GEOMETRIC INFLUENCES

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Abstract. We present a new definition of influences in product spaces of continuous distributions. Our definition is geometric, and for monotone sets it is identical with the measure of the boundary with respect to uniform enlargement. We prove analogues of the Kahn-Kalai-Linial (KKL) and Talagrand's influence sum bounds for the new definition. We further prove an analogue of a result of Friedgut showing that sets with small "influence sum" are essentially determined by a small number of coordinates. In particular, we establish the following tight analogue of the KKL bound: for any set in $\mathbb{R}^n$ of Gaussian measure t, there exists a coordinate i such that the i-th geometric influence of the set is at least $ct(1-t)\sqrt{\log n}/n$, where c is a universal constant. This result is then used to obtain an isoperimetric inequality for the Gaussian measure on $\mathbb{R}^n$ and the class of sets invariant under transitive permutation group of the coordinates.

1. Introduction

Definition 1.1. Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. The influence of the i-th coordinate on f is

$$I_i(f) := \mathbb{P}[f(x) \neq f(x \oplus e_i)],$$

where $x \oplus e_i$ denotes the point obtained from x by replacing x_i by $1-x_i$ and leaving the other coordinates unchanged.

The notion of influences of variables on Boolean functions is one of the central concepts in the theory of discrete harmonic analysis. In the last two decades it found several applications in diverse fields, including Combinatorics, Theoretical Computer Science, Statistical Physics, Social Choice Theory, etc. (see, for example, the survey article [KS06]). The influences have numerous properties that allow to use them in applications. The following three properties are amongst the most fundamental ones:

(1) **Geometric Meaning.** The influences on the discrete cube $\{0,1\}^n$ have a clear geometric meaning. $I_i(f)$ is the size of the *edge boundary in the i-th direction* of the set $A = \{x \in \{0,1\}^n : f(x) = 1\}$.
(2) **The KKL Theorem.** In the remarkable paper [KKL88], Kahn, Kalai, and Linial proved that for any Boolean function $f : \{0,1\}^n \to \{0,1\}$, there

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exists a variable i whose influence is at least $ct(1-t)\log n/n$, where $t = \mathbb{E}f$ and c is a universal constant. Many applications of influences make use of the KKL theorem or of related results such as [Tal94, Fri98] in one way or another.

(3) **The Russo Lemma.** Let μ_p denote the Bernoulli measure where 0 is given weight $1-p$ and 1 is given weight p. Clearly if $A \subseteq \{0,1\}^n$ is increasing then $\mu_p^{\otimes n}(A)$ is monotone increasing as function of p. The question of understanding how $\mu_p^{\otimes n}(A)$ varies with p has important applications in the theory of random graphs and in percolation theory. Russo's Lemma [Mar74, Rus82] asserts that the derivative of $\mu_p^{\otimes n}(A)$ with respect to p is the sum of influences of $f = 1_A$.

The basic results on influences were obtained for functions on the discrete cube, but some applications required generalization of the results to more general product spaces. Unlike the discrete case, where there exists a single natural definition of influence, for general product spaces several definitions were presented in different papers, see for example [BKK+92, Hat09, Kel, MOO09]. While each of these definitions has its advantages, in general all of them lack geometric interpretation for continuous probability spaces.

In this paper we present a new definition of the influences in product spaces of continuous random variables, that has a clear geometric meaning. Moreover, we show that for important classes of product measures, including the Gaussian measure, our definition allows to obtain analogues of the KKL theorem and Russo-type formulas.

Definition 1.2. Let ν be a probability measure on $\mathbb{R}$. Given a Borel-measurable set $A \subseteq \mathbb{R}$, its lower Minkowski content $m_\nu(A)$ is defined as

$$m_\nu(A) := \liminf_{r\downarrow 0} \frac{\nu(A+[-r,r]) - \nu(A)}{r}.$$

Consider the product measure $\nu_1 \otimes \nu_2 \otimes \ldots \otimes \nu_n$ on $\mathbb{R}^n$. Then for any Borel-measurable set $A \subseteq \mathbb{R}^n$, for each $1 \le i \le n$ and an element $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, the restriction of A along the fiber of x in the i-th direction is given by

$$A_i^x := \{y \in \mathbb{R} : (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \in A\}.$$

The *geometric influence* of the i-th coordinate on A is

$$I_i^{\mathcal{G}}(A) := \mathbb{E}_x[m_{\nu_i}(A_i^x)].$$

In order to make the measure we take the influence with respect to clear, we sometimes denote the influence as $I_i^{\mathcal{G}}(A)\Big|_\nu$.

The geometric meaning of the influence is that for a monotone set A, the sum of influences of A is equal to the size of its boundary with respect to a uniform enlargement, that was studied in e.g., [Bob96, Bob97, Bar04].

Proposition 1.3. *Let ν be a probability measure on $\mathbb{R}$ with C^1 density λ and cumulative distribution function Λ. Assume further that $\lambda(z) > 0$ for all $z \in \mathbb{R}$, that $\lim_{|z|\to\infty} \lambda(z) = 0$, and that λ' is bounded. Let $A \subset \mathbb{R}^n$ be a monotone set. Then*

$$\lim_{r\downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n) - \nu^{\otimes n}(A)}{r} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).$$

We show that for the Gaussian measure on $\mathbb{R}^n$, the geometric influences satisfy the following analogue of the KKL theorem:

Theorem 1.4. *Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. Then for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\mu^{\otimes n}(A) = t$ there exists $1 \leq i \leq n$ such that*

$$I_i^{\mathcal{G}}(A) \geq ct(1-t)\frac{\sqrt{\log n}}{n},$$

where $c > 0$ is a universal constant.

The result extends to a larger set of log-concave measures called Boltzmann measures (see Definition 3.8), and is tight up to the constant factor. The proof uses the relation between geometric influences and the h-influences defined in [Kel], combined with isoperimetric estimates for the underlying probability measures.

Using the same methods, we obtain analogues of Talagrand's bound on the vector of influences [Tal94], and of Friedgut's theorem stating that a function with a low sum of influences essentially depends on a few coordinates [Fri98].

Theorem 1.5. *Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. For any Borel-measurable set $A \subset \mathbb{R}^n$, we have:*

(1) *If $\mu^{\otimes n}(A) = t$, then*

$$\sum_{i=1}^{n} \frac{I_i^{\mathcal{G}}(A)}{\sqrt{-\log I_i^{\mathcal{G}}(A)}} \geq c_1 t(1-t),$$

(2) *If A is monotone and $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)\sqrt{-\log I_i^{\mathcal{G}}(A)} = s$, then there exists a set $B \subset \mathbb{R}^n$ such that 1_B is determined by at most $\exp(c_2 s/\epsilon)$ coordinates and $\mu^{\otimes n}(A \triangle B) \leq \epsilon$,*

where c_1 and c_2 are universal constants.

We also show that the geometric influences can be used in Russo-type formulas for location families.

Proposition 1.6. *Let ν be a probability measure on $\mathbb{R}$ with continuous density λ and cumulative distribution function Λ. Let $\{\nu_\alpha : \alpha \in \mathbb{R}\}$ denote a family of probability measures which is obtained by translating ν, that is, ν_α has a density λ_α satisfying $\lambda_\alpha(x) = \lambda(x - \alpha)$.*

Assume that λ is bounded and satisfies $\lambda(z) > 0$ on (κ_L, κ_R), the interior of the support of ν. Let A be an increasing subset of $\mathbb{R}^n$. Then the function $\alpha \to {\nu_\alpha}^{\otimes n}(A)$ is differentiable and its derivative is given by

$$\frac{d{\nu_\alpha}^{\otimes n}(A)}{d\alpha} = \sum_{i=1}^{n} I_i^{\mathcal{G}}(A),$$

where the influences are taken w.r.t. the measure $\nu_\alpha^{\otimes n}$.

Theorem 1.4 and Proposition 1.6 can be combined to get the following corollary which is the Gaussian analogue of the sharp threshold result obtained by Friedgut and Kalai [FK96] for the product Bernoulli measure on the hypercube. We call a set transitive if it is invariant under the action of some transitive subgroup of the permutation group S_n.

Corollary 1.7. *Let μ_α denote the Gaussian measure on the real line with mean α and variance 1. Let $A \subset \mathbb{R}^n$ be an increasing transitive set. For any $\delta > 0$, denote by $\alpha_A(\delta)$ the unique value of α such that $\mu_\alpha^{\otimes n}(A) = \delta$. Then for any $\epsilon > 0$,*

$$\alpha_A(1-\epsilon) - \alpha_A(\epsilon) \leq c \log(1/2\epsilon)/\sqrt{\log n},$$

where c is a universal constant.

We now use the geometric influences to obtain an isoperimetric result for the Gaussian measure on $\mathbb{R}^n$:

Theorem 1.8. *Consider the product spaces $\mathbb{R}^n$ endowed with the product Gaussian measure $\mu^{\otimes n}$. Then for any transitive Borel-measurable set $A \subset \mathbb{R}^n$ we have*

$$\liminf_{r \downarrow 0} \frac{\mu^{\otimes n}(A + [-r,r]^n) - \mu^{\otimes n}(A)}{r} \geq ct(1-t)\sqrt{\log n},$$

where $t = \mu^{\otimes n}(A)$ and $c > 0$ is a universal constant .

This result also extends to all Boltzmann measures.

Since the Gaussian measure is rotation invariant, it is natural to consider the influence sum of rotations of sets. Of particular interest are families of sets that are closed under rotations. In Section 5 we study the effect of *rotations* on the geometric influences, and show that under mild regularity condition of being in a certain class $\mathcal{J}_n$ (see Definition 5.1), the sum of geometric influences of a convex set can be increased up to $\Omega(\sqrt{n})$ by (a random) orthogonal rotation:

Theorem 1.9. *Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. For any convex set $A \in \mathcal{J}_n$ with $\mu^{\otimes n}(A) = t$, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^n I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log(t(1-t))} \times \sqrt{n},$$

where $c > 0$ is a universal constant. Moreover,

$$\mathbb{E}_{M \sim \nu}\left[\sum_{i=1}^n I_i^{\mathcal{G}}(M(A))\right] \geq c\sqrt{n}t(1-t)\sqrt{-\log(t(1-t))},$$

where M is drawn according to the Haar measure ν over the orthogonal group of rotations.

The paper is organized as follows: In Section 2 we prove Proposition 1.3 and Proposition 1.6, thus establishing the geometric meaning of the new definition. In Section 3 we discuss the relation between the geometric influences and the h-influences, and prove Theorem 1.4. In Section 4 we apply Theorem 1.4 to establish a lower bound on the size of the boundary of transitive sets with respect to uniform enlargement, proving Theorem 1.8. Finally, in Section 5 we study the effect of *rotations* on the geometric influences. We conclude the introduction with a brief statistical application of the results established here.

1.1. **A statistical application.** Let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. $\mathrm{N}(\theta, 1)$. Suppose we want to test the hypothesis: $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$ $(\theta_1 > \theta_0)$ with level significance at most β.

The remarkable classical result by Neyman and Pearson [NP33] says that the most powerful test for the above problem is based on the sample average $\bar{Z}_n = n^{-1}\sum_{i=1}^n Z_i$ and the critical region of the test is given by $\mathcal{C}_{\mathrm{mp}} = \{\bar{Z}_n > K\}$ where the constant K is chosen is such that $\mathbb{P}_{\theta_0}\{\mathcal{C}_{\mathrm{mp}}\} = \beta$. It can be easily checked that to achieve power at least $1 - \beta$ for this test, we need the parameters θ_0 and θ_1 to be separated by at least $|\theta_1 - \theta_0| > C(\beta)/\sqrt{n}$ for some appropriate constant $C(\beta)$.

Consider the following setup where the test statistics is given by $f(Z_1, \ldots, Z_n)$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a measurable function which is non-degenerate, transitive and increasing in each of its coordinates. The transitivity of f ensures equal weight is given to each data point while constructing the test and the monotonicity of f implies that the distribution of f depends on θ in a monotone fashion. Note that we do not assume any smoothness property of f. In general the test statistics $f(Z_1, \ldots, Z_n)$, in contrast to the sample average which is a sufficient statistics for this problem, may be resulted from an 'inefficient compression' of the data and we have only access to the compressed data.

In this case the critical region would be of the form $\mathcal{C} = \{f(Z_1, \ldots, Z_n) > K\}$ where K is chosen so that $\mathbb{P}_{\theta_0}\{\mathcal{C}\} = \beta$.

Note that the regions $\mathcal{C}$ satisfy

(i) $\mathbb{P}_{\theta_0}\{\mathcal{C}\} = \beta$.
(ii) $\mathcal{C}$ is transitive,
(iii) $\mathcal{C}$ is an increasing set.

Clearly, the most powerful test belongs to this class but in general a test of above type can be of much less power. An interesting open question will be to find the worst test (that is, having lowest power) among all tests satisfying (i), (ii) & (iii). Intuitively if θ_1 and θ_0 are far apart, even a very weak test can detect the difference between the null and the alternative. Corollary 1.7 gives us a quantitative estimate of how far apart the parameters need to be so that we can safely distinguish them no matter what test we use. Indeed any test satisfying (i), (ii) and (iii) still has power at least $1 - \beta$ as long as $|\theta_1 - \theta_0| > c\log(1/2\beta)/\sqrt{\log n}$ for some absolute constant c.

For the test $\{\max_i Z_i > K\}$, the dependence on n in the above bound is tight up to constant factors.

We briefly note that the statistical reasoning introduced here may be combined with Theorem 2.1 in [FK96]. Thus a similar statement holds when $Z_1, Z_2, \ldots, Z_n$ are i.i.d. Bernoulli(p) and we want to test the hypothesis: $H_0 : p = p_0$ vs $H_1 : p = p_1$ $(1 > p_1 > p_0 > 0)$. In this case, the power of any test satisfying (i), (ii) and (iii) is at least $1 - \beta$ as long as $|p_1 - p_0| > c\log(1/2\beta)/\log n$ for some absolute constant c.

2. Boundary Under Uniform Enlargement and Derivatives

In this section we provide the geometric interpretation of the influence. We begin by proving Proposition 1.3.

2.1. **Proof of Proposition 1.3.** In our proof we use the following simple lemma:

Lemma 2.1. *Let λ be as given in Proposition 1.3. Given $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for all $x, y \in \mathbb{R}$,*

$$|\lambda(x) - \lambda(y)| \le C_\varepsilon |\Lambda(x) - \Lambda(y)| + \varepsilon/4.$$

Proof. Since $\lim_{|z|\to\infty} \lambda(z) = 0$, there exist $0 < z_2 < z_1$ such that $\sup_{|z|\ge z_1} \lambda(z) \le \varepsilon/8$ and $\sup_{|z|\ge z_2} \lambda(z) \le \varepsilon/4$. We consider several cases.

(1) **Case I.** $|x| > z_2, |y| > z_2$. In this case, by the choice of z_2, we have $|\lambda(x) - \lambda(y)| \le \varepsilon/4$.
(2) **Case II.** $|x| \le z_1, |y| \le z_1$. Since the function λ'/λ is continuous, there exists K such that $|\lambda'(z)|/\lambda(z) \le K$ for all $|z| \le z_1$. Hence,
$$|\lambda(x) - \lambda(y)| = |\int_y^x \lambda'(z)dz| \le K|\int_y^x \lambda(z)dz| = K|\Lambda(x) - \Lambda(y)|.$$
(3) **Case III a.** $x > z_1, |y| < z_2$. In this case,
$$|\lambda(x) - \lambda(y)| \le 2\|\lambda\|_\infty \le \frac{2\|\lambda\|_\infty}{\Lambda(z_1) - \Lambda(z_2)}(\Lambda(x) - \Lambda(y)).$$
(4) **Case III b.** $x < -z_1, |y| < z_2$. Similarly,
$$|\lambda(x) - \lambda(y)| \le 2\|\lambda\|_\infty \le \frac{2\|\lambda\|_\infty}{\Lambda(-z_2) - \Lambda(-z_1)}(\Lambda(y) - \Lambda(x)).$$

This completes the proof of the lemma, by taking
$$C_\varepsilon = \max\left(K, \frac{2\|\lambda\|_\infty}{\Lambda(z_1) - \Lambda(z_2)}, \frac{2\|\lambda\|_\infty}{\Lambda(-z_2) - \Lambda(-z_1)}\right).$$
□

Now we are ready to present the proof of Proposition 1.3.

Proof. Without loss of generality, assume that A is decreasing. Thus, $\nu^{\otimes n}(A + [-r, r]^n) = \nu^{\otimes n}(A + [0, r]^n)$. We decompose $\nu^{\otimes n}(A + [0, r]^n) - \nu^{\otimes n}(A)$ as

$$\sum_{i=1}^{n} \nu^{\otimes n}(A + [0, r]^i \times \{0\}^{n-i}) - \nu^{\otimes n}(A + [0, r]^{i-1} \times \{0\}^{n-i+1}). \tag{2.1}$$

It follows immediately from (2.1) that it is sufficient to show that given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $1 \le i \le n$ and for all $0 < r < \delta$,

$$\left|\frac{\nu^{\otimes n}(A + [0, r]^{i-1} \times [0, r] \times \{0\}^{n-i}) - \nu^{\otimes n}(A + [0, r]^{i-1} \times \{0\}^{n-i+1})}{r} - I_i^{\mathcal{G}}(A)\right| \le \varepsilon. \tag{2.2}$$

For a fixed i, define
$$B_r^i = A + [0, r]^{i-1} \times \{0\}^{n-i+1}.$$
Obviously, B_r^i is a decreasing set. Note that $A + [0, r]^{i-1} \times [0, r] \times \{0\}^{n-i} = B_r^i + \{0\}^{i-1} \times [0, r] \times \{0\}^{n-i}$. Hence, Equation (2.2) can be rewritten as

$$\left|\frac{\nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [0, r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r^i)}{r} - I_i^{\mathcal{G}}(A)\right| \le \varepsilon. \tag{2.3}$$

For any decreasing set $D \subset \mathbb{R}^n$ and for any $x \in \mathbb{R}^n$, define
$$t_i(D; x) := \sup\{y : y \in D_i^x\} \in [-\infty, \infty],$$

with the convention that the supremum of the empty set is $-\infty$. We use two simple observations:

(1) For any decreasing set D (and in particular, for A and for B_r^i), it is clear that $\nu^{\otimes n}(D) = \mathbb{E}_x \Lambda(t_i(D;x))$.
(2) For a decreasing set D, we have $I_i^{\mathcal{G}}(D) = \mathbb{E}_x \lambda(t_i(D;x))$. This follows from a known property of the lower Minkowski content: In the case when ν has a continuous density λ and L is a semi-infinite ray, that is, $L = [\ell, \infty)$ or $L = (-\infty, \ell]$, we have $m_\nu(L) = \lambda(\ell)$.

We further observe that

(2.4)
$$\left| \frac{\nu^{\otimes n}(B_r^i + \{0\}^{i-1} \times [0,r] \times \{0\}^{n-i}) - \nu^{\otimes n}(B_r^i)}{r} - \mathbb{E}_x \lambda(t_i(B_r^i;x)) \right| \leq r||\lambda'||_\infty.$$

Indeed, by Observation (1), the l.h.s. of (2.4) is equal to

$$\left| \mathbb{E}_x \Big[\frac{\Lambda(t_i(B_r^i;x)+r) - \Lambda(t_i(B_r^i;x))}{r} - \lambda(t_i(B_r^i;x)) \Big] \right|. \tag{2.5}$$

By the Mean Value Theorem, there exists $h \in [0,r]$ such that

$$\frac{\Lambda(t_i(B_r^i;x)+r) - \Lambda(t_i(B_r^i;x))}{r} = \lambda(t_i(B_r^i;x)+h),$$

and thus,

$$(2.5) = \left| \mathbb{E}_x [\lambda(t_i(B_r^i;x)+h) - \lambda(t_i(B_r^i;x))] \right| \leq r||\lambda'||_\infty.$$

Combining Equations (2.3) and (2.4), and ensuring that $r < \varepsilon/(2||\lambda'||_\infty)$, it is sufficient to show that

$$\left| \mathbb{E}_x \lambda(t_i(B_r^i;x)) - I_i^{\mathcal{G}}(A) \right| \leq \varepsilon/2,$$

and by Observation (2), this is equivalent to

$$\left| \mathbb{E}_x \lambda(t_i(B_r^i;x)) - \mathbb{E}_x \lambda(t_i(A;x)) \right| \leq \varepsilon/2. \tag{2.6}$$

By Lemma 2.1 and Observation (1), we have

$$\begin{aligned} |\mathbb{E}_x \lambda(t_i(B_r^i;x)) - \mathbb{E}_x \lambda(t_i(A;x))| &\leq C_\varepsilon \mathbb{E}_x |\Lambda(t_i(B_r^i;x)) - \Lambda(t_i(A;x))| + \varepsilon/4 \\ &= C_\varepsilon \mathbb{E}_x \Big(\Lambda(t_i(B_r^i;x)) - \Lambda(t_i(A;x)) \Big) + \varepsilon/4 \\ &= C_\varepsilon (\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A)) + \varepsilon/4. \end{aligned}$$

It thus remains to show that there exists $\delta > 0$ sufficiently small such that for all $0 < r < \delta$,

$$\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A) \leq \frac{\varepsilon}{4C_\varepsilon}. \tag{2.7}$$

We can write

$$\nu^{\otimes n}(B_r^i) - \nu^{\otimes n}(A) = \sum_{j=1}^{i-1} \Big(\nu^{\otimes n}(A + [0,r]^j \times \{0\}^{n-j}) - \nu^{\otimes n}(A + [0,r]^{j-1} \times \{0\}^{n-j+1}) \Big),$$

and thus it is sufficient to find $\delta > 0$ such that for all $0 < r < \delta$ and for all $1 \leq j \leq i-1$,

$$\nu^{\otimes n}(A + [0,r]^j \times \{0\}^{n-j}) - \nu^{\otimes n}(A + [0,r]^{j-1} \times \{0\}^{n-j+1}) \leq \frac{\varepsilon}{4nC_\varepsilon}.$$

Since for any decreasing $D \subset \mathbb{R}^n$,

$$|\nu^{\otimes n}(D + \{0\}^{j-1} \times [0,r] \times \{0\}^{n-j}) - \nu^{\otimes n}(D)| \le \|\lambda\|_\infty r,$$

we can choose $\delta = \min\{\frac{\varepsilon}{4nC_\varepsilon\|\lambda\|_\infty}, \frac{\varepsilon}{2||\lambda'||_\infty}\}$. This completes the proof. □

Remark 2.2. We note that the same proof (with minor modifications) holds for any convex set A. The only non-obvious change is noting that the Minkowski content of a segment $[a,b]$ is $m_\nu([a,b]) = \lambda(a) + \lambda(b)$, where λ is the density of the measure ν. On the other hand, it is clear that the statement of Proposition 1.3 does not hold for general measurable sets. For example, if $A = \mathbb{Q}^n$ where $\mathbb{Q}$ is the set of rational numbers, then the L^∞-boundary of A is ∞, while the sum of geometric influences of A is zero. It seems an interesting question to determine to which classes of measurable sets Proposition 1.3 applies.

2.2. **Proof of Proposition 1.6.** Define a function $\Pi : \mathbb{R}^n \to [0,\infty)$ by

$$\Pi(\alpha_1, \dots, \alpha_n) = \nu_{\alpha_1} \otimes \dots \otimes \nu_{\alpha_n}(A).$$

The partial derivative of Π w.r.t. the i-th coordinate can be written as

$$\frac{\partial \Pi(\alpha_1,\dots,\alpha_n)}{\partial \alpha_i} = \lim_{r\downarrow 0} \frac{\mathbb{E}_x \nu_{\alpha_i+r}(A_i^x) - \mathbb{E}_x \nu_{\alpha_i}(A_i^x)}{r}. \tag{2.8}$$

For $x \in \mathbb{R}^n$, define

$$s_i(A;x) := \inf\{y : y \in A_i^x\} \in [-\infty, \infty].$$

Since A is monotone increasing, for any $x \in \mathbb{R}^n$ we have

$$\begin{aligned}\frac{\nu_{\alpha_i+r}(A_i^x) - \nu_{\alpha_i}(A_i^x)}{r} &= \frac{\nu_{\alpha_i+r}([s_i(A;x),\infty)) - \nu_{\alpha_i}([s_i(A;x),\infty))}{r}\\ &= \frac{1}{r}\int_{s_i(A;x)-r}^{s_i(A;x)} \lambda_{\alpha_i}(z)dz,\end{aligned} \tag{2.9}$$

and by the Fundamental Theorem of Calculus, this expression converges to $\lambda_{\alpha_i}(s_i(A;x))$ as $r \to 0$. Moreover, (2.9) is uniformly bounded by $\|\lambda_{\alpha_i}\|_\infty = \|\lambda\|_\infty$. Therefore, by the Dominated Convergence Theorem, it follows that the first order partial derivatives of Π exist and are given by

$$\frac{\partial \Pi(\alpha_1,\dots,\alpha_n)}{\partial \alpha_i} = \mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} \lambda_{\alpha_i}(s_i(A;x)) = I_i^{\mathcal{G}}(A),$$

where the influence is w.r.t. the measure $\nu_{\alpha_1} \otimes \cdots \otimes \nu_{\alpha_n}$. (For the last equality, see Observation (2) in the proof of Proposition 1.3 above. Here we use the convention that $\lambda_{\alpha_i}(-\infty) = \lambda_{\alpha_i}(\infty) = 0$).

Hence, by the chain rule, it is sufficient to check that all the partial derivatives of Π are continuous at $(\alpha, \dots, \alpha)$. Without loss of generality, we assume that $\alpha = 0$. Note that

$$\mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} \lambda_{\alpha_i}(s_i(A;x)) = \mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu} \left(\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)}\right) \lambda_{\alpha_i}(s_i(A;x)). \tag{2.10}$$

For each $x \in \mathbb{R}^n$,

$$\prod_{j=1}^n \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} \lambda_{\alpha_i}(s_i(A;x)) \to \prod_{j=1}^n \lambda(s_i(A;x)) \tag{2.11}$$

as $\max|\alpha_i| \to 0$. Hence, the continuity of the partial derivatives would follow from the Dominated Convergence Theorem if (2.11) was uniformly bounded. In order to obtain such bound, we consider a compact subset.

There exist $\kappa_L < K_L < K_R < \kappa_R$ and $\delta > 0$ such that $\nu([K_L+\delta, K_R-\delta]) \geq 1-\varepsilon$. Let $c := \min_{z\in[K_L,K_R]} \lambda(z)$. Note that $c > 0$. If $|\alpha_j| \leq \delta$ for all j, then

(2.12)
$$\left|(2.10) - \mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\left(\prod_{j=1}^{n} \frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L\leq x_j\leq K_R\}}\right)\lambda_{\alpha_i}(s_i(A;x))\right| \leq \varepsilon\cdot n\cdot\|\lambda\|_\infty.$$

Indeed, denoting $S = \{x\in\mathbb{R}^n : \exists j, x_j \notin [K_L, K_R]\}$ and using Equation (2.10), we have

$$(2.12) = \left|\mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} 1_S \lambda_{\alpha_i}(s_i(A;x))\right| \leq ||\lambda||_\infty \mathbb{E}_{x\sim\nu_{\alpha_1}\otimes\cdots\otimes\nu_{\alpha_n}} 1_S \leq \varepsilon\cdot n\cdot\|\lambda\|_\infty,$$

where the last inequality is a union bound using the choice of K_L and K_R.

Similarly, by a union bound we have

(2.13)
$$\left|\mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\lambda(s_i(A;x)) - \mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu} 1_{\{K_L\leq x_j\leq K_R\ \forall j\}}\lambda(s_i(A;x))\right| \leq \varepsilon\cdot n\cdot\|\lambda\|_\infty.$$

Combining (2.12) with (2.13), it is sufficient to prove that

$$\mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\prod_{j=1}^{n}\frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L\leq x_j\leq K_R\}}\lambda_{\alpha_i}(s_i(A;x)) \to$$
$$\mathbb{E}_{x\sim\nu\otimes\cdots\otimes\nu}\prod_{j=1}^{n} 1_{\{K_L\leq x_j\leq K_R\}}\lambda(s_i(A;x)).$$

This indeed follows from the Dominated Convergence Theorem, since for each $x \in \mathbb{R}^n$,

$$\prod_{j=1}^{n}\frac{\lambda_{\alpha_j}(x_j)}{\lambda(x_j)} 1_{\{K_L\leq x_j\leq K_R\}}\lambda_{\alpha_i}(s_i(A;x)) \to \prod_{j=1}^{n} 1_{\{K_L\leq x_j\leq K_R\}}\lambda(s_i(A;x))$$

as $\max|\alpha_i| \to 0$ and is uniformly bounded by $c^{-n}\|\lambda\|_\infty^{n+1}$. This completes the proof. □

3. Relation to h-influences and a general lower bound on geometric influences

In this section we analyze the geometric influences by reduction to problems concerning h-influences introduced in a recent paper by the first author [Kel]. First we describe and extend the results on h-influences, and then we show their relation to geometric influences.

3.1. h-Influences.

Definition 3.1 (h-influences, [Kel])**.** Let $h : [0,1] \to [0,\infty)$ be a measurable function. For a measurable subset A of X^n equipped with a product measure $\nu^{\otimes n}$, the h-influence of the i-th coordinate on A is

$$I_i^h(A) := \mathbb{E}_x[h(\nu(A_i^x))].$$

The two main results concerning h-influences are a monotonization lemma and an analogue of the KKL theorem.

Lemma 3.2 ([Kel])**.** *Let $h : [0,1] \to [0,1]$ be a concave continuous function. For every Borel measurable set $A \subseteq [0,1]^n$, there exists a monotone set $B \subseteq [0,1]^n$ such that:*

1. $u^{\otimes n}(A) = u^{\otimes n}(B)$.
2. *For all $1 \le i \le n$, we have $I_i^h(A) \ge I_i^h(B)$.*

Theorem 3.3 ([Kel])**.** *Denote the entropy function as $\mathrm{Ent}(x) := -x \log x - (1-x)\log(1-x)$. Consider the product space $[0,1]^n$, endowed with the product Lebesgue measure $u^{\otimes n}$. Let $h : [0,1] \to [0,1]$ such that $h(x) \ge \mathrm{Ent}(x)$ for all $0 \le x \le 1$. Then for every measurable set $A \subseteq [0,1]^n$ with $u^{\otimes n}(A) = t$, there exists $1 \le i \le n$ such that the h-influence of the i-th coordinate on A satisfies*

$$I_i^h(A) \ge ct(1-t)\log n/n,$$

where $c > 0$ is a universal constant.

Other results on h-influences which we shall use later include analogues of several theorems concerning influences on the discrete cube: Talagrand's lower bound on the vector of influences [Tal94], a variant of the KKL theorem for functions with low influences [FK96], and Friedgut's theorem asserting that a function with a low influence sum essentially depends on a few coordinates [Fri98].

In the application to geometric influences we would like to use h-influences for certain functions h that do not dominate the entropy function. In order to overcome this problem, we use the following lemma, that allows to relate general h-influences to the Entropy-influence (i.e., h-influence for $h(x) = \mathrm{Ent}(x)$).

Lemma 3.4. *Consider the product space $(\mathbb{R}^n, \nu^{\otimes n})$, where ν has a continuous cumulative distribution function Λ. Let $h : [0,1] \to [0,\infty)$, and let $A \subseteq \mathbb{R}^n$ be a Borel-measurable set. For all $1 \le i \le n$,*

$$I_i^h(A) \ge \frac{1}{2}\delta \cdot I_i^{\mathrm{Ent}}(A), \tag{3.1}$$

where

$$\delta = \delta(A,i) = \inf_{x \in [\vartheta(I_i^{\mathrm{Ent}}(A)/2),\, 1-\vartheta(I_i^{\mathrm{Ent}}(A)/2)]} \frac{h(x)}{\mathrm{Ent}(x)}, \tag{3.2}$$

and $\vartheta(y) = y/(-2\log y)$.

Proof. Set $f = 1_A$. Let u be the Lebesgue measure on $[0,1]$. Define $g(x_1,\dots,x_n) := f(\Lambda^{-1}(x_1),\dots,\Lambda^{-1}(x_n))$ and write B for the set $\{x \in \mathbb{R}^n : g(x) = 1\}$. Since $\Lambda^{-1}(u) \overset{d}{=} \nu$, the set B satisfies $u^{\otimes n}(B) = \nu^{\otimes n}(A) = t$ and

$$I_i^h(B)\big|_{u^{\otimes n}} = I_i^h(A)\big|_{\nu^{\otimes n}} \qquad \text{for each } \ 1 \le i \le n.$$

Denote $\alpha := \mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2)$. It is clear that for any $x \notin [\alpha, 1-\alpha]$,

$$\mathrm{Ent}(x) \le \mathrm{Ent}\left(\mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2)\right) = I_i^{\mathrm{Ent}}(A)/2,$$

and thus,

$$\begin{aligned}\mathbb{E}_x\Big[\mathrm{Ent}(u(B_i^x))1_{\{u(B_i^x)\in[\alpha,1-\alpha]\}}\Big] &= I_i^{\mathrm{Ent}}(B)\big|_{u^{\otimes n}} - \mathbb{E}_x\Big[\mathrm{Ent}(u(B_i^x))1_{\{u(B_i^x)\notin[\alpha,1-\alpha]\}}\Big] \\ &\ge I_i^{\mathrm{Ent}}(A)/2.\end{aligned}$$

Therefore, by (3.2),

$$\begin{aligned} I_i^h(A)\big|_{\nu^{\otimes n}} = I_i^h(B)\big|_{u^{\otimes n}} &\geq \mathbb{E}_x\Big[h(u(B_i^x))1_{\{u(B_i^x)\in[\alpha,1-\alpha]\}}\Big] \\ &\geq \left(\inf_{x\in[\mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2),1-\mathrm{Ent}^{-1}(I_i^{\mathrm{Ent}}(A)/2)]}\frac{h(x)}{\mathrm{Ent}(x)}\right) I_i^{\mathrm{Ent}}(A)/2 \\ &\geq \delta\cdot I_i^{\mathrm{Ent}}(A)/2, \end{aligned}$$

where the last step follows from the fact that $\vartheta(x) \leq \mathrm{Ent}^{-1}(x)$ for $x \leq 1/2$ which is easy to verify. □

3.2. **Relation between geometric influences and h-influences for log-concave measures.** It is straightforward to check the following relation between the geometric influences and the h-influences for monotone sets. The proof follows immediately from Observation (2) in the proof of Proposition 1.3.

Lemma 3.5. *Consider the product space $(\mathbb{R}^n, \nu^{\otimes n})$ where ν has a continuous density λ. Let Λ denote the cumulative distribution function of ν. Then for any monotone set $A \subseteq \mathbb{R}^n$,*

$$I_i^{\mathcal{G}}(A) = I_i^h(A) \qquad \forall 1 \leq i \leq n,$$

where $h(t) = \lambda(\Lambda^{-1}(t))$ when A is decreasing and $h(t) = \lambda(\Lambda^{-1}(1-t))$ when A is increasing. Here Λ^{-1} denotes the unique inverse of the function Λ.

Using Lemma 3.2 and Lemma 3.5, we can obtain a monotonization lemma for geometric influences that holds if the underlying measure has a log-concave density. In order to show this, we use the following isoperimetric inequality satisfied by log-concave distributions (see, for example, [Bob96]).

Theorem 3.6. *Let ν have a log-concave density λ and let Λ be the corresponding cumulative distribution function. Denote the (unique) inverse of the function Λ by Λ^{-1}. Fix any $t \in (0,1)$. Then in the class of all Borel-measurable sets of ν-measure t, the extremal sets are intervals of the form $(-\infty, a]$ or $[a,\infty)$ for some $a \in \mathbb{R}$. That is, for $t \in (0,1)$ and for every Borel-measurable set $A \subseteq R$ with $\nu(A) = t$,*

$$\nu(A + [-r,r]) \geq \min\left\{\Lambda(\Lambda^{-1}(t)+r), 1-\Lambda(\Lambda^{-1}(1-t)-r)\right\} \qquad \forall r > 0. \tag{3.3}$$

If λ is symmetric (around the median), then the above expression is simplified to

$$\nu(A + [-r,r]) \geq \Lambda(\Lambda^{-1}(t)+r) \qquad \forall r > 0. \tag{3.4}$$

Now we are ready to present the monotonization lemma.

Lemma 3.7. *Consider the product measure $\nu^{\otimes n}$ on $\mathbb{R}^n$ where ν is a probability distribution with a continuous symmetric log-concave density λ satisfying $\lim_{|z|\to\infty} \lambda(z) = 0$. Then for any Borel set $A \subset \mathbb{R}^n$,*

(i) *$I_i^{\mathcal{G}}(A) \geq I_i^h(A)$ for all $1 \leq i \leq n$, where $h(t) = \lambda(\Lambda^{-1}(t))$.*
(ii) *There exists an increasing set B such that $\nu^{\otimes n}(B) = \nu^{\otimes n}(A)$ and*

$$I_i^{\mathcal{G}}(B) \leq I_i^{\mathcal{G}}(A) \qquad \text{for all } \; 1 \leq i \leq n.$$

Proof. Let Λ be the cumulative distribution of ν. Fix $x \in \mathbb{R}^n$. By Theorem 3.6, we have, for all $r > 0$,

$$\frac{\nu(A_i^x + [-r,r]) - \nu(A_i^x)}{r} \geq \frac{\Lambda(\Lambda^{-1}(\nu(A_i^x)) + r) - \Lambda(\Lambda^{-1}(\nu(A_i^x)))}{r}.$$

Taking limit of the both sides as $r \to 0^+$, we obtain

$$m_\nu(A_i^x) \geq \lambda(\Lambda^{-1}(\nu(A_i^x))) = h(\nu(A_i^x)),$$

which implies the first part of the lemma.

For a proof of the second part, we start by noting that the assumptions on ν imply that h is concave and continuous. Thus we can invoke Lemma 3.2 to find an increasing set B such that $\nu^{\otimes n}(B) = \nu^{\otimes n}(A)$ and $I_i^h(B) \leq I_i^h(A)$ for all $1 \leq i \leq n$. By the first part of the lemma, $I_i^h(A) \leq I_i^{\mathcal{G}}(A)$ for all $1 \leq i \leq n$. On the other hand, it follows from Lemma 3.5 that $I_i^{\mathcal{G}}(B) = I_i^h(B)$ for all $1 \leq i \leq n$. Hence,

$$I_i^{\mathcal{G}}(B) = I_i^h(B) \leq I_i^h(A) \leq I_i^{\mathcal{G}}(A),$$

as asserted. □

To keep our exposition simple, we will restrict our attention to an important family of log-concave distributions known as Boltzmann measures for the rest of the section. We mention in passing that some of the techniques that we are going to develop can be applied to other log-concave measures with suitable isoperimetric properties.

3.3. Lower bounds on geometric influences for Boltzmann measures.

Definition 3.8 (Boltzmann Measure)**.** The density of the Boltzmann measure μ_ρ with parameter $\rho \geq 1$ is given by

$$\phi_\rho(x) := \frac{1}{2\Gamma(1+1/\rho)} e^{-|x|^\rho} dx, \quad x \in \mathbb{R}.$$

Note that $\rho = 2$ corresponds to the Gaussian measure with variance 1/2 while $\rho = 1$ gives the two-sided exponential measure.

The following estimates on the tail probability of Boltzmann measures are well-known and easy to verify.

Lemma 3.9. *Let Φ_ρ denote the cumulative distribution function of the Boltzmann distribution with parameter ρ. Then for $z > 0$, we have*

$$\frac{1}{2\rho\Gamma(1+1/\rho)}\left(\frac{1}{z^{\rho-1}} - \frac{\rho-1}{z^\rho}\right)e^{-|z|^\rho} \leq 1 - \Phi_\rho(z) \leq \frac{1}{2\rho\Gamma(1+1/\rho)}\frac{1}{z^{\rho-1}}e^{-|z|^\rho}.$$

In particular,

$$\phi_\rho(\Phi_\rho^{-1}(x)) \asymp x(1-x)(-\log(x(1-x)))^{(\rho-1)/\rho} \qquad (3.5)$$

for x close to zero or one.

It follows from Lemma 3.7(i) and Lemma 3.9 that for Boltzmann measures, the geometric influences lie between previously studied h-influences. On the one hand, they are greater than Variance-influences (i.e., h-influences with $h(t) = t(1-t)$), that were studied in, e.g., [Hat09, MOO09]. On the other hand, for monotone sets they are smaller than the Entropy-influences.

It is well-known that there is no analogue of the KKL influence bound for the Variance-influence, and a tight lower bound on the maximal Variance-influence is the trivial bound:

$$\max_{1 \leq i \leq n} I_i^{\mathrm{Var}}(A) \geq ct(1-t)/n,$$

where t is the measure of the set A. This inequality is an immediate corollary of the Efron-Stein inequality (see, e.g., [Ste86]). On the other hand, the analogue of the

KKL bound proved in [Kel] holds only for h-influences with $h(t) \geq \mathrm{Ent}(t)$. In order to show KKL-type lower bounds for geometric influences, we use the following two results.

The first result is a dimension-free isoperimetric inequality for the Boltzmann measures.

Lemma 3.10 ([Bar04]). *Fix $\rho > 1$ and let μ_ρ denote the Boltzmann measure with parameter ρ. Then there exists a constant $k = k(\rho) > 0$ such that for any $n \geq 1$ and any measurable $A \in \mathbb{R}^n$, we have*

$$\mu_\rho^{\otimes n}(A + [-r, r]^n) \geq \mu_\rho\{(-\infty, \Phi_\rho^{-1}(t) + kr]\}, \quad t = \mu_\rho^{\otimes n}(A).$$

The second key ingredient is a simple corollary of Lemma 3.4.

Lemma 3.11. *Consider the product spaces $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, where μ_ρ denotes the Boltzmann measure with parameter $\rho > 1$. For any $A \subset \mathbb{R}^n$ and for all $1 \leq i \leq n$,*

$$I_i^{\mathcal{G}}(A) \geq c I_i^{\mathrm{Ent}}(A) \left(-\log(I_i^{\mathrm{Ent}}(A))\right)^{-1/\rho},$$

where $c = c(\rho) > 0$ is a universal constant.

Proof. In view of Lemma 3.7, it is sufficient to prove that

$$I_i^{h}(A) \geq c I_i^{\mathrm{Ent}}(A) \left(-\log(I_i^{\mathrm{Ent}}(A))\right)^{-1/\rho},$$

for $h(x) := \phi_\rho(\Phi_\rho^{-1}(x))$. This indeed follows immediately from Lemma 3.4 using the estimate on $h(x)$ given in equation (3.5). □

Now we are ready to prove the KKL-type lower bounds. We start with an analogue of the KKL theorem [KKL88].

Theorem 3.12. *Consider the product spaces $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, where μ_ρ denotes the Boltzmann measure with parameter $\rho > 1$. There exists a constant $c = c(\rho) > 0$ such that for all $n \geq 1$ and for any Borel-measurable set $A \subset \mathbb{R}^n$ with $\nu^{\otimes n}(A) = t$, we have*

$$\max_{1 \leq i \leq n} I_i^{\mathcal{G}}(A) \geq ct(1-t)\frac{(\log n)^{1-1/\rho}}{n}.$$

Proof. The proof is divided into two cases, according to $\nu^{\otimes n}(A) = t$. If $t(1-t)$ is not very small, the proof uses Lemma 3.7 and Lemma 3.11. If $t(1-t)$ is very small, the proof relies on Lemma 3.7 and Lemma 3.10. We note that the same division into cases appears in the proof of the KKL theorem [KKL88]: the core of the proof is the case where $t(1-t)$ is not "too small", and the other case follows immediately from the Edge Isoperimetric Inequality on the cube.

Case A: $t(1-t) > n^{-1}$. By Theorem 3.3, there exists $1 \leq i \leq n$, such that

$$I_i^{\mathrm{Ent}}(A) \geq ct(1-t)\frac{\log n}{n}.$$

Since $t(1-t) > 1/n$, it follows from Lemma 3.11 that

$$I_i^{\mathcal{G}}(A) \geq c I_i^{\mathrm{Ent}}(A) \left(-\log(I_i^{\mathrm{Ent}}(A))\right)^{-1/\rho} \geq c' t(1-t)\frac{\log n}{n} \cdot (\log n)^{-1/\rho},$$

where c' is a universal constant, as asserted.

Case B: $t(1-t) \le n^{-1}$. In view of Lemma 3.7, we can assume w.l.o.g. that the set A is increasing. In that case, by Proposition 1.3, we have

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) = \liminf_{r\to 0+} \frac{\mu_\rho^{\otimes n}(A+[-r,r]^n) - \mu_\rho^{\otimes n}(A)}{r}.$$

By Lemma 3.10,

$$\liminf_{r\to 0+} \frac{\mu_\rho^{\otimes n}(A+[-r,r]^n) - \mu_\rho^{\otimes n}(A)}{r} \ge k\phi_\rho(\Phi_\rho^{-1}(t)). \tag{3.6}$$

Since in this case $t(1-t) \le n^{-1}$, it follows from Lemma 3.9 that

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) \ge k\phi_\rho(\Phi_\rho^{-1}(t)) \ge k' t(1-t)(\log n)^{(\rho-1)/\rho},$$

for some constant $k'(\rho) > 0$. This completes the proof. □

Theorem 1.4 is an immediate consequence of Theorem 3.12. The derivation of Corollary 1.7 from Theorem 1.4 and Proposition 1.6 is exactly the same as the proof of Theorem 2.1 in [FK96] (which is the analogous result for Bernoulli measures on the discrete cube), and thus is omitted here.

We conclude this section with several analogues of results for influences on the discrete cube. In the theorem below, Part (1) corresponds to Talagrand's lower bound on the vector of influences [Tal94], Part (2) corresponds to a variant of the KKL theorem for functions with low influences established in [FK96], Part (3) corresponds to Friedgut's characterization of functions with a low influence sum [Fri98], and Part (4) corresponds to Hatami's characterization of functions with a low influence sum in the continuous case [Hat09]. Statements (1), (3), and (4) of the theorem follow immediately using Lemma 3.11 from the corresponding statements for the Entropy-influence proved in [Kel], and Statement (2) is an immediate corollary of Statement (1).

Theorem 3.13. *Consider the product spaces $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, where μ_ρ denotes the Boltzmann measure with parameter $\rho > 1$. For all $n \ge 1$, for any Borel-measurable set $A \subset \mathbb{R}^n$, and for all $\alpha > 0$, we have:*

(1) *If $\mu_\rho^{\otimes n}(A) = t$, then*
$$\sum_{i=1}^{n} \frac{I_i^{\mathcal{G}}(A)}{(-\log I_i^{\mathcal{G}}(A))^{1-1/\rho}} \ge c_1 t(1-t),$$
(2) *If $\mu_\rho^{\otimes n}(A) = t$ and $\max_{1\le i\le n} I_i^{\mathcal{G}}(A) \le \alpha$, then*
$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(A) \ge c_1 t(1-t)(-\log \alpha)^{1-1/\rho},$$
(3) *If A is monotone and $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)(-\log I_i^{\mathcal{G}}(A))^{1/\rho} = s$, then there exists a set $B \subset \mathbb{R}^n$ such that 1_B is determined by at most $\exp(c_2 s/\epsilon)$ coordinates and $\mu_\rho^{\otimes n}(A \triangle B) \le \epsilon$,*
(4) *If $\sum_{i=1}^{n} I_i^{\mathcal{G}}(A)(-\log I_i^{\mathcal{G}}(A))^{1/\rho} = s$, then there exists a set $B \subset \mathbb{R}^n$ such that 1_B can be represented by a decision tree of depth at most $\exp(c_3 s/\epsilon^2)$ and $\mu_\rho^{\otimes n}(A \triangle B) \le \epsilon$,*[1]

[1]See, e.g., [Hat09] for the definition of a decision tree.

where c_1, c_2, *and* c_3 *are positive constants which depend only on* ρ.

Theorem 1.5 is a special case of Statements (1) and (3) of Theorem 3.13 obtained for $\rho = 2$.

3.4. **A remark on geometric influences for more general measures.** It's worth mentioning that Theorem 3.12 and 3.13 hold for any measure ν on $\mathbb{R}$ which is absolutely continuous with respect to the lebesgue measure and there exist constants $\rho \geq 1, a > 0$ such that the isoperimetric function $\mathcal{I}_\nu$ of ν satisfies

$$\mathcal{I}_\nu(t) \geq a \min(t, 1-t)(-\log \min(t, 1-t))^{1-1/\rho}, \quad t \in [0,1].$$

The proofs are exactly similar to those given for Boltzmann measures except the following remarks.

Lemma 3.7(i) now holds with $h(t) = \mathcal{I}_\nu(t)$. Lemma 3.7(ii) does not hold in general but this is not a problem since for the proof of Theorem 3.12 we only need the first part of the lemma. Instead of Lemma 3.10, we now use the the following dimension-free isoperimetric inequality (see [Bar04]) of the product measure $\nu^{\otimes n}$: for all $n \geq 1$ and $A \subseteq \mathbb{R}^n$ measurable,

$$\liminf_{r \to 0+} \frac{\nu^{\otimes n}(A + [-r, r]^n) - \nu^{\otimes n}(A)}{r} \geq \frac{a}{K} \min(t, 1-t)(-\log \min(t, 1-t))^{1-1/\rho},$$

where $t = \nu^{\otimes n}(A)$ and $K > 0$ is a universal constant.

4. Boundaries of transitive sets under uniform enlargement

It follows from the classical Gaussian isoperimetric inequality by Tsirelson and Sudakov [ST74], Borell [Bor75] (see also [Bob96]) that for the Gaussian case, in any dimension, the half spaces are extremal under uniform enlargement, which implies that the boundary measure of any measurable set $A \subset \mathbb{R}^n$ with $\Phi^{\otimes n}(A) = t$ obeys the following lower bound:

$$\liminf_{r \downarrow 0} \frac{\Phi^{\otimes n}(A + [-r, r]^n) - \Phi^{\otimes n}(A)}{r} \geq \phi(\Phi^{-1}(t)), \tag{4.1}$$

and the bound is achieved when A is a half-space.

In this section we consider the same isoperimetric problem under an additional symmetry condition:

Find a lower bound on the boundary measure (under uniform enlargement) of sets in $\mathbb{R}^n$ *that are transitive.*

The invariance under permutation condition rules out candidates like the half-spaces and one might expect that under this assumption, a set should have "large" boundary. This intuition is confirmed by Theorem 1.8. In this section we prove a stronger version of this theorem that holds for all Boltzmann measures.

Theorem 4.1. *Consider the product spaces* $(\mathbb{R}^n, \mu_\rho^{\otimes n})$, *where* μ_ρ *denotes the Boltzmann measure with parameter* $\rho > 1$. *There exists a constant* $c = c(\rho) > 0$ *such that the following holds for all* $n \geq 1$:

For any transitive Borel-measurable set $A \subset \mathbb{R}^n$, *we have*

$$\liminf_{r \downarrow 0} \frac{\mu_\rho^{\otimes n}(A + [-r, r]^n) - \mu_\rho^{\otimes n}(A)}{r} \geq ct(1-t)(\log n)^{1-1/\rho},$$

where $t = \mu_\rho^{\otimes n}(A)$.

The transitivity assumption on A implies that Theorem 4.1 is an immediate consequence of Theorem 3.12, once we establish the following lemma.

Lemma 4.2. *Let λ be a continuous symmetric log-concave density on $\mathbb{R}$. Let A be any Borel-measurable subset of $\mathbb{R}^n$. Then*

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n)-\nu^{\otimes n}(A)}{r} \geq \sum_{i=1}^{n} I_i^h(A),$$

where $h(x)=\lambda(\Lambda^{-1}(x))$ for all $x\in[0,1]$.

Proof. The proof is similar to the proof of Proposition 1.3. For all $1\leq i\leq n$, define

$$B_r^i = A+[-r,r]^{i-1}\times\{0\}^{n-i+1}.$$

Like in the proof of Proposition 1.3, it is sufficient to show that for each i,

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(B_r^i+\{0\}^{i-1}\times[-r,r]\times\{0\}^{n-i})-\nu^{\otimes n}(B_r^i)}{r} \geq I_i^h(A). \tag{4.2}$$

Note that for all $x\in\mathbb{R}^n$, both $\nu^{\otimes n}(B_r^i)$ and $\nu((B_r^i)_i^x)$ are increasing as functions of r, and thus, they tend to some limit as $r\searrow 0$. Furthermore, we can assume that $\nu^{\otimes n}(\bar{A}\setminus A)=0$, since otherwise,

$$\liminf_{r\downarrow 0} \frac{\nu^{\otimes n}(A+[-r,r]^n)-\nu^{\otimes n}(A)}{r} \geq \liminf_{r\downarrow 0}\frac{\nu^{\otimes n}(\bar{A}\setminus A)}{r}\to\infty.$$

Therefore,

$$\nu^{\otimes n}(B_r^i)\searrow \nu^{\otimes n}(\bar{A})=\nu^{\otimes n}(A),$$

and

$$\nu((B_r^i)_i^x)\searrow \nu(A_i^x), \tag{4.3}$$

for almost every $x\in\mathbb{R}^n$ (w.r.t. the measure $\nu^{\otimes n}$).

Now observe that by the one-dimensional isoperimetric inequality for symmetric log-concave distributions (Theorem 3.6),

$$\begin{aligned}\nu^{\otimes n}(B_r^i+\{0\}^{i-1}\times[-r,r]\times\{0\}^{n-i}) &= \mathbb{E}_x\nu((B_r^i)_i^x+[-r,r]) \\ &\geq \mathbb{E}_x\Lambda(\Lambda^{-1}(\nu((B_r^i)_i^x))+r).\end{aligned}$$

Therefore, using the Mean Value Theorem like in the proof of Lemma 1.3, we get

$$\begin{aligned}\liminf_{r\downarrow 0} &\frac{\nu^{\otimes n}(B_r+\{0\}^{i-1}\times[-r,r]\times\{0\}^{n-i})-\nu^{\otimes n}(B_r)}{r} \\ &\geq \liminf_{r\downarrow 0}\mathbb{E}_x \inf_{z\in[\Lambda^{-1}(\nu((B_r^i)_i^x)),\Lambda^{-1}(\nu((B_r^i)_i^x))+r]}\lambda(z).\end{aligned} \tag{4.4}$$

Finally, by (4.3), for almost every $x\in\mathbb{R}^n$,

$$\lim_{r\downarrow 0}\inf_{z\in[\Lambda^{-1}(\nu((B_r^i)_i^x)),\Lambda^{-1}(\nu((B_r^i)_i^x))+r]}\lambda(z)=\lambda(\Lambda^{-1}(\nu(A_i^x))),$$

and thus, by the Dominated Convergence Theorem,

$$\liminf_{r\downarrow 0}\mathbb{E}_x \inf_{z\in[\Lambda^{-1}(\nu((B_r^i)_i^x)),\Lambda^{-1}(\nu((B_r^i)_i^x))+r]}\lambda(z)=\mathbb{E}_x\lambda(\Lambda^{-1}(\nu(A_i^x)))=I_i^h(A).$$

This completes the proof of the lemma, and thus also the proof of Theorem 4.1. □

4.1. **Tightness of Theorems 3.12, 3.13, and 4.1.** We conclude this section with showing that Theorems 3.12, 3.13, and 4.1 are tight (up to constant factors) among sets with constant measure, which we set for convenience to be 1/2. We demonstrate this by choosing an appropriate sequence of '*one-sided boxes*'.

Proposition 4.3. *Consider the product spaces* $(\mathbb{R}^n, \mu_\rho^{\otimes n})$*, where* μ_ρ *denotes the Boltzmann measure with parameter* $\rho \geq 1$*. Let* $B_n := (-\infty, a_n]^n$ *where* a_n *is chosen such that* $\Phi_\rho(a_n)^n = 1/2$*. Then there exists a constant* $c = c(\rho)$ *such that*

$$I_i^{\mathcal{G}}(B_n) \leq c \cdot \frac{(\log n)^{1-1/\rho}}{n},$$

for all $1 \leq i \leq n$*.*

Proof. Fix an i. By elementary calculation,

$$I_i^{\mathcal{G}}(B_n) = \Phi_\rho(a_n)^{n-1}\phi_\rho(a_n) = (1/2)^{(n-1)/n}\phi_\rho(a_n).$$

Note that $1-\Phi_\rho(a_n) \asymp n^{-1}$, and thus, by Lemma 3.9, $a_n \asymp (\log n)^{1/\rho}$. Furthermore, since by Lemma 3.9, $\phi_\rho(z) \asymp z^{\rho-1}(1-\Phi_\rho(z))$ for large z, we have $I_i^{\mathcal{G}}(B_n) \asymp n^{-1}(\log n)^{1-1/\rho}$, as asserted. □

The tightness of Theorem 3.12 and Theorem 3.13 (1) follows immediately from Proposition 4.3. The tightness of Theorem 4.1 follows using Proposition 1.3 since B is monotone. The tightness of Theorem 3.13 (2) and the tightness in s in Theorem 3.13 (3) and Theorem 3.13 (4) follows by considering the subset $B_k \times \mathbb{R}^{n-k} \subset \mathbb{R}^n$.

5. Geometric influences under rotation

Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. In Section 3 we obtained lower bounds on the sum of geometric influences, and in particular we showed that for a transitive set $A \subset \mathbb{R}^n$, the sum is at least $\Omega(t(1-t)\sqrt{\log n})$, where $t = \mu^{\otimes n}(A)$.

In this section we consider a different symmetry group, the group of rotations of $\mathbb{R}^n$. The interest in this group comes from the fact that the Gaussian measure is invariant under rotations while the influence sum is not.

Indeed, a half space of measure 1/2 may have influence sum as small as of order 1 when it is aligned with one of the axis and as large as of order $\sqrt{n}$ when it is aligned with the diagonal direction $(1, 1, \ldots, 1)$.

In this section we show that under some mild conditions (that do not contain any invariance assumption), rotation allows to increase the sum of geometric influences up to $\Omega(t(1-t)\sqrt{-\log(t(1-t))}\sqrt{n})$. The dependence on n in this lower bound is tight for several examples, including half-spaces and L^2-balls. We note that on the other extreme, rotation cannot decrease the sum of geometric influences below $\Omega\left(t(1-t)\sqrt{-\log(t(1-t))}\right)$, as follows from a combination of Proposition 1.3, Lemma 3.7(ii) and the isoperimetric inequality (4.1).

Definition 5.1. Let $B(x,r) := \{y \in \mathbb{R}^n : \|y-x\|_2 < r\}$ be the open ball in $\mathbb{R}^n$ with center at x and radius r and let $\bar{B}(x,r)$ be the corresponding closed ball. For $\varepsilon > 0$ and $A \subseteq R^n$, define

$$A_\varepsilon := \{x \in A : \bar{B}(x,\varepsilon) \cap A^c = \emptyset\}, \qquad \text{and} \qquad A^\varepsilon := \{x \in \mathbb{R}^n : B(x,\varepsilon) \cap A \neq \emptyset\}.$$

Finally, denote by $\mathcal{J}_n$ the collection of all measurable sets $B \subseteq \mathbb{R}^n$ for which there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$, we have

$$(B_\varepsilon)^{2\varepsilon} \supseteq B. \tag{5.1}$$

The crucial ingredient in the proof Theorem 1.9 is a lemma asserting that under the conditions of the theorem, an enlargement of A by a *random* rotation of the cube $[-r,r]^n$ increases $\mu^{\otimes n}(A)$ significantly.

Notation 5.2. *Let $O = O(n,\mathbb{R})$ be the set of all orthogonal transformations on $\mathbb{R}^n$, and let ν be the (unique) Haar measure on O. Denote by M a random element of O distributed according to the measure ν.*

Lemma 5.3. *There exists a constant $K > 0$ such that for any $A \in \mathcal{J}_n$, we have*

$$\mathbb{E}_{M\sim\nu}\left[\mu^{\otimes n}(A + M^{-1}(Kn^{-1/2}[-r,r]^n))\right] \geq \mu^{\otimes n}(A) + \frac{1}{2}\mu^{\otimes n}(A^{r/3} \setminus A),$$

for all sufficiently small $r > 0$ (depending on A).

First we show that Lemma 5.3 implies Theorem 1.9.

Proof. Note that for any $g \in O$, $g(A)$ is convex, and that $\mu^{\otimes n}$ is invariant under g. Thus by Proposition 1.3,[2] we have

$$\begin{aligned}\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) &= \lim_{r\to 0+} \frac{\mu^{\otimes n}(g(A) + [-r,r]^n) - \mu^{\otimes n}(g(A))}{r} \\ &= \lim_{r\to 0+} \frac{\mu^{\otimes n}(A + g^{-1}([-r,r]^n)) - \mu^{\otimes n}(A)}{r}.\end{aligned}$$

Furthermore, note that for any $g \in O$,

$$\begin{aligned}\lim_{r\to 0+} \frac{\mu^{\otimes n}(A + g^{-1}([-r,r]^n)) - \mu^{\otimes n}(A)}{r} &\leq \lim_{r\to 0+} \frac{\mu^{\otimes n}(A + \sqrt{n}[-r,r]^n) - \mu^{\otimes n}(A)}{r} \\ &= \sqrt{n} \times \sum_{i=1}^{n} I_i^{\mathcal{G}}(A).\end{aligned}$$

Therefore, by the Dominated Convergence Theorem,

$$\mathbb{E}_{M\sim\nu}\left[\sum_{i=1}^{n} I_i^{\mathcal{G}}(M(A))\right] = \lim_{r\to 0+} \frac{\mathbb{E}_{M\sim\nu}\left[\mu^{\otimes n}(A + M^{-1}([-r,r]^n))\right] - \mu^{\otimes n}(A)}{r}. \tag{5.2}$$

By Lemma 5.3, we have (for a sufficiently small r):

$$\mathbb{E}_{M\sim\nu}\left[\mu^{\otimes n}(A + M^{-1}(Kn^{-1/2}[-r,r]^n))\right] - \mu^{\otimes n}(A) \geq \frac{1}{2}\mu^{\otimes n}(A^{r/3} \setminus A).$$

By the standard Gaussian isoperimetric inequality,

$$\mu^{\otimes n}(A^{r/3} \setminus A) \geq \mu((-\infty, \Phi^{-1}(t) + r/3]).$$

Substituting into equation (5.2), we get

$$\begin{aligned}\mathbb{E}_{M\sim\nu}\left[\sum_{i=1}^{n} I_i^{\mathcal{G}}(M(A))\right] &\geq \limsup_{r\to 0+} \frac{\mu((-\infty, \Phi^{-1}(t) + K^{-1}n^{1/2}r/3])}{2r} \\ &\geq \frac{\sqrt{n}}{6K}\phi(\Phi^{-1}(t)) \geq c\sqrt{n}t(1-t)\sqrt{-\log(t(1-t))},\end{aligned}$$

[2]Note that by Remark 2.2, Proposition 1.3 holds for convex sets.

for some constant $c > 0$. Thus, there exists at least one orthogonal transformation $g \in O$ such that

$$\sum_{i=1}^{n} I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log(t(1-t))} \times \sqrt{n},$$

as asserted. □

Now we present the proof of Lemma 5.3.

Proof. By Fubini's theorem, we have

$$\begin{aligned} &\mathbb{E}_{M\sim\nu}\left[\mu^{\otimes n}(A + M^{-1}(Kn^{-1/2}[-r,r]^n))\right] \\ &= \mathbb{E}_{M\sim\nu}\left[\mu^{\otimes n}\{x \in \mathbb{R}^n : x = y + z, y \in A, z \in M^{-1}(Kn^{-1/2}[-r,r]^n)\}\right] \\ &= \mathbb{E}_{x\sim\mu^{\otimes n}}\left[\nu\{g \in O : x = y + z, y \in A, z \in g^{-1}(Kn^{-1/2}[-r,r]^n)\}\right]. \end{aligned} \tag{5.3}$$

Since each $x \in A$ can be trivially represented as $y + z$ with $y = x \in A, z = 0 \in g^{-1}(Kn^{-1/2}[-r,r]^n)$ for any $g \in O$, the assertion of the lemma would follow immediately from equation (5.3) once we show that for all $x \in A^{r/3} \setminus A$,

$$\nu\{g \in O : x = y + z, y \in A, z \in g^{-1}(Kn^{-1/2}[-r,r]^n)\} \geq 1/2. \tag{5.4}$$

Since $A \in \mathcal{J}_n$, we can choose r sufficiently small such that $A \subset (A_{r/3})^{2r/3}$, and thus $A^{r/3} \subset (A_{r/3})^r$. Therefore, for any $x \in A^{r/3} \setminus A$, there exists $y \in A_{r/3}$, such that $||x-y||_2 < r$. If there exists $y' \in B(y, r/3)$ such that $x - y' \in g^{-1}(Kn^{-1/2}[-r,r]^n)$, then x can be represented as $y' + (x - y')$, as required in the left hand side of equation (5.4). Therefore, it is sufficient to prove the following claim:

Claim 5.4. *For any $x, y \in \mathbb{R}^n$ such that $||x-y||_2 < r$,*

$$\nu\Big\{g \in O : \exists\, y' \in B(y, r/3) \text{ such that } x - y' \in g^{-1}(Kn^{-1/2}[-r,r]^n)\Big\} \geq 1/2.$$

Proof of the claim. Fix $x, y \in \mathbb{R}^n$ such that $||x-y||_2 < r$. We have

$$\begin{aligned} &\Big\{g \in O : \exists\, y' \in B(y, r/3) \text{ such that } x - y' \in g^{-1}(Kn^{-1/2}[-r,r]^n)\Big\} \\ &= \Big\{g \in O : \exists\, y' \in B(y, r/3) \text{ such that } g(x - y') \in Kn^{-1/2}[-r,r]^n\Big\} \\ &= \Big\{g \in O : \exists\, y'' \in B(0, r/3) \text{ such that } g(x - y) - y'' \in Kn^{-1/2}[-r,r]^n\Big\} \\ &= \Big\{g \in O : \inf_{y'' \in B(0,r/3)} ||g(x-y) - y''||_\infty \leq Kn^{-1/2}r\Big\}. \end{aligned}$$

Note that

$$\nu\Big\{g \in O : \inf_{y'' \in B(0,r/3)} ||g(x-y) - y''||_\infty \leq Kn^{-1/2}r\Big\} \tag{5.5}$$

is invariant under rotation of the vector $(x - y)$, and in particular,

$$(5.5) = \nu\Big\{g \in O : \inf_{y'' \in B(0,r/3)} ||g(||x-y||_2 \times e_1) - y''||_\infty \leq Kn^{-1/2}r\Big\},$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ is the unit vector along the first coordinate axis.

A well-known property of the Haar measure says that if $M \in O$ is distributed according to ν, then any column of M is distributed like a normalized vector of independent standard Gaussians. That is,

$$M_{column} \sim \frac{Z}{\|Z\|_2},$$

where $Z = (Z_1, \ldots, Z_n)$ is a random n-vector with i.i.d. standard Gaussian entries. Thus, $M(||x - y||_2 \times e_1)$ is distributed like $||x - y||_2 \times Z/||Z||_2$. Therefore, we have

$$\begin{aligned}(5.5) &= \mathbb{P}_{Z \sim \mu^{\otimes n}}\Big(\inf_{y'' \in B(0,r/3)} || \; ||x-y||_2 \times \frac{Z}{||Z||_2} - y''||_\infty \leq K n^{-1/2} r \Big) \\ &\geq \mathbb{P}_{Z \sim \mu^{\otimes n}}\Big(\inf_{y''' \in B(0,1/3)} || \frac{Z}{||Z||_2} - y'''||_\infty \leq K n^{-1/2} \Big).\end{aligned}$$

Note that if $Z \in \mathbb{R}^n$ satisfies

$$\frac{\sum_i Z_i^2 1_{|Z_i|/||Z||_2 > Kn^{-1/2}}}{||Z||_2^2} < 1/9,$$

then the vector y''' defined by $y_i''' = (Z_i \cdot 1_{|Z_i|/||Z||_2 > Kn^{-1/2}})/||Z||_2$ satisfies

$$y''' \in B(0, 1/3) \qquad \text{and} \qquad || \frac{Z}{||Z||_2} - y'''||_\infty \leq K n^{-1/2}.$$

Hence,

$$\begin{aligned}(5.5) &\geq \mathbb{P}_{Z \sim \mu^{\otimes n}}\Big(\inf_{y''' \in B(0,1/3)} || \frac{Z}{||Z||_2} - y'''||_\infty \leq K n^{-1/2} \Big) \\ &\geq \mathbb{P}_{Z \sim \mu^{\otimes n}}\Big(\frac{\sum_i Z_i^2 1_{|Z_i|/||Z||_2 > Kn^{-1/2}}}{||Z||_2^2} < 1/9 \Big).\end{aligned}$$

Finally, by the Markov inequality,

$$\mathbb{P}_{Z \sim \mu^{\otimes n}} \left[\sum_{i : |Z_i| > K/2} Z_i^2 \geq \frac{n}{36} \right] \leq \frac{n \times \left[\mathbb{E} Z_1^2 1_{\{|Z_1| > K/2\}} \right]}{n/36} \leq 1/4$$

for sufficiently large $K > 0$, and by the concentration of norm of a Gaussian vector, $\mathbb{P}[\|Z\|_2 > \sqrt{n}/2] \geq 3/4$. Therefore,

$$\begin{aligned}(5.5) &\geq \mathbb{P}_{Z \sim \mu^{\otimes n}}\Big(\frac{\sum_i Z_i^2 1_{|Z_i|/||Z||_2 > Kn^{-1/2}}}{||Z||_2^2} < 1/9 \Big) \\ &\geq \mathbb{P}_{Z \sim \mu^{\otimes n}} \left[\left(\sum_{i : |Z_i| > K/2} Z_i^2 \leq \frac{n}{36} \right) \wedge (\|Z\|_2 > \sqrt{n}/2) \right] \\ &\geq 3/4 + 3/4 - 1 = 1/2.\end{aligned}$$

This completes the proof of the claim and of Lemma 5.3. □

Intuitively, the condition $A \in \mathcal{J}_n$ means that the boundary of A is "sufficiently smooth". One can easily check that if $A \in \mathcal{J}_n$, then the boundary of A is a porous set and thus has Hausdorff dimension strictly less than n (see [Zaj88] and references therein to know more about porous sets). However, this condition is far from being sufficient. Here we give a sufficient condition for a set to belong to $\mathcal{J}_n$ in terms of smoothness of its boundary.

Definition 5.5. Let $A \subset \mathbb{R}^n$ be a measurable set. We write $\partial A \in C^1$ and say that the boundary of A is of class C^1 if for any point $z \in \partial A$, there exists $r = r(z) > 0$ and a one-to-one mapping ψ of $B(z,r)$ onto an open set $D = D \subseteq \mathbb{R}^n$ such that:

- $\psi \in C^1(\bar{B}(z,r))$ and $\psi^{-1} \in C^1(\bar{D})$,
- $\psi(B(z,r) \cap \partial A) = D \cap \{x \in \mathbb{R}^n : x_1 = 0\}$,
- $\psi(B(z,r) \cap \mathrm{int}(A)) \subseteq (0,\infty) \times \mathbb{R}^{n-1}$.

Proposition 5.6. *Let $A \subset \mathbb{R}^n$ be a bounded set with $\partial A \in C^1$. Then $A \in \mathcal{J}_n$.*

Proof. Suppose on the contrary that $A \notin \mathcal{J}_n$. Then there exists a sequence $\{x^m\}_{m=1}^{\infty}$ such that $x^m \in A$ but $x^m \notin (A_{1/m})^{2/m}$. Since A is bounded, the sequence contains a subsequence $\{x^{m_k}\}$ converging to a point x^0. Clearly, $x^0 \in \partial A$.

Since $\partial A \in C^1$, we can define a new set of local coordinates $(y_1, y_2, \ldots, y_n)$ (also denoted by (y_1, y'), where $y' \in \mathbb{R}^{n-1}$), such that:

(1) The point x^0 is the origin with respect to the y-coordinates,
(2) There exists an open neighborhood $(-\delta_0, \delta_0) \times U \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$ containing the origin and a continuously differentiable function $f : U \to \mathbb{R}_+$, such that in the y-coordinates,
$$\partial A \cap [(-\delta_0,\delta_0) \times U] = \{(f(y'), y') : y' \in U\},$$
and
$$\mathrm{int} A \cap [(-\delta_0,\delta_0) \times U] = \{(y_1, y') : y' \in U, f(y') < y_1 < \delta_0\}. \tag{5.6}$$

By the construction of the new coordinates, $f(y') \geq 0$ for all $y' \in U$, and $f(0) := f(0,0,\ldots,0) = 0$. Since $f \in C^1(U)$, it follows that $\nabla f(0) = 0$. Hence, by the continuity of the partial derivatives of f, there exists $r_0 > 0$ such that $\|\nabla f(y')\|_\infty \leq 1/(3\sqrt{n})$ for all $y' \in B_{n-1}(0, r_0) \subseteq U$.

Let $y^m = (y_1^m, (y^m)')$ be the representation of the point x^m in the y-coordinates. Find m large enough such that $1/m < \min\{\delta_0/10, r_0/10\}$, and y^m lies within $A \cap [0, \delta_0/2] \times B_{n-1}(0, r_0/2)$. Define
$$z = (z_1, z_2, \ldots, z_n) = y^m + (1.5m^{-1}, 0, \ldots, 0, 0).$$
We claim that $B(z, 1/m) \subseteq A$. This would be a contradiction to the hypothesis $y^m \notin (A_{1/m})^{2/m}$.

Note that by the choice of m, we have $z \in A$, and moreover,
$$\begin{aligned} \mathrm{dist}(z, \partial A) &\geq \mathrm{dist}\left(z, \partial A \cap [(-\delta_0,\delta_0) \times B_{n-1}(0,r_0)]\right) \\ &= \inf_{y' \in B_{n-1}(0,r_0)} \left\| \left(y_1^m + 1.5m^{-1}, (y^m)'\right) - (f(y'), y') \right\|_2. \end{aligned} \tag{5.7}$$

We would like to show that if $||(y^m)' - y'||_2$ is "small" then $|y_1^m + 1.5m^{-1} - f(y')|$ is "big", and thus in total, the right hand side of equation (5.7) cannot be "too small".

Define $w_1 := y_1^m + 1.5m^{-1} - f((y^m)')$. Note that since $y^m \in A$, it follows from equation (5.6) that $w_1 \geq 1.5m^{-1}$. By the Mean Value Theorem, for each $y' \in B_{n-1}(0, r_0)$,
$$\begin{aligned} |f((y^m)') - f(y')| &\leq \left(\sup_{y'' \in B_{n-1}(0,r_0)} \|\nabla f(y'')\|_\infty \right) \|(y^m)' - y'\|_1 \\ &\leq \frac{\|(y^m)' - y'\|_1}{3\sqrt{n}} \leq \frac{\|(y^m)' - y'\|_2}{3}, \end{aligned}$$

and thus,

$$|y_1^m + 1.5m^{-1} - f(y')| = |w_1 - (f(y') - f((y^m)'))| \geq 1.5m^{-1} - \frac{\|(y^m)' - y'\|_2}{3}.$$

Consequently, if $\|(y^m)' - y'\|_2 \geq 4.5m^{-1}$, then

$$\|\big(y_1^m + 1.5m^{-1}, (y^m)'\big) - (f(y'), y')\|_2 \geq \|(y^m)' - y'\|_2 \geq 4.5m^{-1},$$

and if $\|(y^m)' - y'\|_2 < 4.5m^{-1}$, then

$$\|\big(y_1^m + 1.5m^{-1}, (y^m)'\big) - (f(y'), y')\|_2 \geq \sqrt{\|(y^m)' - y'\|_2^2 + \left(1.5m^{-1} - \frac{\|(y^m)' - y'\|_2}{3}\right)^2}$$
$$= \min_{0 \leq s < 4.5m^{-1}} \sqrt{s^2 + (1.5m^{-1} - s/3)^2} = \sqrt{\frac{81}{40}}m^{-1}.$$

Combining the two cases, we get

$$\mathrm{dist}(z, \partial A) \geq \inf_{y' \in B_{n-1}(0, r_0)} \|\big(y_1^m + 1.5m^{-1}, (y^m)'\big) - (f(y'), y')\|_2$$
$$\geq \min\left(4.5m^{-1}, \sqrt{\frac{81}{40}}m^{-1}\right) > 1/m.$$

This completes the proof. □

If the condition $A \in \mathcal{J}_n$ is removed, we can prove only a weaker lower bound on the maximal sum of geometric influences that can be obtained by rotation.

Proposition 5.7. *Consider the product Gaussian measure $\mu^{\otimes n}$ on $\mathbb{R}^n$. For any convex set A with $\mu^{\otimes n}(A) = t$, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that*

$$\sum_{i=1}^n I_i^{\mathcal{G}}(g(A)) \geq ct(1-t)\sqrt{-\log(t(1-t))}\frac{\sqrt{n}}{\sqrt{\log n}},$$

where $c > 0$ is a universal constant.

The proof of Proposition 5.7 uses a weaker variant of Lemma 5.3:

Lemma 5.8. *Let M be as defined in Notation 5.2. There exists a constant $K > 0$ such that for any $A \subset \mathbb{R}^n$ and for any $r > 0$, we have*

$$\mathbb{E}_{M \sim \nu}\left[\mu^{\otimes n}(A + M^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n))\right] \geq \mu^{\otimes n}(A) + \frac{1}{2}\mu^{\otimes n}(A^r \setminus A).$$

Proof of the lemma. By Fubini's theorem, we have

$$\mathbb{E}_{M \sim \nu}\left[\mu^{\otimes n}(A + M^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n))\right]$$
$$= \mathbb{E}_{x \sim \mu^{\otimes n}}\left[\nu\{g \in O : x \in A + g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\}\right].$$

Thus, it is sufficient to prove that for any $x \in A^r \setminus A$,

$$\nu\{g \in O : x \in A + g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\} \geq 1/2.$$

Equivalently, it is sufficient to prove that for any $x \in B(0, r)$,

$$\nu\{g \in O : x \in g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r, r]^n)\} \geq 1/2.$$

We can assume w.l.o.g. that $x = r' \cdot e_1$ for some $r' < r$. By the argument used in the proof of Lemma 5.3, if $M \in O$ is distributed according to ν, then $M(r' \cdot e_1)$ is distributed like $r' \cdot Z/||Z||_2$, where $Z = (Z_1, \dots, Z_n)$ is a random n-vector with i.i.d. standard Gaussian entries. Hence,

$$\begin{aligned}
&\nu\{g \in O : x \in g^{-1}(K\sqrt{\log n} \cdot n^{-1/2}[-r,r]^n)\} \\
&= \mathbb{P}_{Z\sim\mu^{\otimes n}}\Big(||r' \frac{Z}{||Z||_2}||_\infty \le K\sqrt{\log n} \cdot n^{-1/2} r\Big) \\
&\ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\Big(||\frac{Z}{||Z||_2}||_\infty \le K\sqrt{\log n} \cdot n^{-1/2}\Big) \\
&\ge \mathbb{P}_{Z\sim\mu^{\otimes n}}\Big[\Big(||Z||_\infty \le K\sqrt{\log n}/2\Big) \wedge (\|Z\|_2 \ge \sqrt{n}/2)\Big]. \qquad (5.8)
\end{aligned}$$

We have

$$\mathbb{P}_{Z\sim\mu^{\otimes n}}(||Z||_\infty \le K\sqrt{\log n}/2) \ge 1 - n\mathbb{P}(Z_i > K\sqrt{\log n}/2) \ge 1 - \frac{n}{\sqrt{2\pi}} \cdot n^{-K^2/8} \ge 3/4,$$

for a sufficiently big K. Therefore,

$$(5.8) \ge 3/4 + 3/4 - 1 = 1/2,$$

and this completes the proof of the lemma. □

The derivation of Proposition 5.7 from Lemma 5.8 is the same as the derivation of Theorem 1.9 from Lemma 5.3.

Note that the convexity assumption on A is used only to apply Proposition 1.3 that relates the sum of influences to the size of the boundary w.r.t. uniform enlargement. Thus, our argument also shows that for *any* measurable set A with $\mu^{\otimes n}(A) = t$, there exists an orthogonal transformation g on $\mathbb{R}^n$ such that

$$\lim_{r\to 0+} \frac{\mu^{\otimes n}(g(A) + [-r,r]^n) - \mu^{\otimes n}(g(A))}{r} \ge ct(1-t)\sqrt{-\log(t(1-t))}\frac{\sqrt{n}}{\sqrt{\log n}},$$

where $c > 0$ is a universal constant.

Finally, we note that apparently the assertion of Proposition 5.7 is not optimal, and the lower bound asserted in Theorem 1.9 should hold for general convex sets.

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