On The Probability of a Rational Outcome for Generalized Social Welfare Functions on Three Alternatives

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Abstract

In [10], Kalai investigated the probability of a rational outcome for a generalized social welfare function (GSWF) on three alternatives, when the individual preferences are uniform and independent. In this paper we generalize the results of [10] to a wider class of distributions of the individual preferences, and obtain new lower and upper bounds for the probability of a rational outcome in several classes of GSWFs. In particular, we show that if the GSWF is monotone and balanced and the distribution of the preferences is uniform, then the probability of a rational outcome is at least 3/4, proving a conjecture raised in [10]. The tools used in the paper are analytic: the Fourier-Walsh expansion of Boolean functions on the discrete cube, properties of Beckner's noise operator, and the FKG inequality.

1 Introduction

Consider a situation in which a society of n members selects a ranking amongst m alternatives. In the election process, each member of the society gives a ranking of the alternatives (the ranking is a full linear ordering; that is, indifference between alternatives is not allowed). The set of the rankings given by the individual members is called a *profile*. Given the profile, the ranking of the society is determined according to some function, called a *generalized social welfare function* (GSWF).

The GSWF is a function $F : L^n \rightarrow \{0,1\}^{\binom{m}{2}}$, where L is the set of linear orderings on m elements. In other words, given the profile consisting of linear orderings supplied by the voters, the function determines the preference of the society between each of the $\binom{m}{2}$ pairs of alternatives. If the output of F can be represented as a full linear ordering of the m alternatives, then F is called a *social welfare function* (SWF).

Throughout this paper we consider GSWFs satisfying the Independence of Irrelevant Alternatives (IIA) condition: For every two alternatives A and B, the preference of the entire society between A and B depends only on the preference of each individual voter between A and B. This natural condition on GSWFs can be traced back to Condorcet [4].

The Condorcet's paradox demonstrates that if the number of alternatives is at least three and the GSWF is based on the majority rule between every pair of alternatives, then there exist profiles for which the voting procedure cannot yield a full order relation. That is, there exist alternatives A, B, and C, such that the majority of the society prefers A over B, the majority prefers B over C, and the majority prefers C over A. Such situation is called *irrational choice* of the society. Arrow's impossibility theorem asserts that under mild conditions, for every non-dictatorial choice function between the pairs of alternatives there exists a profile for which the choice of the society is irrational.

Since the existence of profiles leading to an irrational choice has significant implications on voting procedures, an extensive research has been conducted in order to evaluate the probability of irrational choice for various GSWFs. Most of the results in this area are summarized in [8]. In addition to its significance in Social Choice theory, this area of research leads to interesting questions in probabilistic and extremal combinatorics (see [10, 13]).

In [10], Kalai used analytic tools to study the probability of rational choice for GSWFs on three alternatives. The probability of rational choice was expressed by a Fourier-theoretic formula, and as a result, upper and lower bounds on the probability for various classes of GSWFs were obtained. For example, it was shown that if the GSWF is neutral (i.e., invariant under permutations of the alternatives), symmetric (i.e., invariant under a transitive group of permutations of the voters), and satisfies the IIA condition, then the probability of a rational choice is at most 0.9192.

In this paper we generalize the results of [10] in several directions. First, we consider the distribution of the individual preferences. In [10] it is assumed that the individual preferences are independent and uniformly distributed. We show that the results of [10] are valid (under some modifications) also for non-uniform distributions of the profiles, as long as the voters are independent, and for each ordering of the alternatives, the probability of the ordering is equal to the probability of the inverse ordering. We call such distributions *even product distributions*. In particular, we prove the following generalization of Theorem 5.1 of [10]:

Theorem 1 *Consider a GSWF on three alternatives satisfying the IIA condition. If the distribution of the preferences is an even product distribution such that the probability of each preference is positive, and the GSWF is neutral and symmetric, then the probability of irrational choice is bounded away from zero (independently of the number of the voters).*

Further, we obtain new lower and upper bounds on the probability of rational choice for several classes of GSWFs. Our main result is:

Theorem 2 *Consider a GSWF on three alternatives satisfying the IIA condition. If the individual preferences are independent and uniformly distributed, and the GSWF is monotone and balanced then the probability of a rational choice is at least 3/4.*

The proof of this result relies on properties of Beckner's noise operator and uses the FKG inequality.

Finally, we consider the "stability" version of Arrow's theorem presented in [10]. Arrow's theorem (for three alternatives) can be restated as asserting that if the GSWF satisfies the IIA condition, is not a dictatorship, and has all the three alternatives in its range, then the probability of irrational choice is positive. In [10] it is shown (under the IIA condition) that if a neutral GSWF is "far" from being a dictatorship, then the probability of irrational choice is not negligible. In other words, there exists a constant K such that if the distance between the GSWF and any dictatorship is at least ϵ, then the probability of irrational choice is at least $K \cdot \epsilon$.

We show that a similar stability version for the distance of the GSWF from having only two alternatives in its range does not hold.

Theorem 3 *For all $\epsilon, K > 0$ and $n = n(\epsilon, K)$ big enough, there exists a GSWF on three alternatives satisfying the IIA condition, such that:*

1. *Amongst any pair of alternatives, the probability of each alternative to be preferred by the society over the other alternative is at least $\eta = 2^{-\epsilon n}/(n+1)$.*
2. *The probability of irrational choice is less than η/K.*

It appears that the "most" rational GSWF is not obtained by almost constant choice functions between every pair of alternatives, but rather by almost constant functions between two pairs of alternatives and a majority rule between the remaining pair of alternatives.

We don't know whether a weaker version of the stability result holds in this case. It seems possible that there exists a constant K such that if the distance of the GSWF from having only two alternatives in its range is at least ϵ, then the probability of irrational choice is at least $K \cdot \epsilon^2$, but we weren't able to prove such result.

The paper is organized as follows: In Section 2 we recall some basic notions on the Fourier-Walsh expansion of functions on the discrete cube. In Section 3 we generalize the results of [10] to even product distributions of the profiles and prove Theorem 1. In Section 4 we establish lower bounds for the probability of rational choice for several classes of GSWFs and prove Theorem 2. In Section 5 we discuss upper bounds for the probability of rational choice and prove Theorem 3.

2 Fourier-Walsh Expansion of Functions on the Discrete Cube

Consider the discrete cube $\{0,1\}^n$ endowed with the uniform measure μ. Denote the set of all real-valued functions on the discrete cube by X. The inner product of functions $f, g \in X$ is defined as usual as

$$\langle f, g \rangle = \int fg d\mu = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(x)g(x).$$

This inner product induces a norm on X:

$$||f||_2 = \sqrt{\langle f, f \rangle} = \sqrt{\int f^2 d\mu}.$$

Consider the Rademacher functions $\{r_i\}_{i=1}^n$, defined as:

$$r_i(x_1, \ldots, x_n) = 2x_i - 1.$$

These functions constitute an orthonormal system in X. Moreover, this system can be completed to an orthonormal basis in X by defining

$$r_S = \prod_{i \in S} r_i$$

for all $S \subset \{1, \ldots, n\}$. Every function $f \in X$ can be represented by its Fourier expansion with respect to the system $\{r_S\}_{S \subset \{1,\ldots,n\}}$:

$$f = \sum_{S \subset \{1,\ldots,n\}} \langle f, r_S \rangle r_S.$$

This representation is called the Fourier-Walsh expansion of f. The coefficients in this expansion are denoted by

$$\hat{f}(S) = \langle f, r_S \rangle.$$

By the Parseval identity, for all $f \in X$,

$$\sum_{S \subset \{1,\ldots,n\}} \hat{f}(S)^2 = ||f||_2^2.$$

More generally, for all $f, g \in X$,

$$\langle f, g \rangle = \sum_{S \subset \{1,\ldots,n\}} \hat{f}(S)\hat{g}(S).$$

Following [10], we will be also interested in a biased version of the inner product, defined as follows:

Definition 4 *Let f, g be two real-valued functions on the discrete cube, and let $-1 \leq \delta \leq 1$. Define*

$$\langle\langle f, g \rangle\rangle_\delta = \sum_{S \neq \emptyset} \hat{f}(S)\hat{g}(S)\delta^{|S|}.$$

Note that this definition slightly differs from the definition of [10]. Finally, we note that for all $f \in X$,

$$\hat{f}(\emptyset) = \int (f r_\emptyset) d\mu = \int (f \cdot 1) d\mu = \int f d\mu.$$

3 The Probability of a Rational Choice for a Non-Uniform Distribution of the Profiles

Throughout the paper we assume that the number of alternatives is three and denote the alternatives by A, B, and C. Since (by assumption) the GSWF satisfies the IIA condition, the preference of the society between every pair of alternatives can be represented by a Boolean function on the discrete cube. Formally, given a profile, we consider the pair of alternatives (A, B) and construct a binary vector $(x_1, \ldots, x_n)$ such that $x_i = 1$ if the i-th voter prefers A over B, and $x_i = 0$ if the i-th voter prefers B over A. We set $f(x_1, \ldots, x_n) = 1$ if the entire society prefers A over B and $f(x_1, \ldots, x_n) = 0$ if the society prefers B over A. Note that the preference of the society between A and B is determined by $(x_1, \ldots, x_n)$, and hence f is well defined. Similarly, we define the Boolean functions g and h that represent the preferences between the pairs (B, C) and (C, A), respectively.

Every profile is uniquely represented by the binary vector $(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n)$, where (x_i, y_i, z_i) represent the preferences of the i-th voter between $(A, B), (B, C)$, and (C, A). We assume that the vectors (x_i, y_i, z_i) for different values of i are independent (i.e., the preferences of the individual voters are independent), and that these vectors do not assume the values $(0, 0, 0)$ and $(1, 1, 1)$ (since otherwise the preferences of the i-th voter do not constitute an order relation). In [10], the distribution over the six possible values of (x_i, y_i, z_i) was assumed to be uniform. In our analysis, we consider the following distribution:

$Pr((x_i, y_i, z_i) = (1, 1, 0)) = \alpha \qquad Pr((x_i, y_i, z_i) = (0, 1, 1)) = \beta \qquad Pr((x_i, y_i, z_i) = (1, 0, 1)) = \gamma$

$$Pr((x_i,y_i,z_i)=(0,0,1))=\alpha \qquad Pr((x_i,y_i,z_i)=(1,0,0))=\beta \qquad Pr((x_i,y_i,z_i)=(0,1,0))=\gamma,$$

where $\alpha+\beta+\gamma=1/2$. We call this distribution an *even product distribution.* The intuition behind the restrictions will be explained later.

Theorem 5 *Consider a GSWF on three alternatives satisfying the IIA condition where the choice functions between the pairs of alternatives $(A,B),(B,C)$, and (C,A) are f,g, and h, respectively. If the distribution of the individual preferences is an even product distribution (as described above) then the probability of irrational choice is given by the formula:*

$$W(f,g,h)=p_1p_2p_3+(1-p_1)(1-p_2)(1-p_3)+\langle\langle f,g\rangle\rangle_{4\alpha-1}+\langle\langle g,h\rangle\rangle_{4\beta-1}+\langle\langle h,f\rangle\rangle_{4\gamma-1}, \quad (1)$$

where p_1,p_2, and p_3 are the expectations of f,g, and h, respectively.

Remark Theorem 3.1 of [10] is a partial case of Theorem 5, obtained for $\alpha=\beta=\gamma=1/6$.

Proof For a profile $(x,y,z)=(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)$, the choice of the society is rational if and only if

$$f(x)g(y)h(z)+(1-f(x))(1-g(y))(1-h(z))=0.$$

Therefore, the probability of irrational choice is

$$W(f,g,h)=\sum_{(x,y,z)\in\{0,1\}^{3n}} Pr((x,y,z))(f(x)g(y)h(z)+(1-f(x))(1-g(y))(1-h(z))),$$

where $Pr((x,y,z))=\prod_i Pr((x_i,y_i,z_i))$, as defined above.

Consider the functions $F_1,F_2,F_3:\{0,1\}^{3n}\rightarrow\mathbb{R}$ defined by

$$F_1(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)=f(x)g(y)h(z),$$

$$F_2(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)=(1-f(x))(1-g(y))(1-h(z)),$$

$$F_3(x_1,\ldots,x_n,y_1,\ldots,y_n,z_1,\ldots,z_n)=Pr((x,y,z)).$$

We have

$$W(f,g,h)=2^{3n}\langle F_3,F_1+F_2\rangle,$$

and hence by the Parseval identity,

$$W(f,g,h)=2^{3n}\sum_{S\subset\{1,\ldots,3n\}}\hat{F}_3(S)(\hat{F}_1(S)+\hat{F}_2(S)). \quad (2)$$

Therefore, in order to compute the probability of rational choice it is sufficient to compute the Fourier-Walsh expansions of F_1,F_2, and F_3.

In order to compute the expansions, we use the fact that if a function is a multiplication of functions on disjoint sets of variables, then its Fourier-Walsh expansion also has the same structure. Hence, if we denote $S=(S_1,S_2,S_3)$, where S_1 represents $(x_1,\ldots,x_n)$, S_2 represents $(y_1,\ldots,y_n)$, and S_3 represents $(z_1,\ldots,z_n)$, then

$$\hat{F}_1(S)=\hat{f}(S_1)\hat{g}(S_2)\hat{h}(S_3).$$

Similarly, since the individual preferences are independent, the Fourier-Walsh expansion of F_3 is determined by the Fourier-Walsh expansion of the functions $F_4^i : \{0,1\}^3 \to \mathbb{R}$ defined by

$$F_4^i((x_i, y_i, z_i)) = Pr((x_i, y_i, z_i)).$$

This expansion (presented below) can be found by direct computation.

$$\hat{F_4^i}(\emptyset) = 1/8, \qquad \hat{F_4^i}(\{1\}) = 0 \qquad \hat{F_4^i}(\{2\}) = 0, \qquad \hat{F_4^i}(\{3\}) = 0, \qquad \hat{F_4^i}(\{1,2\}) = (4\alpha - 1)/8,$$
$$\hat{F_4^i}(\{2,3\}) = (4\beta - 1)/8, \qquad \hat{F_4^i}(\{1,3\}) = (4\gamma - 1)/8, \qquad \hat{F_4^i}(\{1,2,3\}) = 0.$$

Since the Fourier-Walsh coefficients of F_3 are multiplications of the corresponding coefficients of $\{F_4^i\}_{i=1}^n$, we have $\hat{F_3}(S) = 0$, unless $S = (S_1, S_2, S_3)$ has a special structure: Each $1 \leq i \leq n$ is contained in either none or two of the sets (S_1, S_2, S_3). For such special sets S, the coefficients are given by the formula

$$\hat{F_3}(S) = (1/8)^{t_1}((4\alpha - 1)/8)^{t_2}((4\beta - 1)/8)^{t_3}((4\gamma - 1)/8)^{t_4},$$

where

$$\begin{aligned} t_1 &= \text{ the number of triples } (x_i, y_i, z_i) \text{ equal to } (0,0,0), \\ t_2 &= \text{ the number of triples } (x_i, y_i, z_i) \text{ equal to } (1,1,0), \\ t_3 &= \text{ the number of triples } (x_i, y_i, z_i) \text{ equal to } (0,1,1), \\ t_4 &= \text{ the number of triples } (x_i, y_i, z_i) \text{ equal to } (1,0,1). \end{aligned}$$

Finally, we note that for all $S_1 \neq \emptyset$ we have $\hat{f}(S_1) = -(\widehat{1-f}(S_1))$ and the same for g and h, and hence if $S_1, S_2, S_3 \neq \emptyset$, then

$$\hat{F_1}(S) + \hat{F_2}(S) = 0.$$

Combining the results above, we get that the term

$$\hat{F_3}(S)(\hat{F_1}(S) + \hat{F_2}(S))$$

vanishes unless $S = (S_1, S_2, S_3)$ has the following special structure: At least one of S_1, S_2, S_3 is empty, and each i is contained in either none or two of S_1, S_2, S_3.

Assume that $S_3 = \emptyset$, and thus $S_1 = S_2$ (otherwise, $\hat{F_3}(S)(\hat{F_1}(S) + \hat{F_2}(S)) = 0$). Assume also that $S_1 \neq \emptyset$. We note that $\widehat{1-h}(\emptyset) = 1 - \hat{h}(\emptyset)$, and hence by the calculations above

$$\hat{F_3}(S)(\hat{F_1}(S) + \hat{F_2}(S)) = (1/8)^{n-|S_1|}((4\alpha - 1)/8)^{|S_1|}\hat{f}(S_1)\hat{g}(S_1) = (1/8)^n(4\alpha - 1)^{|S_1|}\hat{f}(S_1)\hat{g}(S_1).$$

If $S_1 = S_2 = S_3 = \emptyset$, then

$$\hat{F_3}(S)(\hat{F_1}(S) + \hat{F_2}(S)) = (1/8)^n(p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3)).$$

Therefore, summing over all the possible values of S we get

$$\sum_{S \subset \{1,\ldots,3n\}} \hat{F_3}(S)(\hat{F_1}(S) + \hat{F_2}(S)) = (1/8)^n(p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3) +$$
$$\sum_{S_1 \neq \emptyset} (4\alpha - 1)^{|S_1|}\hat{f}(S_1)\hat{g}(S_1) + \sum_{S_2 \neq \emptyset} (4\beta - 1)^{|S_2|}\hat{g}(S_2)\hat{h}(S_2) + \sum_{S_3 \neq \emptyset} (4\gamma - 1)^{|S_3|}\hat{f}(S_3)\hat{h}(S_3)) =$$
$$(1/8)^n(p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3) + \langle\langle f, g\rangle\rangle_{4\alpha-1} + \langle\langle g, h\rangle\rangle_{4\beta-1} + \langle\langle h, f\rangle\rangle_{4\gamma-1}),$$

and thus the assertion of the theorem follows from Equation 2. ■

Using Theorem 5, some of the results of [10] can be generalized to even product distributions of the preferences. We present here two of the results. The first result contains the assertion of Theorem 1.

Corollary 6 *Consider a GSWF on three alternatives satisfying the IIA condition. If the distribution of the profiles is an even product distribution such that the probability of each preference is positive, and the GSWF is neutral and symmetric, then the probability of an irrational choice is bounded away from zero (independently of the number of voters).*

Moreover, if the probabilities of the preferences satisfy $\alpha, \beta, \gamma \leq 1/4$, then the probability of irrational choice satisfies the inequality

$$W(f,g,h) \geq (\frac{1}{4} - d_m)(1 + (4\alpha-1)^3 + (4\beta-1)^3 + (4\gamma-1)^3 > 0, \tag{3}$$

where $d_m = 1/2\pi$ is the sum of squares of the first-level Fourier-Walsh coefficients of the majority function.

Remark Theorem 5.1 in [10] is a partial case of Corollary 6, obtained for $\alpha = \beta = \gamma = 1/6$.

Proof By the assumption, the GSWF is neutral, and hence, balanced. Therefore, by Theorem 5, the probability of irrational choice in our case is

$$W(f,g,h) = 1/4 + \langle\langle f,g \rangle\rangle_{4\alpha-1} + \langle\langle g,h \rangle\rangle_{4\beta-1} + \langle\langle h,f \rangle\rangle_{4\gamma-1}.$$

Since the GSWF is neutral and symmetric, we have $f = g = h$, and all the Fourier coefficients of f on the even non-zero levels vanish (see [10], Proof of Theorem 5.1). Thus,

$$W(f,g,h) = 1/4 + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\alpha-1)^{|S|} + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\beta-1)^{|S|} + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\gamma-1)^{|S|} =$$

$$= 1/4 + \sum_{k=0}^{\lceil n/2 \rceil - 1} ((4\alpha-1)^{2k+1} + (4\beta-1)^{2k+1} + (4\gamma-1)^{2k+1}) \sum_{|S|=2k+1} \hat{f}(S)^2 =$$

$$= 1/4 - \sum_{|S|=1} \hat{f}(S)^2 + \sum_{k=1}^{\lceil n/2 \rceil - 1} ((4\alpha-1)^{2k+1} + (4\beta-1)^{2k+1} + (4\gamma-1)^{2k+1}) \sum_{|S|=2k+1} \hat{f}(S)^2.$$

Since for every k the expression $\sum_{|S|=2k+1} \hat{f}(S)^2$ is non-negative, and since by the Parseval identity,

$$\sum_{k=1}^{\lceil n/2 \rceil - 1} \sum_{|S|=2k+1} \hat{f}(S)^2 = \sum_{S \neq \emptyset} \hat{f}(S)^2 - \sum_{|S|=1} \hat{f}(S)^2 = 1/4 - \sum_{|S|=1} \hat{f}(S)^2,$$

the first assertion of the corollary follows from the claim

$$\inf_{k \geq 1} ((4\alpha-1)^{2k+1} + (4\beta-1)^{2k+1} + (4\gamma-1)^{2k+1}) > -1.$$

In order to prove this claim we first note that since (by the assumption) $0 < \alpha, \beta, \gamma < 1/2$, we have

$$\lim_{k \to \infty} ((4\alpha - 1)^{2k+1} + (4\beta - 1)^{2k+1} + (4\gamma - 1)^{2k+1}) = 0,$$

and hence it is sufficient to prove that for each $k \geq 1$,

$$(4\alpha - 1)^{2k+1} + (4\beta - 1)^{2k+1} + (4\gamma - 1)^{2k+1} > -1. \tag{4}$$

This claim is trivial for $\alpha, \beta, \gamma \leq 1/4$, since in that case

$$(4\alpha - 1)^{2k+1} + (4\beta - 1)^{2k+1} + (4\gamma - 1)^{2k+1} > (4\alpha - 1) + (4\beta - 1) + (4\gamma - 1) = -1.$$

Hence, assume that $\gamma > 1/4$, and write $\gamma = 1/2 - \alpha - \beta$ and hence $4\gamma - 1 = 1 - 4\alpha - 4\beta$. Inequality 4 is equivalent to

$$(1 - 4\alpha)^{2k+1} + (1 - 4\beta)^{2k+1} < 1 + (1 - 4\alpha - 4\beta)^{2k+1},$$

which follows from the strict convexity of the function $F(t) = t^{2k+1}$ on $[0, 1]$. This completes the proof of the first assertion of the corollary.

In order to prove the second assertion, we note that if $\alpha, \beta, \gamma \leq 1/4$ then for all $k \geq 1$,

$$(4\alpha - 1)^{2k+1} + (4\beta - 1)^{2k+1} + (4\gamma - 1)^{2k+1} \geq (4\alpha - 1)^3 + (4\beta - 1)^3 + (4\gamma - 1)^3, \tag{5}$$

and hence

$$W(f, g, h) \geq (1/4 - \sum_{|S|=1} \hat{f}(S)^2)(1 + ((4\alpha - 1)^3 + (4\beta - 1)^3 + (4\gamma - 1)^3)).$$

Finally, since amongst the symmetric neutral functions, the expression $\sum_{|S|=1} \hat{f}(S)^2$ is maximized for the majority function (see proof of Theorem 5.1 in [10]), the second assertion of the corollary follows. ■

Remark It seems possible that Inequality 3 holds also if one of the probabilities of the preferences is greater than 1/4. In order to prove this, it is sufficient to prove that Inequality 4 holds without the restriction $\alpha, \beta, \gamma \leq 1/4$. While this claim seems correct, we weren't able to prove it.

The second result is a combination of Theorem 5 with the following proposition, which is an easy consequence of the "Majority is stablest" theorem [13]:

Proposition 7 *Let $0 \leq \rho \leq 1$ and let $\epsilon > 0$. There exists $n_0 = n_0(\rho, \epsilon)$ such that for all $n > n_0$, if $f : \{0, 1\}^n \to [0, 1]$ is symmetric and balanced then*

$$\langle\langle f, f \rangle\rangle_\rho = \sum_{S \neq \emptyset} \hat{f}(S)^2 \rho^{|S|} \leq \frac{1}{2\pi} \arcsin \rho + \epsilon.$$

Corollary 8 *Consider a GSWF on three alternatives, where the distribution of the profiles is an even product distribution satisfying* $\alpha, \beta, \gamma \leq 1/4$. *Then for all* $\epsilon > 0$ *there exists* $n_0 = n_0(\epsilon, \alpha, \beta, \gamma)$ *such that if the number of voters is* $n > n_0$ *and the GSWF is neutral, symmetric, and satisfies the IIA condition, then the probability of a rational choice is at most* $p + \epsilon$, *where* p *is the probability of a rational choice for the majority GSWF on* n *voters and three alternatives.*

Proof Similarly to the proof of Corollary 6, if $\alpha, \beta, \gamma \leq 1/4$ then

$$W(f,g,h) = 1/4 + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\alpha - 1)^{|S|} + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\beta - 1)^{|S|} + \sum_{|S| \text{ odd}} \hat{f}(S)^2 (4\gamma - 1)^{|S|} =$$

$$= 1/4 - \sum_{|S| \text{ odd}} \hat{f}(S)^2 |4\alpha - 1|^{|S|} - \sum_{|S| \text{ odd}} \hat{f}(S)^2 |4\beta - 1|^{|S|} - \sum_{|S| \text{ odd}} \hat{f}(S)^2 |4\gamma - 1|^{|S|} =$$

$$= 1/4 - \langle\langle f, f \rangle\rangle_{|4\alpha-1|} - \langle\langle f, f \rangle\rangle_{|4\beta-1|} - \langle\langle h, f \rangle\rangle_{|4\gamma-1|}.$$

Hence, by Proposition 7, for every $\epsilon > 0$ there exists $n_0 = n_0(\epsilon, \alpha, \beta, \gamma)$ such that for every GSWF on $n > n_0$ voters satisfying the assumptions of the corollary,

$$W(f,g,h) \geq 1/4 - \frac{1}{2\pi} \arcsin(|4\alpha - 1|) - \frac{1}{2\pi} \arcsin(|4\beta - 1|) - \frac{1}{2\pi} \arcsin(|4\gamma - 1|) - \epsilon.$$

Finally, since for the majority GSWF F_n on n voters we have, for all $0 \leq \rho \leq 1$,

$$\lim_{n \to \infty} \langle\langle F_n, F_n \rangle\rangle_\rho = \frac{1}{2\pi} \arcsin \rho$$

(see [13], Section 4), the assertion of the corollary follows. ■

Remark Corollary 8 was proved in [13] in the partial case of a uniform distribution of the preferences, as a corollary of the "Majority is Stablest" theorem.

Remark Conjecture 8.1 of [10] asserts that for every distribution of the profiles (and even for more than three alternatives), the probability of a rational choice for GSWFs that are neutral, symmetric, and satisfy the IIA condition, is maximized for the majority function. Hence, Corollary 8 proves *in the asymptotic sense* (i.e., for a sufficiently large n) a partial case of the conjecture.

We conclude this section by explaining the restriction on the distribution of the individual preferences. The proof of Theorem 5 crucially depends on the fact that $\hat{F}_4^i(\{j\})$ vanishes for $j = 1, 2, 3$. This condition holds if and only if the probabilities of the preferences satisfy the following three equations:

$$Pr(1,0,0) + Pr(1,1,0) + Pr(1,0,1) - Pr(0,1,0) - Pr(0,0,1) - Pr(0,1,1) = 0,$$

$$Pr(0,1,0) + Pr(1,1,0) + Pr(0,1,1) - Pr(1,0,0) - Pr(0,0,1) - Pr(1,0,1) = 0,$$

$$Pr(0,0,1) + Pr(1,0,1) + Pr(0,1,1) - Pr(1,0,0) - Pr(0,1,0) - Pr(1,1,0) = 0,$$

where $Pr(a, b, c)$ is a shorthand for $Pr((x_i, y_i, z_i) = (a, b, c))$. Summing the first two equations we get

$$2Pr(1, 1, 0) - 2Pr(0, 0, 1) = 0,$$

and similarly by summing the two other pairs of equations we get $Pr(1, 0, 1) = Pr(0, 1, 0)$ and $Pr(0, 1, 1) = Pr(1, 0, 0)$. Finally, since all the probabilities sum up to one, we get $Pr(1, 0, 0) + Pr(0, 1, 0) + Pr(0, 0, 1) = 1/2$, and this completes the restrictions described above. It seems challenging to generalize Theorem 5 to more general distributions on the preferences, but the expression $\sum_{S\subset\{1,\ldots,3n\}} \hat{F}_3(S)(\hat{F}_1(S) + \hat{F}_2(S))$ seems hard to compute in the general case.

4 Lower Bounds on the Probability of Rational Choice

In this section we establish lower bounds on the probability of rational choice for two classes of GSWFs: monotone balanced functions and general balanced functions.

4.1 Monotone Balanced GSWFs

Definition 9 *A function* $f : \{0, 1\}^n \to \mathbb{R}$ *is monotone if for all* $x = (x_1, \ldots, x_n)$ *and* $y = (y_1, \ldots, y_n)$*,*

$$(\forall i : x_i \leq y_i) \Rightarrow (f(x) \leq f(y)).$$

Theorem 2 is a partial case of the following, slightly more general, result:

Theorem 10 *Consider a GSWF on three alternatives satisfying the IIA condition where the choice functions between the pairs of alternatives* $(A, B), (B, C)$*, and* (C, A)*, denoted by* f, g*, and* h*, respectively, are monotone. If the profiles are uniformly distributed then the probability of irrational choice satisfies:*

$$W(f, g, h) \leq p_1p_2p_3 + (1 - p_1)(1 - p_2)(1 - p_3), \tag{6}$$

where p_1, p_2*, and* p_3 *are the expectations of* f, g*, and* h*, respectively.*

Remark The assertion of Theorem 10 is tight, as can be seen in the following example: Assume that f depends only on the first voter, g depends only on the second voter, and h depends only on the third voter. Then clearly, for all $-1 \leq \delta \leq 1$,

$$\langle\langle f, g\rangle\rangle_\delta = \langle\langle g, h\rangle\rangle_\delta = \langle\langle h, f\rangle\rangle_\delta = 0,$$

and hence

$$W(f, g, h) = p_1p_2p_3 + (1 - p_1)(1 - p_2)(1 - p_3),$$

where p_1, p_2, and p_3 are the expectations of f, g, and h, respectively.

By Theorem 5, the assertion of Theorem 10 follows immediately from the following proposition:

Proposition 11 *For any two monotone Boolean functions* f *and* g*, and for every* $-1 \leq \delta \leq 1$*,*

$$\frac{1}{\delta}\langle\langle f, g\rangle\rangle_\delta \geq 0. \tag{7}$$

The proof of Proposition 11 is composed of three stages:

1. First, we reduce the proposition to the case $0 \leq \delta \leq 1$, using the relation between the Fourier-Walsh expansion of a function and the Fourier-Walsh expansion of its dual function.
2. Then, we use Beckner's noise operator [2, 3] to reduce the problem to a correlation inequality for monotone non-Boolean functions.
3. Finally, we use the FKG inequality [6] to conclude the proof of the proposition.

4.1.1 Reduction to positive δ-s

Definition 12 *Let f be a Boolean function. The dual function of f, which we denote by f', is defined by $f'(x) = 1 - f(\overline{x})$, where if $x = (x_1, \ldots, x_n)$ then $\overline{x} = (1 - x_1, \ldots, 1 - x_n)$.*

It is easy to see that a Boolean function f is monotone if and only if its dual function f' is monotone.

Lemma 13 *Consider the Fourier-Walsh expansions of a Boolean function f and its dual function f'. For all $S \subset \{1, \ldots, n\}$ with $|S| \geq 1$,*

$$\hat{f}'(S) = (-1)^{|S|-1} \hat{f}(S).$$

Proof First, we compute the Fourier-Walsh expansion of the function f'' defined by $f''(x) = f(\overline{x})$. Recall that the characters $\{r_S\}$ satisfy

$$r_S(x) = (-1)^{|S \cap x|},$$

where x is considered as a subset of $\{1, \ldots, n\}$. Hence,

$$\hat{f}''(S) = \langle f'', r_S \rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f''(x)(-1)^{|S \cap x|} = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} f(\overline{x})(-1)^{|S \cap x|}. \tag{8}$$

We note that

$$(-1)^{|S \cap x|} = (-1)^{|S \cap \overline{x}|}(-1)^{|S|},$$

and hence substituting $y = \overline{x}$ into Equation 8 we get

$$\hat{f}''(S) = \frac{1}{2^n} \sum_{y \in \{0,1\}^n} f(y)(-1)^{|S \cap y|}(-1)^{|S|} = (-1)^{|S|} \langle f, r_S \rangle = (-1)^{|S|} \hat{f}(S).$$

Finally, by the linearity of the Fourier transform, for every $S \subseteq \{1, \ldots, n\}$ with $|S| \geq 1$,

$$\hat{f}'(S) = -\hat{f}''(S),$$

and hence

$$\hat{f}'(S) = (-1)^{|S|-1} \hat{f}(S),$$

as asserted. ■

Corollary 14 *It is sufficient to prove Proposition 11 for the case* $0 \leq \delta \leq 1$*.*

Proof Consider the expression

$$\frac{1}{\delta}\langle\langle f, g\rangle\rangle_\delta = \sum_{S\neq\emptyset} \hat{f}(S)\hat{g}(S)\delta^{|S|-1}$$

for $-1 \leq \delta \leq 0$ and monotone Boolean functions f and g. Denote by f' the dual function of f. Note that f' is also monotone. By Lemma 13,

$$\sum_{S\neq\emptyset} \hat{f}(S)\hat{g}(S)\delta^{|S|-1} = \sum_{S\neq\emptyset} \hat{f}'(S)\hat{g}(S)\delta^{|S|-1}(-1)^{|S|-1} = \sum_{S\neq\emptyset} \hat{f}'(S)\hat{g}(S)(-\delta)^{|S|-1},$$

and hence the assertion

$$\frac{1}{\delta}\langle\langle f, g\rangle\rangle_\delta \geq 0$$

for any two monotone Boolean functions f and g follows from the same assertion where $(-\delta)$ is substituted instead of δ. ■

4.1.2 Reduction to a Correlation Inequality

We use the following operator, defined in [2, 3]:

Definition 15 *Consider a function* f *on the discrete cube with a Fourier-Walsh expansion* $f(x) = \sum_S \hat{f}(S)r_S$*. For* $0 \leq \epsilon \leq 1$*, the noise operator* T_ϵ *applied to* f *is defined as*

$$T_\epsilon f = \sum_S \epsilon^{|S|}\hat{f}(S)r_S.$$

It is well-known that one can arrive from f to $T_\epsilon f$ by the following process: For any vector x,

$$T_\epsilon f(x) = Exp(f(x \oplus y)), \tag{9}$$

where $\oplus$ denotes bitwise addition modulo 2 and each bit of y is chosen independently according to the distribution $Pr(y_i = 0) = (1+\epsilon)/2, Pr(y_i = 1) = (1-\epsilon)/2$. That is, each bit of x is left unchanged with probability ϵ and replaced by a random value with probability $1-\epsilon$, and then f is evaluated on the result. Thus, $T_\epsilon f$ represents a noisy variant of f, and for this reason $T_\epsilon f$ is called "the noise operator".

Lemma 16 *Let* $0 \leq \epsilon \leq 1$ *and let* T_ϵ *be the corresponding noise operator. If* $f : \{0,1\}^n \to \mathbb{R}$ *is a monotone function then* $T_\epsilon f$ *is also monotone.*

Proof Without loss of generality, it is sufficient to prove that for each vector $(x_2, \ldots, x_n)$,

$$T_\epsilon f(0, x_2, \ldots, x_n) \leq T_\epsilon f(1, x_2, \ldots, x_n). \tag{10}$$

Using the equivalent definition of the noise operator presented above,

$$T_\epsilon f(x_1, x_2, \ldots, x_n) = Exp(f((x_1, x_2, \ldots, x_n) \oplus (y_1, \ldots, y_n))),$$

where each y_i is distributed according to the distribution $Pr(y_i = 0) = (1+\epsilon)/2, Pr(y_i = 1) = (1-\epsilon)/2$, independently of other y_i's. Hence,

$$T_\epsilon f(x_1, x_2, \ldots, x_n) = \sum_{z \in \{0,1\}^n} (\frac{1-\epsilon}{2})^{\sum_{k=1}^n |x_k - z_k|} (\frac{1+\epsilon}{2})^{n - \sum_{k=1}^n |x_k - z_k|} f(z_1, \ldots, z_n).$$

We partition the sum in the right hand side into pairs of values according to the last $n-1$ coordinates. We get

$$T_\epsilon f(x_1, x_2, \ldots, x_n) = \sum_{(z_2, \ldots, z_n) \in \{0,1\}^{n-1}} (\frac{1-\epsilon}{2})^{\sum_{k=2}^n |x_k - z_k|} (\frac{1+\epsilon}{2})^{n-1-\sum_{k=2}^n |x_k - z_k|} \times$$
$$\times ((\frac{1-\epsilon}{2})^{x_1} (\frac{1+\epsilon}{2})^{1-x_1} f(0, z_2, \ldots, z_n) + (\frac{1-\epsilon}{2})^{1-x_1} (\frac{1+\epsilon}{2})^{x_1} f(1, z_2, \ldots, z_n)).$$

Therefore, in order to prove Inequality 10, it is sufficient to prove that for every $(z_2, \ldots, z_n) \in \{0,1\}^{n-1}$ we have

$$(\frac{1-\epsilon}{2})^0 (\frac{1+\epsilon}{2})^1 f(0, z_2, \ldots, z_n) + (\frac{1-\epsilon}{2})^1 (\frac{1+\epsilon}{2})^0 f(1, z_2, \ldots, z_n) \leq$$
$$(\frac{1-\epsilon}{2})^1 (\frac{1+\epsilon}{2})^0 f(0, z_2, \ldots, z_n) + (\frac{1-\epsilon}{2})^0 (\frac{1+\epsilon}{2})^1 f(1, z_2, \ldots, z_n),$$

or equivalently

$$\epsilon f(0, z_2, \ldots, z_n) \leq \epsilon f(1, z_2, \ldots, z_n),$$

which indeed follows from the monotonicity of f. ■

Corollary 17 *In order to prove Proposition 11 it is sufficient to prove that for every two monotone non-negative functions f and g,*

$$\sum_{S \neq \emptyset} \hat{f}(S) \hat{g}(S) \geq 0.$$

Proof By Corollary 14, it is sufficient to prove that for every two monotone Boolean functions f and g and for every $0 \leq \delta \leq 1$,

$$\sum_{S \neq \emptyset} \hat{f}(S) \hat{g}(S) \delta^{|S|-1} \geq 0. \tag{11}$$

Consider the function $T_\delta f$. We have

$$\sum_{S \neq \emptyset} \hat{f}(S) \hat{g}(S) \delta^{|S|-1} = \frac{1}{\delta} \sum_{S \neq \emptyset} \widehat{T_\delta f}(S) \hat{g}(S),$$

and hence in order to prove Inequality 11 it is sufficient to prove the inequality

$$\sum_{S \neq \emptyset} \widehat{T_\delta f}(S) \hat{g}(S) \geq 0.$$

Since $T_\delta f$ is monotone by Lemma 16 and non-negative by Equation 9, the assertion follows. ■

4.1.3 Application of the FKG Inequality

We conclude the proof of Proposition 11 by deducing the claim of Corollary 17 from the FKG inequality [6]. First we recall the assertion of the FKG inequality in the case of functions on the discrete cube endowed with the uniform measure:

Theorem 18 ([6]) *Let $f, g : \{0,1\}^n \to \mathbb{R}$ be monotone non-negative functions. Then*

$$Exp(fg) \geq Exp(f)Exp(g).$$

Now we return to the proof of our result. Let $f, g : \{0,1\}^n \to \mathbb{R}$ be monotone non-negative functions. By the Parseval identity,

$$\sum_S \hat{f}(S)\hat{g}(S) = \langle f, g \rangle = Exp(fg),$$

and by the definition, $\hat{f}(\emptyset) = Exp(f)$ and $\hat{g}(\emptyset) = Exp(g)$. Hence, by the FKG inequality,

$$\sum_{S \neq \emptyset} \hat{f}(S)\hat{g}(S) = \sum_{S \in \{0,1\}^n} \hat{f}(S)\hat{g}(S) - \hat{f}(\emptyset)\hat{g}(\emptyset) = Exp(fg) - Exp(f)Exp(g) \geq 0.$$

This proves the claim of Corollary 17, and thus concludes the proof of Proposition 11.

4.1.4 Non-uniform distribution of the preferences

If the distribution of the preferences is not uniform, the lower bound on the probability of a rational choice is much weaker:

Proposition 19 *Consider a GSWF on three alternatives satisfying the IIA condition. If the choice functions f, g, and h are monotone and balanced and the distribution of the profiles is an even product distribution, then $W(f, g, h) \leq 1/2$.*

Proof By Theorem 5, in our case

$$W(f, g, h) = 1/4 + \langle\langle f, g \rangle\rangle_{4\alpha-1} + \langle\langle g, h \rangle\rangle_{4\beta-1} + \langle\langle h, f \rangle\rangle_{4\gamma-1}. \tag{12}$$

By Proposition 11, an expression of the form $\langle\langle f, g \rangle\rangle_{4\alpha-1}$ is non-negative if and only if $\alpha \geq 1/4$. Therefore, since in our distribution $\alpha + \beta + \gamma = 1/2$, at most one of the expressions of this form appearing in Equation 12 is non-negative. If all the expressions are non-positive, then $W(f, g, h) \leq 1/4$. Hence, we assume w.l.o.g. that $\langle\langle f, g \rangle\rangle_{4\alpha-1} \geq 0$. Define new functions $\tilde{f}, \tilde{g}$ by

$$\tilde{f} = \sum_{S \neq \emptyset} |\hat{f}(S)| r_S,$$

and similarly for $\tilde{g}$. Since f and g are balanced, using the Parseval identity we get

$$||\tilde{f}||_2^2 = \sum_{S \subset \{1,\ldots,n\}} \widehat{\tilde{f}}(S)^2 = \sum_{S \neq \emptyset} |\hat{f}(S)|^2 = ||f||_2^2 - \hat{f}(\emptyset)^2 = 1/2 - 1/4 = 1/4,$$

and similarly $||\tilde{g}||_2^2 = 1/4$. Thus, by the Cauchy-Schwarz inequality,

$$\langle\langle f, g\rangle\rangle_{4\alpha-1} = \sum_{S\neq\emptyset} \hat{f}(S)\hat{g}(S)(4\alpha-1)^{|S|} \leq \sum_{S\subset\{1,\dots,n\}} \widehat{\tilde{f}}(S)\widehat{\tilde{g}}(S)1^{|S|} = \langle\tilde{f}, \tilde{g}\rangle \leq ||\tilde{f}||_2||\tilde{g}||_2 = 1/4.$$

Finally, since $\langle\langle f, g\rangle\rangle_{4\beta-1}$ and $\langle\langle f, g\rangle\rangle_{4\gamma-1}$ are non-positive,

$$W(f, g, h) = 1/4 + \langle\langle f, g\rangle\rangle_{4\alpha-1} + \langle\langle g, h\rangle\rangle_{4\beta-1} + \langle\langle h, f\rangle\rangle_{4\gamma-1} \leq 1/4 + 1/4 + 0 + 0 = 1/2.$$

This completes the proof of the proposition. ■

Remark The assertion of Proposition 19 is tight, as can be seen in the following example: Assume that the distribution on the preferences is $Pr((x_i, y_i, z_i) = (1, 1, 0)) = 1/2$ and $Pr((x_i, y_i, z_i) = (0, 0, 1)) = 1/2$, while the probability of the other preferences is zero (i.e., $\alpha = 1/2$ and $\beta = \gamma = 0$). The choice functions f and g are a dictatorship of the first voter, and h is a dictatorship of the second voter. Then it is easy to see that $W(f, g, h) = 1/2$.

4.2 General Balanced GSWFs

In [10] it is stated (Proposition 5.2) that if the preferences are uniformly distributed, then the lower bound for the probability of rational choice for general balanced GSWFs is 2/3. However, the proof presented in [10] is incorrect, and it is not even sure that the lower bound itself is correct. In this subsection we prove a weaker lower bound, and discuss its tightness.

Theorem 20 *Consider a GSWF on three alternatives satisfying the IIA condition such that the choice functions between the pairs of alternatives are balanced. If the profiles are uniformly distributed then the probability of rational choice is at least* 5/8.

Proof Consider the Fourier-Walsh expansions of the choice functions f, g, and h. Let

$$\sum_{i=1}^{n} \hat{f}(\{i\})^2 = a, \qquad \sum_{i=1}^{n} \hat{g}(\{i\})^2 = b, \qquad \sum_{i=1}^{n} \hat{h}(\{i\})^2 = c.$$

Since f, g, and h are balanced, then by the Parseval identity

$$\sum_{|S|>1} \hat{f}(S)^2 = 1/4 - a, \qquad \sum_{|S|>1} \hat{g}(S)^2 = 1/4 - b, \qquad \sum_{|S|>1} \hat{h}(S)^2 = 1/4 - c.$$

Recall that by Theorem 5, in our case

$$W(f, g, h) = 1/4 + \langle\langle f, g\rangle\rangle_{-1/3} + \langle\langle g, h\rangle\rangle_{-1/3} + \langle\langle h, f\rangle\rangle_{-1/3}. \qquad (13)$$

We have

$$\langle\langle f, g\rangle\rangle_{-1/3} + \langle\langle g, h\rangle\rangle_{-1/3} + \langle\langle h, f\rangle\rangle_{-1/3} = \sum_{|S|>0} (\hat{f}(S)\hat{g}(S) + \hat{g}(S)\hat{h}(S) + \hat{h}(S)\hat{f}(S))(-1/3)^{|S|} =$$

$$= (-1/3)\sum_{|S|=1}(\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S))+\sum_{|S|>1}(\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S))(-1/3)^{|S|} \leq$$

$$\leq (-1/3)\sum_{|S|=1}(\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S))+(1/9)\sum_{|S|>1}|\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S)|.$$

In order to bound the first summand, we use the elementary inequality

$$-(xy+yz+xz) \leq (x^2+y^2+z^2)/2.$$

We get

$$(-1/3)\sum_{|S|=1}(\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S)) = (-1/3)\sum_{i=1}^{n}(\hat{f}(\{i\})\hat{g}(\{i\})+\hat{g}(\{i\})\hat{h}(\{i\})+\hat{h}(\{i\})\hat{f}(\{i\})) \leq$$

$$\leq (1/6)\sum_{i=1}^{n}(\hat{f}(\{i\})^2+\hat{g}(\{i\})^2+\hat{h}(\{i\})^2) = (a+b+c)/6.$$

In order to bound the second summand, we use the Cauchy-Schwarz inequality and the inequality between the arithmetic and the geometric means. Let

$$\tilde{f} = \sum_{|S|>1}|\hat{f}(S)|r_S, \qquad \tilde{g} = \sum_{|S|>1}|\hat{g}(S)|r_S.$$

Applying the Cauchy-Schwarz inequality and the Parseval identity we get

$$\sum_{|S|>1}|\hat{f}(S)\hat{g}(S)| = \langle\tilde{f},\tilde{g}\rangle \leq ||\tilde{f}||_2||\tilde{g}||_2 = \sqrt{(1/4-a)(1/4-b)} \leq 1/4-(a+b)/2,$$

where the last inequality follows from the Mean Values inequality. Applying the same inequalities to the pairs (g,h) and (h,f), we get

$$(1/9)\sum_{|S|>1}|\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S)| \leq$$

$$\leq (1/9)(1/4-(a+b)/2+1/4-(b+c)/2+1/4-(c+a)/2) = 1/12-(a+b+c)/9.$$

Combining the bounds obtained above, we get

$$(-1/3)\sum_{|S|=1}(\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S))+(1/9)\sum_{|S|>1}|\hat{f}(S)\hat{g}(S)+\hat{g}(S)\hat{h}(S)+\hat{h}(S)\hat{f}(S)| \leq$$

$$\leq (a+b+c)/6+1/12-(a+b+c)/9 = 1/12+(a+b+c)/18.$$

Substituting to Equation 13 we get

$$W(f,g,h) \leq 1/4+1/12+(a+b+c)/18 = 1/3+(a+b+c)/18.$$

Finally, since by the Parseval identity $0 \leq a,b,c \leq 1/4$, the maximum in the right hand side is obtained for $a=b=c=1/4$, and thus

$$W(f,g,h) \leq 1/3+(3/4)/18 = 3/8,$$

as asserted. ■

The tightness of the lower bound in Theorem 20 is not clear to us. The example presented in [10] yields the value $W(f, g, h) = 1/3$, where all the weight of the Fourier-Walsh coefficients of f, g, and h is concentrated on the second level. Another example yielding the same value of $W(f, g, h)$ is

$$f(x_1, \ldots, x_n) = x_i, \qquad g(x_1, \ldots, x_n) = x_i, \qquad h(x_1, \ldots, x_n) = 1 - x_i$$

for any $1 \leq i \leq n$. In this example, all the weight of f, g, and h is concentrated on the first level. It seems possible that the correct lower bound is $2/3$, as asserted in [10]. However, in order to prove this bound, one has to exploit the fact that the choice functions are *Boolean*, as can be seen in the following example:

Example Let f, g, h be defined by $\hat{f}(\emptyset) = \hat{g}(\emptyset) = \hat{h}(\emptyset) = 1/2$ and

$$\hat{f}(i) = \frac{2}{2\sqrt{6}}, \qquad \hat{f}(j) = -\frac{1}{2\sqrt{6}}, \qquad \hat{f}(k) = -\frac{1}{2\sqrt{6}},$$

$$\hat{g}(i) = -\frac{1}{2\sqrt{6}}, \qquad \hat{g}(j) = \frac{2}{2\sqrt{6}}, \qquad \hat{g}(k) = -\frac{1}{2\sqrt{6}},$$

$$\hat{h}(i) = -\frac{1}{2\sqrt{6}}, \qquad \hat{h}(j) = -\frac{1}{2\sqrt{6}}, \qquad \hat{h}(k) = \frac{2}{2\sqrt{6}},$$

for $1 \leq i < j < k \leq n$. The rest of the Fourier-Walsh coefficients of f, g, and h are zero. Since

$$\sum_{S \neq \emptyset} \hat{f}(S) = \sum_{S \neq \emptyset} \hat{g}(S) = \sum_{S \neq \emptyset} \hat{h}(S) = 1/4,$$

the functions f, g, and h "look like" balanced functions from the Fourier-theoretic point of view. Nevertheless, $W(f, g, h) = 3/8$, which agrees with the lower bound of Theorem 20. This shows that the Booleanity of the choice functions is essential if a better lower bound can be obtained.

5 Upper Bounds on the Probability of Rational Choice

In this section we discuss upper bounds on the probability of rational choice for general GSWFs satisfying the IIA condition. Our starting point is the celebrated Arrow's theorem:

Theorem 21 ([1]) *Consider a GSWF on three alternatives, satisfying the following conditions:*

1. *The preference of the entire society between every pair of alternatives depends only on the individual preferences between these two alternatives (the IIA condition).*

2. *For every pair (A, B) of alternatives, if all the individuals prefer A over B then the entire society also prefers A over B (Pareto efficiency).*

3. *The GSWF is not a dictatorship.*

If the probability of every rational profile is positive then the probability of irrational choice is positive.

There are several proofs of Arrow's theorem in the literature, including simple direct combinatorial proofs (see, for example, [7]). In [10], Kalai presented a Fourier-theoretic proof of the theorem in the balanced case (and assuming that the distribution of profiles is uniform), along with a stability result. We start with a discussion of Kalai's proof, and then turn to the stability result.

5.1 Fourier-Theoretic Proof of Partial Cases of Arrow's Theorem

Kalai's proof uses the fact that if the choice functions are balanced and the preferences are uniformly distributed, then the probability of irrational choice is given by the equation

$$W(f,g,h) = 1/4 + \langle\langle f,g\rangle\rangle_{-1/3} + \langle\langle g,h\rangle\rangle_{-1/3} + \langle\langle h,f\rangle\rangle_{-1/3}.$$

Define

$$\tilde{f} = \sum_{S\neq\emptyset} \hat{f}(S) r_S, \qquad \overline{g} = \sum_{S\neq\emptyset} \hat{g}(S)(-1/3)^{|S|} r_S.$$

Note that since f,g are balanced, by the Parseval identity $||\tilde{f}||_2 = 1/2$ and $||\overline{g}||_2 \leq 1/6$. Therefore, by the Cauchy-Schwarz inequality,

$$|\langle\langle f,g\rangle\rangle_{-1/3}| = |\langle \tilde{f}, \overline{g}\rangle| \leq ||\tilde{f}||_2 ||\overline{g}||_2 \leq 1/12,$$

and it can be shown that equality can hold only if all the Fourier-Walsh coefficients of f and of g are on the first level. Then, it can be further shown that $W(f,g,h)=0$ can hold only if f,g, and h are dictatorships of the same voter, and this completes the proof of the theorem.

It was suggested in [10] to use the same reasoning in the non-balanced case. Such generalization is possible if p_1, p_2, and p_3, the expectations of f, g, and h, satisfy some condition described in [10]. However, this condition is not satisfied in many cases, e.g., for $p_1 = p_2 = 1/5$ and $p_3 = 1$, as noted in [10]. Kalai [11] suggested to improve the upper bound $||\overline{g}||_2 \leq 1/6$ (or, more generally, $||\overline{g}||_2 \leq \sqrt{p_2(1-p_2)}/3$) used in the proof by using the hypercontractive inequality for Beckner's noise operator [2, 3].

We show by an example that this proof strategy, even using the hypercontractive inequality, cannot lead to a complete proof of Arrow's theorem. The example shows that if the biased inner product $\langle\langle f,g\rangle\rangle_{-1/3}$ is replaced by

$$\langle\langle f,g\rangle\rangle'_{-1/3} = -\sum_{S\neq\emptyset} |\hat{f}(S)\hat{g}(S)(-1/3)^{|S|}|,$$

then there exist functions f,g,h such that

$$W'(f,g,h) = p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3) + \langle\langle f,g\rangle\rangle'_{-1/3} + \langle\langle g,h\rangle\rangle'_{-1/3} + \langle\langle h,f\rangle\rangle'_{-1/3} < 0,$$

and hence a proof of Arrow's theorem using Equation 13 cannot ignore the *sign* of the Fourier-Walsh coefficients of the choice functions.

Example Assume that n is odd, $f(x_1,x_2,\ldots,x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is the AND function, $g = f'$ is its dual function, and h is the majority function. The Fourier-Walsh coefficients of

f satisfy $|\hat{f}(S)| = 2^{-n}$ for all $S \subset \{1, \ldots, n\}$. The first-level Fourier-Walsh coefficients of the majority function satisfy

$$\hat{h}(\{i\}) = \binom{n-1}{(n-1)/2} 2^{-n} \approx \sqrt{\frac{1}{2\pi n}}$$

for all $1 \leq i \leq n$. Hence,

$$\langle\langle h, f\rangle\rangle'_{-1/3} \leq (-1/3) \sum_{i=1}^{n} |\hat{h}(\{i\})\hat{f}(\{i\})| \approx (-1/3) n 2^{-n} \sqrt{\frac{1}{2\pi n}} = -\frac{1}{3\sqrt{2\pi}} \sqrt{n} 2^{-n}.$$

Therefore,

$$W'(f, g, h) \leq p_1 p_2 p_3 + (1-p_1)(1-p_2)(1-p_3) + \langle\langle h, f\rangle\rangle'_{-1/3} \leq 2^{-n}(1-2^{-n}) - \frac{1}{3\sqrt{2}} \sqrt{n} 2^{-n} < 0,$$

for n large enough.

A possible step towards a Fourier-theoretic proof of Arrow's theorem in the general case is the following lower bound on the biased inner product $\langle\langle f, g\rangle\rangle_\delta$:

Proposition 22 *Let $f, g : \{0,1\}^n \rightarrow \mathbb{R}$ be non-negative functions with $Exp(f) = p_1$ and $Exp(g) = p_2$, and let $-1 \leq \delta \leq 1$. Then*

$$\langle\langle f, g\rangle\rangle_\delta \geq -p_1 p_2,$$

and equality holds only if either $f \equiv 0$ or $g \equiv 0$.

Proof We prove the proposition in the case $\delta < 0$, the case $\delta \geq 0$ is similar. Let $f''(x_1, \ldots, x_n) = f(1-x_1, \ldots, 1-x_n)$. Clearly, $\hat{f''}(\emptyset) = Exp(f'') = p_1$. By the proof of Lemma 13, for all $S \neq \emptyset$,

$$\hat{f''}(S) = (-1)^{|S|} \hat{f}(S).$$

Hence, by the definition of Beckner's noise operator [2, 3],

$$\widehat{T_{-\delta} f''}(S) = (-\delta)^{|S|} (-1)^{|S|} \hat{f}(S) = \delta^{|S|} \hat{f}(S).$$

Therefore, by the Parseval identity,

$$\langle\langle f, g\rangle\rangle_\delta + p_1 p_2 = \sum_{S \neq \emptyset} \hat{f}(S)\hat{g}(S)\delta^{|S|} + p_1 p_2 = \sum_{S \neq \emptyset} \widehat{T_{-\delta} f''}(S)\hat{g}(S) + \widehat{T_{-\delta} f''}(\emptyset)\hat{g}(\emptyset) = \langle T_{-\delta} f'', g\rangle.$$

Finally, by the assumption g is non-negative, and by Equation 9, the function $T_{-\delta} f''$ is strictly positive, unless $f \equiv 0$. Hence,

$$\langle T_{-\delta} f'', g\rangle = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} T_{-\delta} f''(x) g(x) > 0,$$

unless either $f \equiv 0$ or $g \equiv 0$, and in that cases $\langle T_{-\delta} f'', g\rangle = 0$. This completes the proof of the proposition. ■

Corollary 23 *The assertion of Arrow's theorem holds if the distribution of the profiles is an even product distribution and* $p_1 + p_2 + p_3 \leq 1$*, where* p_1, p_2*, and* p_3 *are the mean values of the choice functions* f, g*, and* h*.*

Proof By Proposition 22,

$$\langle\langle f, g\rangle\rangle_\delta + \langle\langle g, h\rangle\rangle_\delta + \langle\langle h, f\rangle\rangle_\delta > -(p_1p_2 + p_2p_3 + p_3p_1).$$

(Equality cannot hold since by the assumption of Arrow's theorem, f, g, and h are non-constant). Hence, by Equation 1,

$$W(f, g, h) > p_1p_2p_3 + (1 - p_1)(1 - p_2)(1 - p_3) - (p_1p_2 + p_2p_3 + p_3p_1) = 1 - p_1 - p_2 - p_3 \geq 0,$$

and thus the assertion of Arrow's theorem holds. ■

Another corollary of Proposition 22 uses dual functions:

Corollary 24 *Let* $f, g : \{0, 1\}^n \to \{0, 1\}$ *such that* $Exp(f) = p_1$ *and* $Exp(g) = p_2$*, and let* $-1 \leq \delta \leq 1$*. Then*

$$\langle\langle f, g\rangle\rangle_\delta \geq -(1 - p_1)(1 - p_2),$$

and equality holds only if either $f \equiv 1$ *or* $g \equiv 1$*.*

Proof Denote the dual functions of f and g by f' and g', respectively. By Lemma 13, for all $S \neq \emptyset$,

$$\hat{f}'(S)\hat{g}'(S) = (-1)^{|S|-1}\hat{f}(S)(-1)^{|S|-1}\hat{g}(S) = \hat{f}(S)\hat{g}(S),$$

and hence

$$\langle\langle f', g'\rangle\rangle_\delta = \langle\langle f, g\rangle\rangle_\delta.$$

The functions f', g' are non-negative and satisfy $Exp(f') = 1 - p_1$ and $Exp(g') = 1 - p_2$. Thus, by Proposition 22,

$$\langle\langle f', g'\rangle\rangle_\delta \geq -(1 - p_1)(1 - p_2),$$

and equality holds only if $f' \equiv 0$ or $g' \equiv 0$, or equivalently, only if $f \equiv 1$ or $g \equiv 1$. ■

Proposition 22 and Corollary 24 yield an immediate proof of Arrow's theorem in the case where there exists $1 \leq i \leq 3$ such that $p_i = 0$ or $p_i = 1$. Indeed, two of the biased inner products of the form $\langle\langle f, g\rangle\rangle_\delta$ appearing in Equation 1 vanish, and the third biased inner product is bounded using either Proposition 22 or Corollary 24. This settles the example given in [10]. However, we note that this case is anyway ruled out by the assumption (made in Arrow's theorem) that the choice functions are non-constant.

It seems possible that Proposition 22 can be extended to yield a full proof of Arrow's theorem, but we couldn't find such extension.

5.2 Discussion on a Stability Version of Arrow's Theorem

Throughout this subsection we assume that the preferences are uniformly distributed.

In [10], Kalai proved a stability version of Arrow's theorem:

Theorem 25 ([10]) *For every $\epsilon > 0$ and for every balanced GSWF on three alternatives, if the probability that the social choice is irrational is smaller than ϵ then there is a dictator such that the probability that the output of the GSWF differs from the dictator's choice is smaller than $K \cdot \epsilon$.*

Following Theorem 25, it is natural to ask:

Question 1 *Amongst the GSWFs on three alternatives satisfying the assumptions of Arrow's theorem, what is the "most rational" one (i.e., the one having the highest probability of a rational outcome)?*

Remark The idea behind the question is similar to the idea behind the Hilton-Milner theorem [9] concerning intersecting families. A family of subsets of a given finite set is called *intersecting* if the intersection of any two elements of the family is non-empty. The Erdös-Ko-Rado theorem [5] asserts that an intersecting family of k-element subsets of an n-element set has at most $\binom{n-1}{k-1}$ elements, and that the only maximal families are of the form $\{S \subset \{1, \ldots, n\} : |S| = k, i \in S\}$, for $1 \leq i \leq n$. The Hilton-Milner theorem [9] answers the question: What is the *second largest* intersecting family?

Similarly, in our situation, Arrow's theorem asserts that under some conditions, the only "most rational" GSWFs are the dictatorship functions. Question 1 asks, what is the most rational GSWF except for the dictatorship functions.

One class of natural alternatives for being the most rational GSWF is functions close to a dictatorship. Since the probability that the output of the GSWF differs from a dictatorship is at least 2^{-n}, by Theorem 25 for every balanced function of this class the probability of irrational choice is at least $K^{-1} \cdot 2^{-n}$, where K is a universal constant.

Another class of natural alternatives is almost constant functions. It can be shown that if all the three choice functions are almost constant (e.g., $f(x_1, \ldots, x_n) = 1$ unless $(x_1, \ldots, x_n) = (0, 0, \ldots, 0)$) then the probability of irrational choice is also $\Theta(2^{-n})$.

However, it appears that there exists a GSWF with a much lower probability of irrational outcome:

Example Assume that n is odd, $f(x_1, x_2, \ldots, x_n) = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ is the AND function, $g = f'$ is its dual function, and h is the majority function. Let

$$p_1 = Exp(f) = 2^{-n}, \qquad p_2 = Exp(g) = 1 - 2^{-n}, \qquad p_3 = Exp(h) = 1/2.$$

By the proof of Proposition 22,

$$\langle\langle f, g\rangle\rangle_{-1/3} = \langle T_{1/3} f'', g\rangle - p_1 p_2.$$

By Equation 9,

$$\langle T_{1/3} f'', g\rangle = 2^{-n} \sum_{x \in \{0,1\}^n} (1/3)^{\sum_{i=1}^n x_i} (2/3)^{n - \sum_{i=1}^n x_i} g(x) = 2^{-n}(1 - (2/3)^n),$$

and thus

$$\langle\langle f,g\rangle\rangle_{-1/3} = 2^{-n}(1-(2/3)^n) - 2^{-n}(1-2^{-n}) = -(1/3)^n + (1/4)^n.$$

Similarly,

$$\langle T_{1/3}f'', h\rangle = 2^{-n}\sum_{x\in\{0,1\}^n}(1/3)^{\sum_{i=1}^n x_i}(2/3)^{n-\sum_{i=1}^n x_i}h(x) =$$

$$= 2^{-n}\sum_{\{x:\sum_{i=1}^n x_i>n/2\}}(1/3)^{\sum_{i=1}^n x_i}(2/3)^{n-\sum_{i=1}^n x_i} \leq 2^{-n}\sum_{\{x:\sum_{i=1}^n x_i>n/2\}}(1/3)^{n/2}(2/3)^{n/2} =$$

$$= (1/2)(2/9)^{n/2} \approx (1/2)0.471^n.$$

Hence,

$$\langle\langle f,h\rangle\rangle_{-1/3} = (1/2)0.471^n - (1/2)2^{-n}.$$

Finally, since the dual function of f is g and since h is self-dual,

$$\langle\langle g,h\rangle\rangle_{-1/3} = \langle\langle f,h\rangle\rangle_{-1/3}.$$

Therefore,

$$W(f,g,h) = p_1p_2p_3 + (1-p_1)(1-p_2)(1-p_3) + \langle\langle f,g\rangle\rangle_{-1/3} + \langle\langle g,h\rangle\rangle_{-1/3} + \langle\langle h,f\rangle\rangle_{-1/3} \leq$$

$$\leq 2^{-n}(1-2^{-n}) + 0.471^n - 2^{-n} - (1/3)^n + (1/4)^n \leq 0.471^n.$$

We conjecture that the GSWF in the example is the most rational GSWF under the conditions of Arrow's theorem, but we weren't able to prove this conjecture. It is interesting to note that in the example, f, g, and h are taken from two different classes of Boolean functions: while f and g are almost constant functions, h is a majority function. If all the functions f, g, h are either almost constant or majorities, the resulting GSWF is "less rational".

The example can be generalized to a series of examples that can be used to prove Theorem 3.

Example For $0 < q < 1/2$ and for odd n, assume that

$$f(x) = \begin{cases} 1, & \sum_{i=1}^n x_i \geq (1-q)n \\ 0, & \sum_{i=1}^n x_i < (1-q)n, \end{cases}$$

g is the dual function of f, and h is the majority function. Using the well-known (see, for example, [12], Lemma 9.2) inequality:

$$\frac{2^{nH(q)}}{n+1} \leq \binom{n}{qn} \leq 2^{nH(q)}$$

(where $H(q) = -q\log_2 q - (1-q)\log_2(1-q)$ is the value of the entropy function at q) and considerations similar to those of the previous example (but more tedious), one obtains

$$W(f,g,h) \leq 2^{n(H(q)-1)}(1-2^{n(H(q)-1)}) + 2^{n(q-1.08)} - 2^{n(H(q)-1)} < 2^{n(q-1.08)}.$$

Since for all $q < 1/2$,

$$q - 1.08 < H(q) - 1,$$

for $n = n(q,K)$ big enough we have

$$W(f,g,h) < 2^{n(q-1.08)} < \frac{2^{n(H(q)-1)}}{(n+1)K}.$$

Therefore, substituting $\epsilon = H(q) - 1$, the assertion of Theorem 3 follows.

We conclude the paper with an open question:

Question 2 *Does there exist a universal constant K such that for every GSWF on three alternatives satisfying the assumptions of Arrow's theorem, if the distance of the choice functions f, g, h from constant functions is at least ϵ, then the probability of irrational choice is at least $K\epsilon^2$?*

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