_0}^{\infty} t^2 \cdot \left( \mathbb{E}_{x \in \{0,1\}^I} \left[ \mathbf{1}_{\{|f_j(x)| > t\}} \cdot \left| \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right| \right] \right)^2 dt
\end{aligned}$$

(by Cauchy-Schwarz)

$$\leq \frac{2}{t_0} \cdot \int_{t=t_0}^{\infty} t^2 \cdot \mathbf{Pr}_{x \in \{0,1\}^I} [ |f_j(x)| > t ] \cdot \mathbb{E}_{x \in \{0,1\}^I} \left[ \left( \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 \right] dt$$

(by applying Cauchy-Schwarz on the space  $\{0, 1\}^I$ )

$$\leq \frac{2}{t_0} \cdot \mathbb{E}_{x \in \{0,1\}^I} \left[ \left( \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 \right] \cdot \int_{t=t_0}^{\infty} t^2 \cdot \exp \left( -\frac{(d-1)}{2B(p)e} t^{2/(d-1)} \right) dt$$

(by pulling the expectation outside of the integral, as it does not depend on  $t$ , and bounding the deviation of  $f_j$  using Theorem 10 )

$$\leq 10e \cdot t_0^{2-\frac{2}{d-1}} \cdot B(p) \cdot \exp \left( -\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)} \right) \cdot \mathbb{E}_{x \in \{0,1\}^I} \left[ \left( \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 \right]$$

(using Lemma 12).

**Proving inequality (12).** Since the sum of (18) and (19) bounds the l.h.s. of (16), we have from the above bounds that

$$\begin{aligned}
\sum_{T \subset I, |T|=d-1} \hat{f}(T, j)^2 &\leq 2t_0^2 \cdot I'_j(f)^2 + \\
&+ 10e \cdot t_0^{2-\frac{2}{d-1}} \cdot B(p) \cdot \exp \left( -\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)} \right) \cdot \mathbb{E}_{x \in \{0,1\}^I} \left[ \left( \mathbb{E}_{y \in \{0,1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 \right].
\end{aligned} \tag{20}$$

We now use the following observation: for any fixed  $x \in \{0, 1\}^I$ , let  $f_x : \{0, 1\}^J \rightarrow \{0, 1\}$  be defined by  $f_x(y) = f(x, y)$ . Then

$$\left( \mathbb{E}_{y \in \{0, 1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 = \widehat{f}_x(\{j\})^2.$$

Since  $\|f_x\|_2 \leq 1$ , by Parseval's identity we have that for every  $x \in \{0, 1\}^I$ ,

$$\sum_{j \in J} \left( \mathbb{E}_{y \in \{0, 1\}^J} [\omega_j(x, y) f(x, y)] \right)^2 \leq 1. \quad (21)$$

By summing (20) over  $j \in J$  and substituting (21) inside the expectation, we obtain that

$$\sum_{j \in J} \sum_{\substack{T \subset I \\ |T|=d-1}} \widehat{f}(T, j)^2 \leq 2t_0^2 \cdot \sum_{j \in J} I'_j(f)^2 + 10e \cdot t_0^{2-\frac{2}{d-1}} \cdot B(p) \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)}\right). \quad (22)$$

We now choose  $t_0$  so that

$$\exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)}\right) = \sum_{j \in J} I'_j(f)^2.$$

A simple computation shows that

$$(t_0)^2 = \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \left( \log \left( \frac{1}{\sum_{j \in J} I'_j(f)^2} \right) \right)^{d-1} \quad (23)$$

and by assumption (3), we have that  $(t_0)^{2/(d-1)} \geq 4B(p) \cdot e$ , satisfying the requirement mentioned in the bound on (19). We now substitute  $t_0$  in (22), obtaining the bound

$$\sum_{j \in J} \sum_{\substack{T \subset I \\ |T|=d-1}} \widehat{f}(T, j)^2 \leq 5 \cdot \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \left( \sum_{j \in J} I'_j(f)^2 \right) \cdot \left( \log \left( \frac{1}{\sum_{j \in J} I'_j(f)^2} \right) \right)^{d-1}, \quad (24)$$

thus proving (12).

**Completing the proof of Lemma 6.** Let us choose  $J \subset \{1, \dots, n\}$  to be a random subset, independently containing each coordinate with probability  $1/d$ , and let  $I = [n] \setminus J$ . For each subset  $S \subseteq \{1, \dots, n\}$  of size  $d$ , the probability that it can be represented as a pair  $(T, j)$  where  $T \subseteq I$  and  $j \in J$ , in which case  $\widehat{f}(S)^2$  is included as a summand in the left-hand-side of (12), is  $((d-1)/d)^{d-1} > 1/e$ . Hence

$$\sum_{|S|=d} \widehat{f}(S)^2 \leq e \cdot \mathbb{E}_J \left[ \sum_{j \in J} \sum_{\substack{T \subset I \\ |T|=d-1}} \widehat{f}(T, j)^2 \right]. \quad (25)$$

We wish to apply a similar argument to the r.h.s. of (12), using the fact that each coordinate in  $\{1, \dots, n\}$  appears in the sum with probability  $1/d$ . We observe that the function  $x \log(1/x)^{d-1}$

is concave in the segment  $[0, \exp(-(d-1))]$ , and by assumption (3) the sum  $\sum_{j \in J} I'_j(f)^2$  is in this range for any  $J \subseteq \{1, \dots, n\}$ . We thus have

$$\begin{aligned} & \mathbb{E}_J \left[ 5 \cdot \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \left( \sum_{j \in J} I'_j(f)^2 \right) \cdot \left( \log \left( \frac{1}{\sum_{j \in J} I'_j(f)^2} \right) \right)^{d-1} \right] \\ & \leq \frac{5}{d} \cdot \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \left( \sum_{j \in \{1, \dots, n\}} I'_j(f)^2 \right) \cdot \left( \log \left( \frac{d}{\sum_{j \in \{1, \dots, n\}} I'_j(f)^2} \right) \right)^{d-1} \\ & = \frac{5}{d} \cdot \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \mathcal{W}(f) \cdot \left( \log \left( \frac{d}{\mathcal{W}(f)} \right) \right)^{d-1}. \end{aligned} \quad (26)$$

The combination of (25) and (26) completes the proof.

## 4 Proof of Theorem 7

Let  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  be a function, and let  $\mathcal{W} = \mathcal{W}(f) = p(1-p) \cdot \sum_{j=1}^n I_j(f)^2$ . Our goal is to show that

$$\mathcal{S}_\epsilon(f) \leq (6e+1)\mathcal{W}^{\alpha-\epsilon}, \quad \text{where} \quad \alpha = \frac{1}{\epsilon + \log(2B(p)e) + 3 \log \log(2B(p)e)}. \quad (27)$$

Recall that by Claim 8, we have

$$\mathcal{S}_\epsilon(f) = \sum_{S \neq \emptyset} (1-\epsilon)^{|S|} \hat{f}(S)^2. \quad (28)$$

For some  $L$  that we choose later, we write

$$\mathcal{S}_\epsilon(f) = \sum_{0 < |S| \leq L} (1-\epsilon)^{|S|} \hat{f}(S)^2 + \sum_{|S| > L} (1-\epsilon)^{|S|} \hat{f}(S)^2, \quad (29)$$

and bound each of the terms separately.

**Bounding the high degrees term.** The second term in (29) is dominated by the powers of  $(1-\epsilon)$ . Since  $\|f\|_2^2 \leq 1$ , we have from Parseval's identity that

$$\sum_{|S| > L} (1-\epsilon)^{|S|} \hat{f}(S)^2 \leq (1-\epsilon)^{L+1} \cdot \sum_{|S| > L} \hat{f}(S)^2 \leq (1-\epsilon)^{L+1}. \quad (30)$$

**Bounding the low degrees term.** Here we neglect the powers of  $(1-\epsilon)$  and use Lemma 6 to bound the Fourier coefficients of degree  $d$  for each  $1 < d \leq L$  (for  $d = 1$  we use Equation (6)).

$$\sum_{0 < |S| \leq L} (1-\epsilon)^{|S|} \hat{f}(S)^2 \leq \sum_{d=1}^L \sum_{|S|=d} \hat{f}(S)^2 \leq \mathcal{W} + \sum_{d=2}^L \frac{5e}{d} \cdot \left( \frac{2B(p) \cdot e}{d-1} \right)^{d-1} \cdot \mathcal{W} \cdot \left( \log \left( \frac{d}{\mathcal{W}} \right) \right)^{d-1}. \quad (31)$$

Assume that  $L \leq B(p) \log(1/\mathcal{W})$ . In that case for any  $d$ ,  $1 < d < L$ , the ratio between the  $(d+1)$  term and the  $d$  term in (31) is bounded from below by 2. Indeed, this ratio is

$$\begin{aligned} & \frac{d}{d+1} \cdot 2B(p) \cdot e \cdot \left(\frac{d-1}{d}\right)^{d-1} \cdot \frac{1}{d} \cdot \left(\frac{\log((d+1)/\mathcal{W})}{\log((d)/\mathcal{W})}\right)^{d-1} \cdot \log\left(\frac{d+1}{\mathcal{W}}\right) \\ & \geq \frac{2B(p) \cdot e}{d+1} \cdot \frac{1}{e} \cdot 1 \cdot \log\left(\frac{d+1}{\mathcal{W}}\right) \geq \frac{2}{\log(1/\mathcal{W})} \cdot \log\left(\frac{d+1}{\mathcal{W}}\right) \geq 2. \end{aligned}$$

We can thus replace the sum in (31) by twice its last term, thereby getting

$$\sum_{0 < |S| \leq L} (1-\epsilon)^{|S|} \hat{f}(S)^2 \leq \mathcal{W} + \frac{10e}{L} \cdot \left(\frac{2B(p) \cdot e}{L-1}\right)^{L-1} \cdot \mathcal{W} \cdot \left(\log\left(\frac{L}{\mathcal{W}}\right)\right)^{L-1}. \quad (32)$$

**Choosing the value of  $L$ .** We choose the value of  $L$  such that the bound on  $\mathcal{S}_\epsilon(f)$  obtained from (29), (30) and (32) is minimized. We thus take

$$L = \alpha \cdot \log(1/\mathcal{W}), \quad \text{where} \quad \alpha = \frac{1}{\epsilon + \log(2B(p)e) + 3 \log \log(2B(p)e)}. \quad (33)$$

Theorem 7 is obtained immediately from the following claim.

**Claim 13.** *For  $\alpha$  and  $L$  as chosen in (33), it holds that*

$$(30) \leq \mathcal{W}^{\alpha-\epsilon},$$

and

$$(32) \leq 6e \cdot \mathcal{W}^{\alpha-\epsilon}.$$

**Proof:** The first inequality follows since

$$(30) \leq (1-\epsilon)^L \leq \exp(-\epsilon \cdot \alpha \cdot \log(1/\mathcal{W})) = \mathcal{W}^{\alpha-\epsilon}.$$

To bound (32), we first note that

$$\begin{aligned} (32) &= \mathcal{W} + \frac{10e}{L} \cdot \left(\frac{2B(p) \cdot e}{\alpha}\right)^{L-1} \cdot \left(\frac{\alpha}{L}\right)^{L-1} \cdot \left(\frac{L}{L-1}\right)^{L-1} \cdot \mathcal{W} \cdot \left(\log\left(\frac{L}{\mathcal{W}}\right)\right)^{L-1} \\ &\leq \mathcal{W} + \frac{10e^2}{L} \cdot \left(\frac{1}{\log(1/\mathcal{W})}\right)^{L-1} \cdot \left(\log\left(\frac{L}{\mathcal{W}}\right)\right)^{L-1} \cdot \left[\mathcal{W} \cdot \left(\frac{2B(p) \cdot e}{\alpha}\right)^{L-1}\right] \\ &= \mathcal{W} + \frac{10e^2}{L} \cdot \left(1 + \frac{\log(L)}{\log(1/\mathcal{W})}\right)^{L-1} \cdot \left[\mathcal{W} \cdot \left(\frac{2B(p) \cdot e}{\alpha}\right)^{L-1}\right] \\ &\leq \mathcal{W} + \frac{10e^2}{L} \cdot \left(1 + \frac{\log(L)}{\log(1/\mathcal{W})}\right)^{\log(1/\mathcal{W})} \cdot \left[\mathcal{W} \cdot \left(\frac{2B(p) \cdot e}{\alpha}\right)^{L-1}\right] \\ &\leq \mathcal{W} + 10e^2 \cdot \left[\mathcal{W} \cdot \left(\frac{2B(p) \cdot e}{\alpha}\right)^{L-1}\right], \end{aligned}$$

where the inequality before last holds because  $\alpha \leq 1$  (since  $B(p) \geq 1$  for all  $p$ ).

Since  $\mathcal{W} \leq \mathcal{W}^{\alpha \cdot \epsilon}$ , to finish the claim it remains to prove that

$$\mathcal{W} \cdot \left( \frac{2B(p) \cdot e}{\alpha} \right)^{L-1} \leq \frac{1}{2e} \cdot \mathcal{W}^{\alpha \cdot \epsilon}. \quad (34)$$

Note that

$$\begin{aligned} \mathcal{W} \cdot \left( \frac{2B(p) \cdot e}{\alpha} \right)^{L-1} &\leq \frac{1}{2B(p) \cdot e} \cdot \mathcal{W} \cdot \left( \frac{2B(p) \cdot e}{\alpha} \right)^{\alpha \cdot \log(1/\mathcal{W})} \\ &= \frac{1}{2B(p) \cdot e} \cdot \mathcal{W} \cdot (\mathcal{W})^{-\alpha \cdot \log\left(\frac{2B(p) \cdot e}{\alpha}\right)} \\ &\leq \frac{1}{2e} \cdot \mathcal{W}^{1 - \alpha \cdot \log\left(\frac{2B(p) \cdot e}{\alpha}\right)}. \end{aligned}$$

Hence it is sufficient to show that the exponent above is higher than  $\alpha \cdot \epsilon$ . Since

$$\begin{aligned} 1 - \alpha \cdot \log\left(\frac{2B(p) \cdot e}{\alpha}\right) &= \frac{\epsilon + 3 \log \log(2B(p) \cdot e) - \log(1/\alpha)}{\epsilon + \log(2B(p) \cdot e) + 3 \log \log(2B(p) \cdot e)} \\ &= \alpha \cdot \epsilon + \frac{3 \log \log(2B(p) \cdot e) - \log(1/\alpha)}{\epsilon + \log(2B(p) \cdot e) + 3 \log \log(2B(p) \cdot e)}, \end{aligned}$$

we actually need to prove that  $3 \log \log(2B(p) \cdot e) - \log(1/\alpha) \geq 0$ . Substituting the value of  $\alpha$  and simplifying, this reduces to

$$\epsilon + \log(2B(p) \cdot e) + 3 \log \log(2B(p) \cdot e) \leq (\log(2B(p) \cdot e))^3. \quad (35)$$

It is easy to verify that the function  $t^3 - t - 3 \log t - \epsilon$  is monotone increasing in  $t$  for every  $t \geq 1.5$ , and for  $\epsilon \leq 1$  its value for  $t = 1.5$  is positive. Hence, since  $\log(2B(p)e) \geq 1.5$ , (35) follows. This completes the proof of Claim 13, and thus of Theorem 7.  $\square$

Theorem 4 follows immediately from Theorem 7, substituting in Equation (27)  $B(p) = 1$  for  $p = 1/2$  and bounding  $\epsilon$  from above by 1.

## 5 Tightness of Lemma 6 and Theorem 7

In this section we examine a variant of the “tribes” function presented in [BOL90]. We show that the assertions of Lemma 6 and Theorem 7 are essentially tight for this function.

The tribes function over  $n$  coordinates with tribes of size  $r$  is defined as follows: we partition  $n$  coordinates into sets (tribes) of size  $r$  each, and let the tribes function assume the value 1 if in at least one tribe all the coordinates are equal to 1, and 0 otherwise. In order to make our function approximately balanced with respect to the biased measure  $\mu_p$ , we take  $f$  to be a tribes function with tribes of size

$$r = \frac{\log n - \log \log n + \log \log(1/p)}{\log(1/p)}.$$

This variant of the tribes function was first considered in [Tal94]. It is easy to see that for this choice of  $r$  we have  $\mathbb{E}_{x \sim \mu_p}[f(x)] \approx 1 - 1/e$ . In the following computations we use the symbol  $\approx$  to denote equality up to constant factors.

## 5.1 Tightness of Lemma 6

To simplify the computations, we add a restriction on the range of parameters we consider. We require that

$$d \leq \min(1/p, \sqrt{r}, \log n / \log \log n). \quad (36)$$

We note that a wide range of combinations of the parameters satisfies this restriction. For example, if  $p = 1/\log n$ , then (36) holds for all  $d \leq \sqrt{\log n / (2 \log \log n)}$ .

**Evaluating the right hand side of (4).** For each  $x \in \{0, 1\}^n$ , we have  $f(x) \neq f(x \oplus e_i)$  if and only if in the tribe of  $i$ , all the coordinates of  $x$  other than  $x_i$  are ones, and in each of the other tribes, at least one of the coordinates is zero. Thus, the influences of  $f$  are:

$$I_i(f) = \Pr[f(x) \neq f(x \oplus e_i)] = p^{r-1}(1-p^r)^{(n/r)-1} \approx p^{r-1} = \frac{\log n}{p \log(1/p) \cdot n}.$$

Summing over the values of  $i$ , we get

$$\mathcal{W}(f) = p(1-p) \sum_{i=1}^n I_i(f)^2 \approx \frac{(\log n)^2}{p(\log(1/p))^2 \cdot n}. \quad (37)$$

Since by Assumption (36),  $d \leq \frac{\log n}{\log(1/p)}$ , we have

$$(\log(1/\mathcal{W}(f)))^{d-1} \approx \left(1 - \frac{\log(1/p)}{\log n}\right)^{d-1} \log n \approx (\log n)^{d-1}. \quad (38)$$

It follows that for our function  $f$ , the right hand side of (4) is approximately

$$\frac{5e}{d} \cdot \left(\frac{2B(p) \cdot e}{d-1}\right)^{d-1} \cdot \frac{(\log n)^{d+1}}{p(\log(1/p))^2 \cdot n}. \quad (39)$$

Finally, since by Equation (7),  $B(p) \leq \frac{1}{2p(1-p)\log((1-p)/p)}$ , and since by Assumption (36),  $d \leq 1/p$ , the right hand side of (4) is at most

$$\frac{5e}{d} \cdot \left(\frac{e}{(d-1) \cdot p(1-p)\log((1-p)/p)}\right)^{d-1} \cdot \frac{(\log n)^{d+1}}{p(\log(1/p))^2 \cdot n} \approx \left(\frac{e}{d}\right)^d \cdot \frac{(\log n)^{d+1}}{p^d \cdot (\log(1/p))^{d+1} \cdot n}. \quad (40)$$

**Evaluating the left hand side of (4).** We compute a lower bound on the l.h.s. of (4) by considering only part of the  $d$ -th level Fourier-Walsh coefficients of  $f$ . We compute the coefficients of the form  $\hat{f}(\{i_1, \dots, i_d\})$  where  $\{i_1, \dots, i_d\}$  belong to the same tribe, the idea being that these are the dominant  $d$ -th level coefficients. We want to compute

$$\hat{f}(\{i_1, \dots, i_d\}) = \mathbb{E}[f \cdot \omega_{\{i_1, \dots, i_d\}}].$$

We divide  $\{0, 1\}^n$  into structures of  $2^d$  values each according to the coordinates  $\{i_1, \dots, i_d\}$ . Note that a structure does not contribute to  $\mathbb{E}[f \cdot \omega_{\{i_1, \dots, i_d\}}]$  if the value of  $f$  on all the elements of the structure is the same. Since  $\{i_1, \dots, i_d\}$  are all in the same tribe, the only case in which  $f$  is not constant on a structure is when in the tribe containing  $\{i_1, \dots, i_d\}$ , all the other coordinates

are ones, and in each of the other tribes there is at least one zero element. For such structures,  $f$  assumes the value 1 only when all the coordinates  $\{i_1, \dots, i_d\}$  are ones. Hence,

$$\mathbb{E}[f \cdot \omega_{\{i_1, \dots, i_d\}}] = \left(-\sqrt{(1-p)/p}\right)^d p^r (1-p^r)^{(n/r)-1} \approx (-1)^d p^{-d/2} p^r \approx (-1)^d p^{-d/2} \cdot \frac{\log n}{\log(1/p) \cdot n}. \quad (41)$$

The number of  $d$ -th level coefficients of this type is  $\frac{n}{r} \binom{r}{d}$ . Since by Assumption (36), we have  $d \leq \sqrt{r}$  and  $d \leq \frac{\log \log n}{\log n}$ , it follows that

$$\frac{n}{r} \binom{r}{d} \geq \frac{n}{r} \frac{(r-d)^d}{d!} \approx \frac{nr^{d-1}}{d!} \approx \frac{n(\log n)^{d-1}}{d! (\log(1/p))^{d-1}}.$$

Therefore, using Stirling's approximation  $d! \approx \sqrt{2\pi d} (d/e)^d$ , we get the following lower bound on the left hand side of (4):

$$\sum_{|S|=d} \hat{f}(S)^2 \gtrsim \frac{n(\log n)^{d-1}}{d! (\log(1/p))^{d-1}} \cdot \frac{(\log n)^2}{p^d \cdot (\log(1/p))^2 \cdot n^2} \approx \left(\frac{e}{d}\right)^d \frac{(\log n)^{d+1}}{\sqrt{d} \cdot p^d \cdot (\log(1/p))^{d+1} \cdot n}. \quad (42)$$

Comparing expressions (40) and (42) shows that the assertion of Lemma 6 is tight up to factor  $c\sqrt{d}$ , which is small compared to the other terms in the expressions in both sides of Equation (4).

## 5.2 Tightness of Theorem 7

In order to show that the assertion of Theorem 7 is tight up to a constant factor in the exponent, we have to prove that

$$\log \mathcal{S}_\epsilon(f) \approx \alpha(\epsilon) \cdot \epsilon \cdot \log \mathcal{W}(f). \quad (43)$$

To simplify the computations, we assume that  $p \leq 1/\log n$ , and in particular we have  $r \leq 1/p$  (we also deal separately with the uniform measure case below).

We compute a lower bound on  $(\log \mathcal{S}_\epsilon(f))$  by considering part of the  $r$ -th level Fourier-Walsh coefficients of  $f$ . By Formula (41), each of the coefficients  $\hat{f}(\{i_1, i_2, \dots, i_r\})$  corresponding to a full tribe equals

$$\left(-\sqrt{(1-p)/p}\right)^r p^r (1-p^r)^{(n/r)-1} \approx (-1)^r p^{-r/2} p^r = (-1)^r p^{r/2}.$$

The number of coefficients of this form is  $n/r$ . Thus,

$$\sum_{|S|=r} \hat{f}(S)^2 \gtrsim \frac{np^r}{r} \approx \frac{n \log n}{\log(1/p) \cdot n \cdot r} = \Theta(1).$$

Hence, by Claim 8,

$$\mathcal{S}_\epsilon(f) \geq \sum_{|S|=r} \hat{f}(S)^2 (1-\epsilon)^r \gtrsim (1-\epsilon)^r,$$

and therefore,

$$\log(\mathcal{S}_\epsilon(f)) \gtrsim \frac{\log(1-\epsilon) \log n}{\log(1/p)}. \quad (44)$$

On the other hand, using Formula (38) and the approximation  $\log(2B(p)e) \approx \log(1/p)$ , we get

$$\alpha(\epsilon) \cdot \epsilon \cdot \log \mathcal{W}(f) \approx \frac{-\epsilon \log n}{\log(1/p)}. \quad (45)$$

Comparing Formulas (44) and (45) yields Formula (43). Moreover, it can be seen from the proof that the exponent in the assertion of Theorem 7 is tight up to factor

$$(1 + o(1)) \frac{\epsilon}{-\log(1 - \epsilon)},$$

which tends to 1 for small  $\epsilon$ .

### 5.2.1 The Uniform Measure Case

A similar computation shows the tightness up to constant factor in the exponent in the case  $p = 1/2$  (namely, tightness of Theorem 4). This time we compute a lower bound on  $(\log \mathcal{S}_\epsilon(f))$  by considering the Fourier-Walsh coefficients of  $f$  all of whose coordinates are contained in the same tribe. It is easy to check that for  $p = 1/2$ , all such coefficients are of order  $\log_2 n/n$ . For each of the  $n/r$  tribes, there are  $2^r$  coefficients that correspond to its subsets, and thus,

$$\sum_{\text{S contained in a tribe}} \hat{f}(S)^2 \approx \frac{n}{r} \cdot 2^r \cdot \frac{(\log_2 n)^2}{n^2} = \Theta(1).$$

Hence, like in the previous case we get

$$\mathcal{S}_\epsilon(f) \geq \sum_{\text{S contained in a tribe}} \hat{f}(S)^2 (1 - \epsilon)^r \gtrsim (1 - \epsilon)^r,$$

and the rest of the proof is the same as in the previous case.

We note that in this computation, the value of  $C$  in the exponent tends to 1 as  $\epsilon \rightarrow 0$ . A refined computation can probably improve the value of the constant, but we won't be able to match the value  $C = .234$  asserted in Theorem 4, since it follows from Corollary 12 in [MO03] that for the tribes function with  $p = 1/2$ , we have

$$\mathcal{S}_\epsilon(f) \lesssim \mathcal{W}(f)^{\log_2(\epsilon)/2} = \mathcal{W}(f)^{.721}.$$

## 6 Acknowledgements

We are grateful to Gil Kalai for encouraging us to work on this project, and to Yuval Peres for useful suggestions.

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## 7 Appendix

In this appendix we prove a decoupled variant of Lemma 5, stated as Theorem 16 below. This variant generalizes Theorem 2.4 of [Tal96], that was used by Talagrand to establish a lower bound on the correlation between monotone subsets of the discrete cube. While employing the same basic idea, our proof is shorter than Talagrand’s proof, and applies also to a biased measure  $\mu_p$  on the discrete cube. For the sake of generality, we present the proof in the biased case, thus providing a decoupled variant of Lemma 6. As in the proof of Lemma 6, we use throughout the appendix the normalized variant of the influences:  $I'_i(f) = \sqrt{p(1-p)}I_i(f)$ .

We start with a generalization of Lemma 3.1 in [Tal96]. The proof is a slight modification of the proof of Lemma 6.

**Lemma 14.** Let  $f : \{0, 1\}^n \rightarrow \{0, 1\}$  be a function, and let  $\{I, J\}$  be a partition of  $\{1, \dots, n\}$ . For  $t > 0$ , denote

$$L_t^f = \{j \in J : \sum_{\substack{T \subset I \\ |T|=m-1}} \hat{f}(T, j)^2 > t \cdot I_j'(f)^2\}.$$

Then for all  $t \geq 4 \cdot (4B(p) \cdot e)^{d-1}$ ,

$$\sum_{j \in L_t^f} I_j'(f)^2 \leq 5e \cdot (t/4)^{-\frac{1}{d-1}} \cdot B(p) \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot (t/4)^{1/(d-1)}\right). \quad (46)$$

**Proof:** Exactly the same argument as used in the proof of Lemma 6 (see Equation 22) shows that for all  $t_0 \geq (4B(p) \cdot e)^{(d-1)/2}$ ,

$$\sum_{j \in L_t^f} \sum_{\substack{T \subset I \\ |T|=d-1}} \hat{f}(T, j)^2 \leq 2t_0^2 \cdot \sum_{j \in L_t^f} I_j'(f)^2 + 10e \cdot t_0^{2-\frac{2}{d-1}} \cdot B(p) \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot t_0^{2/(d-1)}\right). \quad (47)$$

On the other hand, by the definition of  $L_t^f$  we have

$$\sum_{j \in L_t^f} \sum_{\substack{T \subset I \\ |T|=d-1}} \hat{f}(T, j)^2 \geq \sum_{j \in L_t^f} t \cdot I_j'(f)^2 = t \cdot \sum_{j \in L_t^f} I_j'(f)^2. \quad (48)$$

Taking  $t_0 = \sqrt{t}/2$  and combining Inequalities (47) and (48), we get

$$t \cdot \sum_{j \in L_t^f} I_j'(f)^2 \leq (t/2) \cdot \sum_{j \in L_t^f} I_j'(f)^2 + 10e \cdot (t/4)^{1-\frac{1}{d-1}} \cdot B(p) \cdot \exp\left(-\frac{(d-1)}{2B(p) \cdot e} \cdot (t/4)^{1/(d-1)}\right), \quad (49)$$

and simplification yields the assertion.  $\square$

We now present a decoupled variant of Lemma 14. The proof is a series of applications of the Cauchy-Schwarz inequality.

**Lemma 15.** Let  $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$  be functions, such that

$$\sum_{j=1}^n I_j'(f)^2 \leq 1, \quad \text{and} \quad \sum_{j=1}^n I_j'(g)^2 \leq 1, \quad (50)$$

and let  $\{I, J\}$  be a partition of  $\{1, \dots, n\}$ . For  $t > 0$ , denote

$$L_t = \{j \in J : \sum_{\substack{T \subset I \\ |T|=d-1}} \hat{f}(T, j) \hat{g}(T, j) > t \cdot I_j'(f) I_j'(g)\}.$$

Then for all  $t \geq 4 \cdot (4B(p) \cdot e)^{d-1}$ ,

$$\sum_{j \in L_t} I_j'(f) I_j'(g) \leq 2 \cdot (5e)^{1/2} \cdot (t/4)^{-\frac{1}{2(d-1)}} \cdot (B(p))^{1/2} \cdot \exp\left(-\frac{(d-1)}{4B(p) \cdot e} \cdot (t/4)^{1/(d-1)}\right). \quad (51)$$

**Proof:** Define the sets  $L_t^f$  and  $L_t^g$  as in Lemma 14 above. By the Cauchy-Schwarz inequality and Assumption (50),

$$\sum_{j \in L_t^f} I'_j(f)I'_j(g) \leq \left( \sum_{j \in L_t^f} I'_j(f)^2 \right)^{1/2} \left( \sum_{j \in L_t^f} I'_j(g)^2 \right)^{1/2} \leq \left( \sum_{j \in L_t^f} I'_j(f)^2 \right)^{1/2},$$

and thus by Lemma 14,

$$\sum_{j \in L_t^f} I'_j(f)I'_j(g) \leq (5e)^{1/2} \cdot (t/4)^{-\frac{1}{2(d-1)}} \cdot (B(p))^{1/2} \cdot \exp\left(-\frac{(d-1)}{4B(p) \cdot e} \cdot (t/4)^{1/(d-1)}\right).$$

The same holds for  $\sum_{j \in L_t^g} I'_j(f)I'_j(g)$ . Now we note that by the Cauchy-Schwarz inequality,  $L_t \subseteq L_t^f \cup L_t^g$ , and thus,

$$\begin{aligned} \sum_{j \in L_t} I'_j(f)I'_j(g) &\leq \sum_{j \in L_t^f} I'_j(f)I'_j(g) + \sum_{j \in L_t^g} I'_j(f)I'_j(g) \\ &\leq 2 \cdot (5e)^{1/2} \cdot (t/4)^{-\frac{1}{2(d-1)}} \cdot (B(p))^{1/2} \cdot \exp\left(-\frac{(d-1)}{4B(p) \cdot e} \cdot (t/4)^{1/(d-1)}\right), \end{aligned}$$

as asserted.  $\square$

Now we are ready to present the decoupled version of Lemma 6.

**Theorem 16.** *For all  $d \geq 2$ , and for any functions  $f, g : \{0, 1\}^n \rightarrow \{0, 1\}$  such that:*

$$\sum_{j=1}^n I'_j(f)^2 \leq 1, \quad \sum_{j=1}^n I'_j(g)^2 \leq 1, \quad \text{and} \quad \sum_{j=1}^n I'_j(f)I'_j(g) \leq \exp(-2(d-1)), \quad (52)$$

we have

$$\sum_{|S|=d} \hat{f}(S)\hat{g}(S) \leq \frac{70e}{d} \cdot \left(\frac{4B(p) \cdot e}{d-1}\right)^{d-1} \cdot \left(\sum_{j=1}^n I'_j(f)I'_j(g)\right) \cdot \left(\log\left(\frac{d}{\sum_{j \leq n} I'_j(f)I'_j(g)}\right)\right)^{d-1}. \quad (53)$$

**Proof:** As in the proof of Lemma 6, we first consider a partition  $\{I, J\}$  of  $\{1, \dots, n\}$ , and prove that

$$\begin{aligned} \sum_{j \in J} \sum_{\substack{T \subset I \\ |T|=d-1}} \hat{f}(T, j)\hat{g}(T, j) &\leq 70 \cdot \left(\frac{4B(p) \cdot e}{d-1}\right)^{d-1} \cdot \left(\sum_{j \in J} I'_j(f)I'_j(g)\right) \\ &\cdot \left(\log\left(\frac{1}{\sum_{j \in J} I'_j(f)I'_j(g)}\right)\right)^{d-1}. \end{aligned} \quad (54)$$

We apply Lemma 11 with  $\Omega = J$  endowed with the uniform measure, and the functions

$$f_1(j) = I'_j(f)I'_j(g), \quad \text{and} \quad f_2(j) = \frac{\sum_{T \subset I, |T|=d-1} \hat{f}(T, j)\hat{g}(T, j)}{I'_j(f)I'_j(g)}.$$

Noting that the set  $L(t) = \{j : f_2(j) > t\}$  is exactly the set  $L_t$  defined in Lemma 15, we get

$$\sum_{j \in J} \sum_{\substack{T \subset I \\ |T|=d-1}} \hat{f}(T, j) \hat{g}(T, j) = \int_{t=0}^{\infty} \left( \sum_{j \in L_t} I'_j(f) I'_j(g) \right) dt. \quad (55)$$

Therefore, using Lemma 15, we have, for all  $t_1 \geq 4 \cdot (4B(p) \cdot e)^{d-1}$ ,

$$\begin{aligned} (55) &= \int_{t=0}^{t_1} \left( \sum_{j \in L_t} I'_j(f) I'_j(g) \right) dt + \int_{t=t_1}^{\infty} \left( \sum_{j \in L_t} I'_j(f) I'_j(g) \right) dt \leq t_1 \sum_{j \in J} I'_j(f) I'_j(g) \\ &+ \int_{t=t_1}^{\infty} \left( 2 \cdot (5e)^{1/2} \cdot (t/4)^{-\frac{1}{2(d-1)}} \cdot (B(p))^{1/2} \cdot \exp \left( -\frac{(d-1)}{4B(p) \cdot e} \cdot (t/4)^{1/(d-1)} \right) \right) dt. \end{aligned}$$

We evaluate the integral in the same way as the integral in Lemma 12, and obtain that for all  $t_1 \geq 4 \cdot (8B(p) \cdot e)^{d-1}$ ,

$$(55) \leq t_1 \sum_{j \in J} I'_j(f) I'_j(g) + 18.74 \cdot (B(p))^{1/2} \cdot (16B(p)e) \cdot (t_1/4)^{1-\frac{3}{2d-2}} \cdot \exp \left( -\frac{(d-1)}{4B(p) \cdot e} \cdot (t_1/4)^{1/(d-1)} \right). \quad (56)$$

We then choose  $t_1$  such that

$$\exp \left( -\frac{(d-1)}{4B(p) \cdot e} \cdot (t_1/4)^{1/(d-1)} \right) = \sum_{j \in J} I'_j(f) I'_j(g).$$

(Note that due to Assumption (52), we have  $t_1 \geq 4 \cdot (8B(p) \cdot e)^{d-1}$ , as required). Substituting  $t_1$  into Inequality (56), we get

$$(55) \leq 70 \cdot \left( \frac{4B(p)e}{d-1} \right)^{d-1} \left( \sum_{j \in J} I'_j(f) I'_j(g) \right) \left( \log \frac{1}{\sum_{j \in J} I'_j(f) I'_j(g)} \right)^{d-1}, \quad (57)$$

proving (54). Finally, the derivation of (53) from (54) is exactly the same as the last step in the proof of Lemma 6.  $\square$