Throwing a Party: Contracting with Type dependent Externalities*

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Abstract

We model situations in which a principal provides incentives to a group of agents to participate in a project (such as a social event or a commercial activity). Agents' benefits from participation depend on the identity of other participating agents. We assume bilateral externalities and characterize the optimal incentive mechanism. Using a graph-theoretic approach we show that the optimal mechanism provides a ranking of incentives for the agents, which can be described as arising from a virtual popularity tournament among the agents (similar to ones carried out by sport associations). Rather then simply ranking agents according to their measure of popularity, the optimal mechanism makes use of more refined two-way comparison between the agents. An implication of our analysis is that higher levels of asymmetry of externalities between the agents enable a reduction of the principal's payment. In addition, contrary to intuition, an increase in the aggregate externalities, does not necessarily decrease principal's payment, nor does it change agents rewards.

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*PRELIMINARY VERSION

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1 Introduction

The success of economic ventures often depends on the participation of a group of agents. One example is the malls industry. An owner of a mall needs to convince store owners to "participate" and open stores in the mall. A second example is an acquisition game. Consider a firm that makes acquisition offers to several owners of target firms. A successful venture would be a gathering of sufficient market power (through acquisitions) by the acquirer firm to maximize her profits, and participation would be an agreement of the target firms to sell. A third example is a standardization agency who succeeds in introducing a new technology standard if it manages to convince a group of firms to adopt the new standard. Throwing a party or organizing a conference are yet other examples, the success of which depends on the participation of the invited guests.

Multilateral contracting scenarios generate externalities which are rarely symmetric. In a mall, a small store subtantialy gains from the presence of an anchor store (such as a national brand name), while the opposite externality, induced by the small store has hardly any effect on the anchor store. The recruitment of a senior star to an academic department can easily attract a young assistant professor to apply to that department but not the other way around. The adoption of a new standard proposed by a standardization agency induces substantial positive externalities among the adopting firms but the level of benefits for a given firm crucially depends on the identity of the other firms who adopt the standard. In such environments, when an agent decides whether or not to participate she takes into account not only how many other agents are expected to participate but, more importantly, *who* is expected to participate. In this paper we focus on this type of type dependent externalities and investigate the consequences of asymmetry between agents in a multilateral contracting environment.

We explore ventures initiated by a certain party (henceforth a principal), in which we refer to as *multi-agent initiatives.* The success of these initiatives depends on the participation of other agents, and thus the principal provides incentive contracts to induce them to participate (such as discounts, gifts or any other benefits) and has to design these contracts optimally in view of the prevailing type dependent externalities between the agents. Any set of participating agents generate some revenue for the principal, and the principal will attempt to maximize his revenue net of the cost of the optimal incentive scheme.

Multi-agent initiatives consist two stages: the *selection stage*, in which the principal selects the target audience for the venture, and the *participation stage*, in which the principal introduces a set of contracts in order to induce the participation of his selected group. To work out the overall solution we will work backwards by first characterizing the optimal mechanism inducing the participation of a given group and this will enable to solve the selection part of the problem. Our analysis is comprehensive in two respects. First, we allow for all type of externalities starting with purely positive externalities, continuing with the case of negative externalities and concluding with the case of mixed externalities, where both positive and negative externalities can coexist in the same problem. Second, we will derive the optimal mechanism for both partial implementation, where the principal sustains agents' participation in some equilibrium, as well as full implementation, where participation is sustained via a unique equilibrium. If the principal cannot coordinate the agents to play her desired equilibrium she will have to pay a premium in terms of higher payments to guarantee that participation is sustained as the unique equilibrium of the underlying game. We will show that this premium varies with the structure of externalities within the group of agents, and depends on level of asymmetry between the agents in the group.

The type dependent externalities among agents are described in our model by a matrix whose entry $w_i(j)$ represents the extent to which agent i is attracted to the initiative when agent j participates. An optimal mechanism is a vector of rewards (offered by the principal to the agents) that sustains full participation at minimal total cost (or maximal total extraction) to the principal. In characterizing the optimal mechanisms we will focus on three main questions: 1. What is the hierarchy of incentives across agents as a function of the externalities; i.e., who should be getting higher incentives and who should be rewarded less? 2. How does the structure of externalities affect the principal's cost of sustaining the group participation? 3. How does a slight change in the externality that an agent induces on the others affect his reward and the principal's benefits?

Under positive externalities the optimal contracts are determined by a virtual popularity tournament among the agents. In this tournament *agent i beats agent j* if agent j is attracted i more than agent i is attracted to j. These relations between the agents give rise to a network described by a graph. We use basic graph theory arguments to characterize the optimal mechanism and show that agents who induce relatively larger externalities (agents who gets more "winnings") receive higher incentives.

The idea that agents who induce relatively stronger externalities receive higher incentives is supported by an empirical paper by Gould et al (2005). This paper demonstrates how externalities between stores in malls affect contracts offered by the malls owners. As in our model, stores are heterogeneous in the externalities they induce on each other. Anchor stores generate large positive externalities by attracting most of the customer traffic to the mall, and therefore increase the sales of non anchor stores. The most noticeable characteristic of mall contracts is that most anchor stores either do not pay any rent or pay only a trivial amount. On average, anchor stores occupy over 58% of the total leasable space in the mall and yet pay only 10% of the total rent collected by the mall's owner.

A key characteristic of the structure of externalities in a certain group of agents is the level of asymmetry between the agents, which we show to decrease the principal's cost. Put differently, the principal gains whenever the attraction between any two agents is distributed more asymmetrically (less mutually). Such greater asymmetry allows the principal more leverage in exploiting the externalities to lower costs. This observation has an important implication on the principal's choice of group for the initiative in the selection stage.

However, we show that an increase in the (positive) externalities among agents does not necessarily entail that the principal will be strictly better off, nor does it imply a change in agents' payoff. Moreover, the structure of the optimal mechanism has some implications for the way in which agents choose to affect the externalities they induce on others. Slight change in the externality that an agent induces on the others can result in a substantial change in the payment that this agent receives from the principal.

When discussing multi-agent participation scenarios one possible and intuitive solution might be to reward agents according to their measure of popularity such that the most popular agents would be rewarded the most. This follows the argument that once a popular agent agrees to participate it is easier to convince the others to join. While the term "popularity" can be defined in many ways, they all come down to the quality of being widely accepted by others . In our context agent i's popularity will be the sum of externalities it induces on the other agents in the group. However, we show that agents' rewards in the optimal mechanism are determined by something more refined than this standard definition of popularity. Agent i's reward depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different tournament than the one imposed by a standard definition of popularity.

This work is part of an extensive literature on multi-agent contracting in which externalities arise between the agents. The structure of our game, in which the principal offers a set of contracts and the agents can either accept or reject the offer, is akin to various applications introduced in the literature. These include vertical contracting models (Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992) in which the principal supplies an intermediate good, which is a fixed input (a license to use the principal's patent) to N identical downstream firms (agents), who then produce substitute consumer goods; exclusive dealing models (Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston 2000) in which the principal is an incumbent monopolist who offers exclusive dealing contracts to N identical buyers (agents) in order to deter entrance of a rival; acquisition for monopoly models (Lewis 1983; Kamien and Zang 1990; Krishna 1993) in which the principal makes acquisition offers to N capacity owners (agents), and these capacities are used to produce homogeneous consumer goods and network externalities models (Katz and Shapiro 1986b).

Our main departure from the above-mentioned literature lies in the fact that we focus on the case of heterogeneous agents with type dependent externalities. The papers mentioned above, and indeed most of the literature, assume that externalities depend on the volume of aggregate trade, and not on the identity of the agents. Our emphasis on heterogeneous agents and type dependent externalities allows us to capture a more realistic ingredient of the multilateral contracts, which are affected by the more complex relationships between the agents. Identity-type externalities were used in Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stachetti (1996), which consider the sale of a single indivisible object by the principal to multiple heterogeneous agents using auctions, when the utilities of the agents depend on which agent ultimately receives the good.

Our general approach is closely related to the seminal papers by Segal (1999, 2003) on contracting with externalities. These papers present a generalized model for the applications mentioned above as well as others. Our approach is also related to the incentive schemes investigated by Winter (2004) in the context of organizations. While we provide a solution for partial implementation, we follow Segal (2003) and Winter (2004) in that we concentrate on situations in which the principal cannot coordinate agents on his preferred equilibrium; that is we are mainly looking for contracts which sustain full implementation**.** Indeed, recent experimental papers (see, for example, Brandt and Cooper 2005) indicate that in an environment of positive externalities players typically are trapped in the bad equilibrium of no-participation. Finally, we point out that since our optimal mechanism is derived by means of a virtual tournament our results surprsingly connect to the literture on two quite distinct topics: 1. Creating a method for ranking sport teams based on tournament results, which has been discussed in the operation research literature and 2. Creating a rule for ranking candidates based on the outcome of binary election. It turns out that Condorcet's (1785) solution to the second problem as well as the method proposed by the OR literature to the first problem are closely related to our solution for the optimal mechanism. We will expand on this later.

The rest of the paper is organized as follows. In Section 2 we provide a simple two-agents example to illustrate some of the key results in the paper. We introduce the general model in section 3 and section 4 provides the solution for participation problem with positive externalities between the agents. In section 4 we examine the influence of some characteristics of the externalities structure on the principal's cost of incentivizing agents. In section 5 we consider the solution of participation problems with negative externalities and show that agents must be fully compensated to sustain a full participation equilibrium. Section 6 provides a solution for the most general case in which positive and negative externalities coexist. In section 7 we demonstrate how this model can be used to solve selection problems. We conclude in section 8. Proofs are presented in the appendix.

2 A Simple Two-Agent Example

The key ideas behind the paper's results can be illustrated by using a simple two-agent example. Suppose a principal would like to convince agents 1 and 2 to take part in his initiative by offering agent i a contract which pays v_i to agent i if he participates. Let's assume the agents have identical outside options in case they decline the principal's offer of $c > 0$. Furthermore, the decision to participate induce an externality on the other agent. If agent 1 is participating, agent 2's benefit (loss) is $w_2(1)$. Equivalently, if agent 2 is participating agent 1's benefit (loss) is $w_1(2)$. The agents will choose to participate if the payoff from the principal and the benefit/loss from other participating agents, taken together, is greater than the outside option.

Suppose first that the externalities $w_1(2)$ and $w_2(1)$ are stricly positive. A simple mechanism that induces the participation of both agents as a Nash Equilibrium is $(v_1, v_2) = (c - w_1(2), c - w_2(1))$, in which agent 1 is offered $c - w_1(2)$ and agent 2 is offered $c - w_2(1)$. However, this mechanism is not satisfactory as it includes an additional equilibrium in which both agents are not participating. We refer to such a mechanism as a partial implementation mechanism. In order to sustain the participation of both agents in a unique equilibrium, it is necessary to provide at least one agent, say agent 1, his entire outside option c. Now agent 1 will participate, even if agent 2 will decline. Given agent 1's participation, it is sufficient to offer agent 2 only $c - w_2(1)$ to induce his participation. Hence the mechanism $(c, c - w_2(1))$, while the total payment under this mechanism is higher here than in the one of partial implementation, it induces participation in a unique equilibrium. We refer to such a mechansim as a full implementation mechanism, and consider full implementation for the rest of the example.

Let's assume further that externalities are symmetric, hence $w_1(2) = w_2(1) > 0$. In this case, the decision of which agent will receive higher payoff is arbitrary, as the cost of both mechanisms $(c, c - w_2(1))$ and $(c - w_1(2), c)$ is identical. Suppose now that externalities are asymmetric, say, $w'_1(2) = w_1(2) + \varepsilon$ and $w'_2(1) = w_2(1) - \varepsilon$, when $\varepsilon > 0$, so that $w'_1(2) > w'_2(1)$. Note that the sum of externalities remained unchanged. In this case, clearly, the principal would prefer to offer agent 2 a higher payoff as the sum of incentives in mechanism $(c - w'_1(2), c)$ is lower than the alternative full implementation mechanism $(c, c - w'_2(1))$. To get a cheaper full implementation mechanism, the principal exploits the fact that agent 1 favors 2 more than agent 2 favors 1, and thus gives preferential treatment to agent 2 by providing him with a higher incentive. We will later provide a general result, and demonstrate that the set of contracts that minimize the principal's cost in full implementation, is based on these bilateral relationships between the agents.

This simple example also demonstrates that the principal benefits from higher asymmetry between agents' externalities (i.e., lower mutuality). Note that the principal's optimal cost in the full implementation is $2c - w'_1(2) = 2c - w_1(2) - \varepsilon$. This observation is extended later in the paper. Moreover, we show that the cost difference between the more expensive full implementation mechanism and the partial implementation, is decreasing with the level of asymmetry. In this example, the difference between the two types of mechanisms is simply $w_2(1) - \varepsilon$. Therefore, the level of asymmetry between the agents becomes an important consideration at the stage of participants selection and the choice of full or partial implementation mechanism.

To conclude this example, consider the case of negative externalities, i.e., $w_i(j) < 0$ and $w_j(i) < 0$. In this case, the principal has to compensate each agent for the damage caused by the participation of the other. Therefore, the optimal mechanism is simply $v_1 = c + |w_1(2)|$ and $v_2 = c + |w_2(1)|$. In the paper we provide a comprehensive solution that includes both positive and negative externalities.

3 The Model

A participation problem is given by a triple (N, w, c) where N is a set of n agents. The agents' decision is binary, participate in the initiative or not. The structure of externalities w is an $n \times n$ matrix specifying the bilateral externalities among the agents. An entry $w_i(j)$ represents the extent to which agent i is attracted to the initiative when agent j is participating. Agents gain no additional benefit from their own participation, so $w_i(i) = 0$. We assume that agents' preferences are additively separable, i.e., agent i's utility from participating jointly with a group of agents M is $\sum_{j \in M} w_i(j)$ for every $M \subseteq N$. We assume that the externality structure w is fixed and exogenous. Finally, c is the vector of the outside options of the agents. For simplicity and in a slight abuse of notation, we assume that c is constant over all agents. Our results can be generalized easily to the case of heterogeneous costs.

We assume that contracts offered by the principal are simple and descriptive in the sense that the principal cannot provide payoffs that are contingent on the participation behavior of other agents. Much of the examples discussed above seem to share this feature. Evidently, in Gould et al (2005), rental contracts in malls include fixed rental component and overage rent provision but exclude any contingencies on participation of other stores. This set of contracts can be described as an incentive mechanism $v = (v_1, v_2, ..., v_n)$ by which agent i receives a payoff of v_i if he decides to participate and zero otherwise. v_i are not constrained in sign and the principal can either pay or charge his agents but he cannot punish agents for not participating (limited liability). Given a mechanism v agents face a normal form game $G(v)$. Each agent has two possible strategies in the game: participation or default. For a given set M of participating agents, each agent $i \in M$ earns $\sum_{j \in M} w_i(j) + v_i$ and each agent $j \notin M$ earns his outside option. We assume agents' participation decisions are taken simultaneously. We focus first on full implementation, i.e., mechanisms that sustain full participation as a unique equilibrium in the game $G(v)$. A mechanism sustaining full participation as a unique equilibrium with minimal total cost for the principal is said to be an optimal mechanism.

We view the participation problem as a reduced form of the global optimization problem faced by the principal which involves both the selection of the optimal group for the initiative and the design of incentives. Specifically, let U be a (finite) universe of potential participants. For each $N \subseteq U$ let $v^*(N)$ be the total payment made in an optimal mechanism that sustains the participation of the set of agents N. The principal will maximize the level of net benefit she can guarantee herself which is given by the following optimization problem: $\max_{N \subseteq U}[u(N) - v^*(N)]$, where $u(N)$ is the principal's gross benefit from the participation of the set N of agents and is assumed to be strictly monotone with respect to inclusion, i.e., if $T \subsetneq S$, then $u(T) < u(S)$. Our full implementation framework assumes that the principal cannot coordinate the agents to play anything better than the worse equilibrium is her respect, while partial implementation assumes that players select the full participation equilibrium in case of multiple equilibria. While most of our analysis will concern the structure of incentives within the selected set N,our results will also shed light on the selection problem.

4 Positive Externalities

Suppose that $w_i(j) > 0$ for all $i, j \in N$, such that $i \neq j$. In this case, agents are more attracted to the initiative the larger the set of participants. We demonstrate how an agent's payment is affected by the externalities that he induces on others as well as by the externalities that others induce on him. We will also refer to how changes in the structure of externalities affect the principal's welfare.

In proposition 1 we show that the optimal mechanism is part of a general set of mechanisms characterized by the *divide and conquer*[1] property. This set of mechanisms is constructed by ordering agents in an arbitrary fashion, and offering each agent a reward that would induce him to participate in the initiative under the belief that all the agents who are before him participate and all the agents who are after him default. Due to positive externalities, "later" agents are induced (implicitly) by the participations of others and thus can be offered smaller (explicit) incentives. More formally, the *divide and conquer* (*DAC*) mechanisms have the following structure:

$$v = (c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), ..., c - \sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary order of agents. We refer to this order as the ranking of the agents and say that v is a DAC mechanism with respect to the ranking φ. The reward for a certain agent i is increasing along with his position in the ranking. More specifically, the higher agent i is located in the ranking, the higher is the payment offered by the principal.

Note that given mechanism v, agent i_1 has a dominant strategy in the game $G(v)$, which is to participate[2] . Given the strategy of agent i_1, agent i_2 has a dominant strategy to participate as well. Agent i_j has a dominant strategy to participate provided that agents i_1 to i_{j-1} participate as well. Therefore, mechanism v sustains full participation through an iterative elimination of dominated strategies. The following proposition provides a necessary condition for the optimal mechanism.

Proposition 1 *If v is an optimal mechanism then it is a divide and conquer mechanism.*

Proof. Let $v = (v_{i_1}, v_{i_2}, ..., v_{i_n})$ be an optimal mechanism of the participation problem (N, w, c). Hence, v generates full participation as a unique Nash equilibrium.

[1] See Segal (2003) and Winter (2004) for a similarly structured optimal incentive mechanism.

[2] Since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate.

Since no-participation is not an equilibrium, at least a single agent, henceforth i_1, receives a reward weakly higher than his outside option c. Due to the optimality of v his payoff would be exactly c. Agent i_1 chooses to participate under any profile of other agents' decisions. Given that agent i_1 participates and an equilibrium of a single participation is not feasible, at least one other agent, henceforth i_2, must receive a reward weakly higher than $c - w_{i_2}(i_1)$. Since v is the optimal mechanism, i_2's reward cannot exceed $c - w_{i_2}(i_1)$, and under any profile of decisions i_2 will participate. Applying this argument iteratively on the first $k-1$ agents, at least one other agent, henceforth i_k, must be incentivized with a payoff weakly higher than $c - \sum_{j=1}^{k-1} w_{i_k}(j)$, but again, since v is optimal, the payoff for agent k must be equal to $c - \sum_{j=1}^{k-1} w_{i_k}(j)$. Hence, the optimal mechanism v must satisfy the *divide and conquer* property and therefore it is a DAC mechanism under a certain ranking φ. ■

4.1 Optimal Ranking

Our construction of the optimal mechanism for the participation problem (N, w, c) relies on proposition 1, which shows that the optimal mechanism is a DAC mechanism. Therefore, we are left to characterize the optimal ranking which is the ranking that yields the lowest payment DAC mechanism. We show that under positive externalities the optimal ranking is determined by a virtual popularity tournament among the agents, in which each agent is challenged by all other agents. The results of these matches between all pairs of agents are described by a directed graph $G(N, A)$, when N is the set of nodes and A is the set of arcs. Hence, N represents the agents, and $A \subset N \times N$ is a binary relation on N that defines the set of arcs. The directed graph are *simple* and *complete*[3]. We refer to such graphs as **tournaments**. Note that we allow both $(i, j) \in A$ and $(j, i) \in A$ unless $i = j$. We define the set of arcs in tournament $G(N, A)$ as follows:

(1) $w_i(j) < w_j(i) \iff (i, j) \in A$

(2) $w_i(j) = w_j(i) \iff (i, j) \in A \text{ and } (j, i) \in A$

The interpretation of a directed arc (i, j) in the tournament G is that agent j values mutual participation with agent i more than agent i values mutual participation with agent j. We also use the term *agent i beats agent j* whenever $w_i(j) < w_j(i)$. In case of a two sided arc, i.e., $w_i(j) = w_j(i)$ we say that *agent i is even with agent j* and the match ends with a tie.

In our solution analysis we distinguish between acyclic and cyclic graphs. We say that a tournament is *cyclic* if there exists at least one node v for which there is

[3] A directed graph $G(N, A)$ is *simple* if $(i, i) \notin A$ for every $i \in N$ and *complete* if for every $i, j \in N$ at least $(i, j) \in A$ or $(j, i) \in A$.

a directed path starting and ending at v, and *acyclic* if no such path exists for all nodes.[4]

4.2 Optimal Ranking for Acyclic Tournaments

A ranking φ is said to be **consistent** with tournament $G(N, A)$ if for every pair $i, j \in N$ if i is ranked before j in φ, then i beats j in the tournament G. In other words, if agent i is ranked higher than agent j in a consistent ranking, then agent j values agent i more than agent i values j. We start with the following lemma:

Lemma 1 *If tournament $G(N, A)$ is acyclic, then there exists a unique ranking that is consistent with $G(N, A)$.*

We refer to the unique consistent ranking proposed in Lemma 1 as the *tournament ranking.*[5] From the consistency property, if agent i is ranked above agent j in the tournament ranking, then i beats j. Moreover, each agent's location in the tournament ranking is determined by the number of his winnings. Hence, the agent ranked first is the agent who won all matches and the agent ranked last lost all matches. As we demonstrate later, there may be multiple solutions when tournament $G(N, A)$ is cyclic. Proposition 2 provides a unique solution for participation problems with acyclic tournaments:

Proposition 2 *Let (N, w, c) be a participation problem for which the corresponding tournament $G(N, A)$ is acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal mechanism of (N, w, c) is given by the DAC mechanism with respect to φ.*

The intuition behind Proposition 2 is based on the notion that if agents $i, j \in N$ satisfy $w_i(j) < w_j(i)$ then the principal should exploit the fact that j favors i more than i favors j by giving preferential treatment to i (putting him higher in the ranking) and using agent i's participation to incentivize agent j. Thus, the principal is able to reduce the cost of incentives by $w_j(i)$ rather then by only $w_i(j)$ if agent j precedes agent i in the ranking. Applying this notion upon all pairs of agents minimizes the principal total payment to the agents. One way to think on tournament $G(N, A)$ is as a set of constraints that the optimal mechanism has to satisfy which eventually leads to a ranking.

[4] By definition, if $(i, j) \in A$ and $(j, i) \in A$, then the tournament is cyclic.

[5] The tournament ranking is actually the ordering of the vertices in the unique hamiltonian path in tournament $G(N, A)$.

The optimal mechanism can be viewed as follows. First the principal pays the outside option c for each one of his agents. Then the agents participate in a tournament that matches each agent against all other agents. The winner of each match is the agent who imposes a higher externality on his competitor. The loser of each match pays the principal an amount equals to the benefit that he acquires from mutually participating with his competitor. Note that if agent i is ranked higher than agent j in the tournament then it is not necessarily the case that j pays back more than i in total. The total amount paid depends on the size of bilateral externalities and not merely on the number of winning matches. However, the higher agent i is located in the tournament, the lower is the total amount paid to the principal.

From the perspective of the agents, their reward is *not* a continuous increasing function of the externalities they imposes on the others. However, a slight change may increase rewards significantly, since a minor change in externalities may change the optimal ranking and thus affect agents' payoffs.

An intuitive solution for the participation problem might be to reward agents according to their level of popularity in the group, such that the most popular agents would be rewarded the most. One possible interpretation of popularity in our context would be the sum of of externalities imposed on other by participation, i.e., $\sum_{j=1}^{n} w_j(i)$. However, as we have seen, agents' rankings in the optimal mechanism are determined by something more refined than this standard definition of popularity. Agent i's position in our ranking depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different ranking than the one imposed by a standard definition of popularity. This can be illustrated in the following example in which agent 3 is ranked first in the optimal mechanism despite of being less "popular" than agent 1.

Example 1 *Consider a group of 4 agents with identical outside option $c = 20$. The externalities structure of the agents is given by matrix w as shown in Figure 1. The tournament G is acyclic and the tournament ranking is $\varphi = (3, 1, 2, 4)$. Consequently, the optimal mechanism is $v = (20, 17, 14, 10)$, which is the divide and conquer mechanism with respect to the tournament ranking. Note that agent 3 who is ranked first is not the agent who has the maximal $\sum_{j=1}^{n} w_j(i)$.*

$$w = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{pmatrix}$$

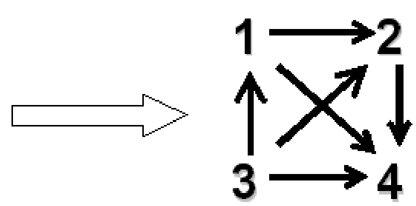

Figure 1

Note that while the principal's cost of incentivizing full participation is weakly decreasing with respect to the entries in the matrix of externalities it is not strictly decreasing. Consider a two-agent example in which $w_i(j) > w_j(i)$. If we increase $w_i(j)$ by a small ε the total payment will decrease by ε. However, if $w_j(i)$ is increased by ε, the total payment in the optimal mechanism will remain unchanged.[6] That is, the principal does not exploit the externality j induces on i since the reverse externality is greater. In general, let V be the optimal sum of payments of a participation problem (N, w, c). If $w_i(j) > w_j(i)$ then V is strictly decreasing with $w_i(j)$.

The total cost of incentives in the optimal mechanism can be simply expressed in a formula without going through the combinatorial problem of identifying the tournament ranking. Two terms play a role in this formula: The first measures the aggregate level of externalities, i.e., $K_{agg} = \sum_{i\ j} w_i(j)$; the second measures the bilateral asymmetry among the agents, i.e., $K_{asym} = \sum_{i<j} |w_i(j) - w_j(i)|$. K_{asym} stands for the extent to which agents induce mutual externalities on each other. The smaller the value of K_{asym} the higher the degree of mutuality of the agents. Proposition 3 shows that the cost of the optimal mechanism is additive and declining in these two measures.

Proposition 3 *Let (N, w, c) be a participation problem and V be the principal's optimal cost of inducing participation. If the corresponding tournament $G(N, A)$ is acyclic then $V = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym})$*

An interesting consequence of Proposition 3 is that for a given level of aggregate externalities, the principal's payment is decreasing with lower levels of mutuality among the agents, as stated in Corollary 3.1. Hence, when the level of asymmetry in the externalities among agents is increasing the principal's payment is lower. The intuition behind this result relates to the virtual tournament discussed above. In each matching that takes place the principal extracts a "fine" from the losing agents. It is clear that these fines are increasing with the level of asymmetry (assuming $w_i(j) + w_j(i)$ is kept constant). Hence, a higher level of asymmetry allows the principal more leverage in exploiting the externalities. This observation may have important implications for the principal's selection stage.

Corollary 3.1 *Let V be the principal's cost of the mechanism for the participation problem (N, w, c) in an acyclic tournament. For a given level of aggregate*

[6] As long as the inequality holds.

externalities, V is strictly decreasing with the asymmetry level of the externalities within the group of agents.

With partial implementation, i.e., with incentive mechanisms that sustain full participation as an equilibrium, not necessarily unique, the cost for the principal in the optimal mechanism is substantially lower. More specifically, in the least costly mechanism that induce full participation, each agent i receives $v_i = c - \sum_j w_i(j)$. However, this mechanism entails no-participation equilibrium as well, hence coordination is required. The total cost of the partial implementation mechanism is $V_{partial} = n \cdot c - \sum_{i\ j} w_i(j)$. In other words, under partial implementation the principal can extract the full revenue generated by the externalities. Our emphasis on full implementation is motivated by the fact that under most circumstances the principal cannot coordinate the agent to play his most-preferred equilibrium. Brandts and Cooper (2005) report experimental results that speak to this effect. Agents' skepticism about the prospects of the participation of others trap the group in the worst possible equilibrium even when the group is small. Nevertheless, one might be interested in evaluating the cost of moving from partial to full implementation. The following corollary points out that for a given level of aggregate externalities, the premium is decreasing with the level of asymmetry. Hence, the asymmetry level is an important factor in choosing between partial and full implementation mechanisms.

Corollary *3.2 Let V be the principal's cost of the optimal full implementation mechanism for the problem (N, w, c) with acyclic tournament and $V_{partial}$* the equivalent partial implementation mechanism. *Then $V - V_{partial} = \frac{1}{2}(K_{agg} - K_{asym})$. For a given level of aggregate externalities, $V - V_{multiple}$ is strictly decreasing with the level of asymmetry.*

If the asymmetry level $K_{asym} = 0$ (equivalently, when $w_i(j) = w_j(i)$ for all pairs), then the cost of moving from partial to full implementation is the most expensive. The other extreme case is when the externalities are always one-sided, i.e., for each pair of agents $i, j \in N$ satisfies that either $w_i(j) = 0$ or $w_j(i) = 0$[7]. In this case, the additional cost is zero and full implementationis as expensive as partial implementation.

[7] Since this section deals with positive externalities, assume that $w_i(j) = \varepsilon$ or $w_j(i) = \varepsilon$ when ε is very small.

4.3 Optimal Ranking of Cyclic Tournaments

In the previous section we have shown that the optimal mechanism for the participation problem can be derived from a virtual tournament among the agents in which agent i beats agent j if $w_i(j) < w_j(i)$. The discussion was based however, on the tournament being acyclic. If the tournament is cyclic, the choice of the optimal DAC mechanism (i.e., the optimal ranking) is more delicate since proposition 1 does not hold anymore. Any ranking is prone to inconsistencies in the sense that there must be a pair i, j such that i is ranked above j although j beats i in the tournament[8].

The inconsistent ranking problem is similar to problems in sports tournaments, which involve bilateral matches that may turn to yield cyclic outcomes. Various sports organizations (such as the National Collegiate Athletic Association - NCAA) nevertheless provide rankings of teams/players based on the cyclic tournament outcome. Extensive literature in operations research suggests solution procedures for determining the "minimum violation ranking" (Kendall 1955, Ali et al. 1986, Cook and Kress 1990, Coleman 2005 are a few examples) that selects the ranking for which the number of inconsistencies is minimized. It can be shown that this ranking can be obtained as follows. Take the cyclic (directed) graph obtained by the tournament and find the smallest set of arcs such that reversing the direction of these arcs results in an acyclic graph. The desired ranking is taken to be the consistent ranking (per Lemma 1) with respect to the resulting acyclic graph[9].

The solution to our problem follows a very similar path. In our framework arcs are not homogeneous and so they will be assigned weights determined by the difference in the bilateral externalities. Instead of looking at the smallest set of arcs which with their reversal the graph becomes acyclic, we will look for the set of arcs with minimal total weight which with their reversal the graph is acyclic. This solution follows the same logic of a different problem of how to rank multiple candidates in a voting game, suggested by Condorcet (1785). In the voting game, the set of nodes is the group of candidates, the arcs' directions are the results of pairwise votings, and the weights are the plurality in the votings. While Young (1988) characterized this method axiomatically, our solution results from a completely different approach, i.e. the design of optimal incentives.

Formaly, for a participation problem (N, w, c) and for each arc $(i, j) \in A$ we define by $t(i, j) = w_j(i) - w_i(j)$ the weight of the arc from i to j. Note that $t(i, j)$

[8] Consider, for example, a three-agent case where agent i beats j, agent j beats k, and agent k beats i. The tournament is cyclic and any ranking of these agents necessarily yields inconsistencies. The ranking $[i, j, k]$, for instance, yields an inconsistency involving the pair (k, i) since k beats i and i is ranked above agent k.

[9] Note that there may be multiple rankings resulting from this method.

is always non-negative because an arc (i,j) refers to a situation in which j favors i more than i favors j.[10] Hence $t(i,j)$ refers to the extent of the one-sidedness of the externalities between the pairs of agents. If an inconsistency involves the arc (i,j), i.e., j precedes i although i beats j, then the additional payment for the principal relative to the consistent ordering of the pair is $t(i,j)$.[11] For each subset of arcs $S = \{(i_1,j_1),(i_2,j_2),...,(i_k,j_k)\}$ we define $t(S) = \sum_{(i,j)\in S} t(i,j)$, which is the total weight of the arcs in S. For each graph G and subset of arcs S we denote by G_{-S} the graph obtained from G by reversing the arcs in the subset S. Consider a cyclic graph G and let S^* be a subset of arcs that satisfies the following:

(1) G_{-S^*} is acyclic

(2) $t(S^*) \leq t(S)$ for all S such that G_{-S^*} is acyclic.

Then, G_{-S^*} is the acyclic graph obtained from G by reversing the set of arcs with the minimal total weight, and S^* is the set of pairs of agents that satisfy inconsistencies in the tournament ranking of G_{-S^*}. Proposition 4 shows that the optimal ranking of G is the tournament ranking of G_{-S^*} since the additional cost from inconsistencies, $t(S^*)$, is the lowest.

Proposition 4 *Let (N,w,c) be a participation problem with a cyclic tournament G. Let φ be the tournament ranking of G_{-S^*}. Then, the optimal mechanism is the DAC mechanism with respect to φ.*

In the following example we demonstrate how the optimal mechanism is obtained in the case of cyclic tournaments with positive externalities:

Example 2 *Consider a group of 4 agents each having identical outside option $c = 20$. The externality structure and the equivalent cyclic tournament are demonstrated in Figure 2. The reversion of the arcs of both subsets $S_1^* = \{(2,4)\}$, $S_2^* = \{(1,2),(3,4)\}$ provide acyclic graphs $G_{-S_1^*}$ and $G_{-S_2^*}$ with minimal weights. The corresponding tournament rankings are $\varphi_1 = (4,3,1,2)$ and $\varphi_2 = (3,2,4,1)$. Hence, the optimal mechanisms are $v_1 = (20,13,13,12)$ and $v_2 = (20,16,10,12)$.*

[10] If the arc is two sided then $t(i,j) = 0$

[11] Consider an inconsistency that arises from a pair of agents (i,j), when i beats j. Since agent j precedes i the payment for agent i is reduced by $w_i(j)$. However, in a consistent order of the agents (in which i precedes j) the payment for agent j is reduced by $w_j(i)$. Since $w_i(j) < w_j(i)$ the principal is forced to pay an additional cost of $w_i(j) - w_j(i)$ relative to the consistent ranking of the pair, which is equivalent to the weight $t(i,k)$.

$$w = \begin{pmatrix} 0 & 1 & 2 & 5 \\ 2 & 0 & 4 & 2 \\ 1 & 2 & 0 & 7 \\ 3 & 4 & 6 & 0 \end{pmatrix}$$

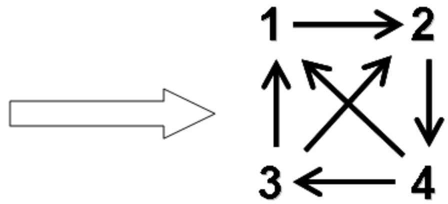

Figure 2

A participation problem is said to be *symmetric* if $w_i(j) = w_j(i)$ for all pairs $i, j \in N$. In the symmetric case, the principal cannot exploit the externalities among the agents, and the total payment made by the principal is identical for all rankings. This follows from Proposition 4 by noting that the tournament has two-way arcs connecting all pairs of agents, and so $t(i, j) = 0$ for all i, j and $t(S)$ is uniformly zero.

Corollary 4.1 *When the externality structure w is symmetric then all DAC mechanisms are optimal.*

Now we can provide the analogue version of proposition 3 for the cyclic case. In this case, the optimal ranking has an additional term $K_{cyclic} = t(S^*)$ representing the cost of making the tournament acyclic.

Proposition 5 *Let (N, w, c) be a participation problem. Let V be the principal's optimal cost of inducing participation. Then $V = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$.*

Corollary 3.1 still holds for pairs of agents that are not in S^*. More specifically, if we decrease the level of mutuality over pairs of agents that are outside S^*, we reduce the total expenses that the principal incurs in the optimal mechanism.

5 Negative Externalities

So far we have limited our discussion to environments in which an agent's participation positively affects the willingness of other agents to participate; i.e., we assumed that externalities are positive. We now turn to the case in which externalities are all negative. Later in Section 5 we discuss the general case of mixed externalities.

Environments of negative externalities are those of congestions. Traffic, market entry, and competition among applicants are all share the property that the more agents that participate, the lower the utility of each participant is. The heterogenous property in our framework seems quite descriptive in some of these examples. In the context of competition it is clear that a more competitive candidate/firm induces a larger externality (in absolute value) than a less competitive one. It is also reasonable to assume, at least for some of these environments, that the principal desires a large number of participants in spite of the negative externalities that they induce on each other.

We show that in order to sustain full participation as a unique Nash equilibrium under negative externalities the principal has fully to compensate all agents for the participation of the others. As we have seen, positive externalities allow the principal to exploit the participation of some agents in order to incentivize others. With negative externalities this is not the case since agents' incentives to participate decline with the participation of the others. Hence it remains for the principal simply to reimburse the agents for the disutility arising from the participation of the others.

Proposition 6 *Let* (N, w, c) *be a participation problem with negative externalities. Then the unique optimal mechanism* v *is given by* $v_i = c + \sum_{i \neq j} |w_i(j)|$

We can solve the negative externalities problem using the results of the previous section. By providing each agent an initial compensation equals to the sum of negative externalities to which he is exposed, we receive a new participation problem in which all externalities are zero (symmetric externalities structure). By Corollary 4.1, all rankings with respect to the DAC mechanism of the new problem are optimal, and thus adding these incentives to the initial compensation yields the optimal incentive scheme.

6 Mixed Externalities

The optimal mechanism for this case is a hybrid solution combining the structure of the optimal mechanisms in the two special cases (positive and negative externalities). Specifically, we show that the optimal mechanism for the mixed case can be derived by decomposing the problem into two separate problems, one with positive externalities and the other with negative externalities. The optimal mechanism for the original (mixed) problem will be obtained by adding agents' compensation payoffs to the solution of the positive participation problem. Formally:

Proposition 7 *Let* v *be the optimal mechanism of a participation problem* (N, w, c)*. Let* (N, q, c) *be an amended participation problem such that* $q_i(j) = w_i(j)$ *if* $w_i(j) > 0$ *and* $q_i(j) = 0$ *if* $w_i(j) < 0$*, and let* u *be the optimal mechanism of* (N, q, c)*. Then,* $v_i = u_i + \sum_{j \in D_i} |w_i(j)|$ *where* $D_i = \{j \mid w_i(j) < 0 \text{ s.t. } i, j \in N\}$.

Proposition 7 implies that the virtual tournament we discussed in earlier sections plays a central role also in the mixed externalities case because it determines payoffs for the positive component of the problem, which the principal can exploit to reduce his costs. In this tournament i beats j whenever (1) $w_j(i) > 0$, and (2) $w_j(i) > w_i(j)$ (where $w_i(j)$ can be either positive or negative). We use the following example to demonstrate how the optimal mechanism is derived in the mixed externalities case.

Example 3 *Consider a group of 4 agents each having identical outside option* $c = 20$*. The externality structure of the agents is demonstrated by matrix* w*, as shown in Figure 3. The positive externality component* (N, q, c) *of the decomposition yields the optimal ranking* $\varphi = (4, 3, 2, 1)$*. The corresponding optimal mechanism of the positive component is* $u = (20, 16, 3, 15)$*. Adding compensations for negative externalities results in the optimal mechanism* $v = (20, 20, 4, 17)$*. Note that* $S^* = \{(1, 3)\}$*.*

$$w = \begin{pmatrix} 0 & 2 & -2 & 3 \\ -1 & 0 & 8 & 9 \\ -4 & 1 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{pmatrix}$$

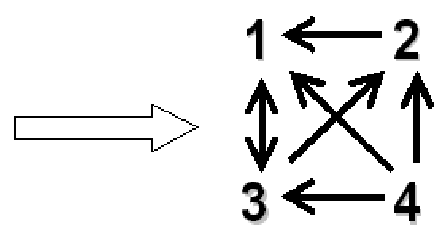

Figure 3

We conclude this section by deriving the analogous result to propositions 3 and 6 in the case of mixed externalities. Lets define $K_{neg} = \sum_{(i,j)\in D} |w_i(j)|$ when $D = \{(i, j) \mid w_i(j) < 0 \ s.t. \ i, j \in N\}$.We show that the principal's cost of incentivizing his agents is decomposed in pretty much the same way as in the positive externalities case, only that now the principal has to add the compensation for the negative externalities. Specifically:

Proposition 8 *Let* (N, w, c) *be a mixed participation problem and* V *be the payment of the optimal mechanism* v*. Let* (N, q, c) *be an amended participation problem such that* $q_i(j) = w_i(j)$ *if* $w_i(j) > 0$ *and* $q_i(j) = 0$ *if* $w_i(j) < 0$*. Let* K_{agg}*,* K_{asym}*, and* K_{cyclic} *be the characteristics of the amended participation problem* (N, q, c)*, and* K_{neg} *be the characteristic of the participation problem* (N, q, c)*. Then,* $V = n \cdot c - K_{agg} - K_{asym} + K_{cyclic} + K_{neg}$

Proposition 8 follows trivially from proposition 5, 6 and 7.

7 Group Identity and Selection

In this section we demonstrate our model as a special case in which externalities assume the values 0 and 1. We interpret it as an environment in which an agent either benefits from the participation of his peer or gains no benefit. We provide three examples of group identity in which the society is partitioned into two groups and agents have hedonic preferences over members in these groups. We demonstrate how the optimal mechanism proposed in previous sections may affect the selection of the agents in the planning of the initiative.

(1) **Segregation** - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the other group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 1$. Otherwise, $w_i(j) = 0$.

(2) **Desegregation**[12] - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 0$. Otherwise, $w_i(j) = 1$.

(3) **Status** - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high status group and enjoys no benefit from participating with members of the low status group. Formally, let B_1 be the high status group and set $w_i(j) = 1$ if and only if $j \in B_1$ (otherwise $w_i(j) = 0$).

Proposition 9 *Let (N, w, c) be a participation problem. Let n_1 and n_2 be the number of agents selected from groups B_1 and B_2 respectively such that $n_1 + n_2 = n$. Denote by $v(n_1, n_2)$ the principal cost of incentivizing agents under the optimal mechanism given that the group composition is n_1 and n_2. The following holds:*
1) under Segregation $v(n_1, n_2)$ is decreasing with $| n_1 - n_2|$.
2) under Desegregation $v(n_1, n_2)$ is increasing with $| n_1 - n_2|$.
3) under Status $v(n_1, n_2)$ is decreasing with n_1.

In a segregated environment the principal's cost of incentives is increasing with the level of mixture of groups, hence in the selection stage the principal would prefer to give precedence to one group over the other. In the desegregation case the principal's

[12] An example could be a singles party.

cost is declining with mixture, hence in the selection stage the principal would like to balance between members of the groups. In the Status case the cost is declining with the number of agents recruited from B_1, which will be strongly preferred over members from B_2.

8 Conclusion

In this paper we analyzed a model of multi-agent initiatives with exogenous externalities, i.e., i's level of attraction of j, $w_i(j)$ is fixed. As we saw, the matrix of bilateral externalities affects agents' payoffs. This may suggest some preliminary game in which agents invest effort to increase the positive externalities that they induce on others. For example, agents can invest in their social skills to make themselves more attractive invitees to social events. A firm may invest to increase its market share in order to improve its ranking position in an acquisition game. Under certain circumstances such an investment may turn out to be quite attractive as we have seen that a slight change in externalities may result in a substantial change in rewards. The preliminary game on externalities can be thought of as a network formation game similar to the ones discussed in the network formation literature (see Jackson 2003 for a comprehensive survey). Specifically, consider a selection[13] of an optimal mechanism function that maps each matrix of externalities to a payoff vector $\Gamma : w \to \pi$ (payoffs for agents include both the transfer from the principal as well as the intrinsic benefits from participation). One can think of the matrix of externalities as a generalized network in the sense that it specifies the intensity[14] of arcs, in contrast to standard networks which only specify whether a link exists. If we assume that agents can increase bilateral externalities according to a given cost function then the externalities become endogenous in the model. The new game will now have two stages. The first one is a network formation game (that determines the externalities) and the second stage is the participation game. The analysis of such a game is beyond the scope of this paper but seems to be a natural next step.

[13] We refer to selection because the optimal mechanism may not be unique.

[14] See Calvo, Lasaga, and van den Nouweland (1999), Calvo-Armengol and Jackson (2001, 2001b), Goyal and Moraga (2001), and Page, Wooders, and Kamat (2001) for such models.

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9 Appendix

9.1 Non Additive Preferences

Here we present an extention of the model in which agents' preferences are non additive. A participation problem is described by a group of agents N, and an outside options vector c as noted previously. Here we assume a general externalities structure, which is given by non additive preferences of the agents over all subsets of agents in group N. More specifically, for each i, $u_i : 2^{N\backslash\{i\}} \longrightarrow R$. Hence, $u_i(S)$ is the benefit of agent i from the participation of subset $S \subseteq N$ in the initiative. Also, we normalize $u(\emptyset) = 0$. We consider the positive externalities case, in which for each i and S, $u_i(S) \geq 0$.

Following the logic of proposition 1, it is clear that the optimal mechanism which sustains full participation as a unique equilibrium is also a divide and conquer mechanism. Hence, to construct the optimal mechanism we need to construct the optimal ranking of the agents.

Consider a 3 agents example, with the following order $\phi = \{i_1, i_2, i_3\}$. The payoff vector in a DAC mechanism with ranking ϕ is $\{c, c - u_{i_2}(i_1), c - u_{i_3}(i_1, i_2)\}$. Hence, the optimal order would maximize $u_{i_2}(i_1) + u_{i_3}(i_1, i_2)$.More generally, the principal has to choose ϕ to solve the following optimization problem:

$$\max_{\phi} \sum_{j=2}^{n} u_{i_{j+1}}(i_1, ..., i_{j-1})$$

We say that agent i beats j if for all $S \subset N$ such that $i, j \notin S$ we have $u_i(S \bigcup j) - u_i(S) < u_j(S \bigcup i) - u_j(S)$[15]. Intuitively, i beats j if i's marginal contribution to the utility of j is greater than j's marginal contribution to the utility of i, regardless of subset S at which marginal contributions are being calculated. Assuming this binary relation is complete (and not necessarily transitive) enables to construct a complete directed graph $G(N, A)$ when N is the set of nodes (which represent the agents), and A is the set of arcs which are defined in the following way: If agent i beats j then $(i, j) \in A$. The following result incorporates the same logic as in proposition 2:

Proposition 10 *Let (N, c) be a participation problem with non additive preferences, for which the corresponding directed graph $G(N, A)$ is complete and acyclic. Let φ be the tournament ranking of $G(N, A)$. The optimal mechanism of (N, c) is given by the DAC mechanism with respect to φ.*

[15]Note that with $S = \emptyset$ we get the condition we had with the additively separarable preferences.

The framework presented in this part is more general than the seperable additive preferences in that the marginal contribution of player i to the utility of player j is not constant but depends on the set of other players who participate in the initiative.

9.2 Proofs

Proof of Lemma 1 Lets demonstrate that there is a single node with $n-1$ outgoing arcs. Since the tournament is complete, directed, and acyclic graph there cannot be two such nodes. If we assume such a node doesn't exist, then all nodes in G have both incoming and outgoing arcs. Since the number of nodes is finite, we get a contradiction for G being acyclic. Let's denote this node as i_1 and place its corresponding agent first in the ranking (hence this agent beats all other agents). Now let's consider a subgraph $G(N^1, A^1)$ which results from the removal of node i_1 and its corresponding arcs. Graph $G(N^1, A^1)$ is directed, acyclic, and complete and, therefore, following the previous argument, has a single node that has exactly $n-2$ outgoing arcs. We denote this node as i_2, and place its corresponding agent at the second place in the ranking. Note that agent i_1 beats agent i_2 and therefore the ranking is consistent so far. After the removal of node i_2 and its arcs we get subgraph $G(N^2, A^2)$ and consequentially node i_3 is the single node that has $n-3$ outgoing arcs in subgraph $G(N^2, A^2)$. Following this construction, we can easily observe that the ranking $\varphi = (i_1, i_2, ..., i_n)$ is consistent among all pairs of agents and due to its construction is unique. ■

Proof of Proposition 2 Due to proposition 1 the optimal mechanism is a DAC mechanism. Hence the optimal mechanism is derived from constructing the optimal ranking and is equivalent to the following optimization problem:

$$
\begin{aligned}
& \min_{(j_1,j_2,\ldots,j_n)} \left[c + [c - w_{j_2}(j_1)] + \ldots + [c - \sum_{k=1}^{n} w_{j_n}(j_k)] \right] \\
= \quad & \min_{(j_1,j_2,\ldots,j_n)} \left[n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + \ldots + \sum_{k=1}^{n} w_{j_n}(j_k) \right\} \right] \\
= \quad & \max_{(j_1,j_2,\ldots,j_n)} \left[\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + \ldots + \sum_{k=1}^{n} w_{j_n}(j_k) \right]
\end{aligned}
$$

Since no externalities are imposed on nonparticipants, the outside options of the agents have no role in the determination of the optimal mechanism. We will next show that the ranking that solves the maximization problem of the principal is the tournament ranking. Let's assume, without loss of generality, that the tournament ranking φ is the identity permutation, hence $\varphi(i) = i$, and $W_\varphi = \sum_{k=1}^{1} w_1(k) + \sum_{k=1}^{2} w_2(k) + ... + \sum_{k=1}^{n} w_n(k)$. W_φ is the principal's revenue extraction. By contradiction let's assume that there exists a different ranking denoted by σ such that $W_\varphi \leq W_\sigma$. First, assume σ is obtained from having two *adjacent* agents i and j in φ trade places such that i precedes j in φ and j precedes i in σ. By Lemma 1, agent i beats agent j. Therefore, $W_\sigma = W_\varphi - w_j(i) + w_i(j)$ and $W_\sigma < W_\varphi$. Consider now the case in which i and j are not adjacent. Since any substitution is a result of a series of adjacent substitutions, using the previous argument iteratively results with $W_\sigma < W_\varphi$. Consider now any arbitrary ranking σ different from φ. Since we can move from σ to φ by a finite number of swaps of the sort described above we get again the result that $W_\sigma < W_\varphi$. Therefore the DAC mechanism with respect to the tournament ranking is unique and optimal. ■

Proof of Proposition 3 Without loss of generality, let's assume that the tournament ranking φ is the identity permutation. Hence, under the optimal mechanism, the principal's payment is $V = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + ... + \sum_{j=1}^{n} w_n(j)\right]$. Denote $s_i(j) = [w_i(j) + w_j(i)]$ and $a_i(j) = [w_i(j) - w_j(i)]$. We can represent K_{agg} and K_{asym} in the following manner: $K_{agg} = \sum_{i\ j} w_i(j) = \sum_{i<\ j} (w_i(j) + w_j(i)) = \sum_{i<j} s_i(j)$ and $K_{asym} = \sum_{i<j} |a_i(j)|$. Since $w_i(j) = \frac{1}{2}(s_i(j) + a_i(j))$ we can rewrite the principal's payment as:

$$
\begin{aligned}
V &= n \cdot c - \frac{1}{2}\left[\sum_{j=1}^{1} \{s_1(j) + a_1(j)\} + ... + \sum_{j=1}^{n} \{s_n(j) + a_n(j)\}\right] \\
&= n \cdot c - \frac{1}{2}\left(\sum_{i>j} s_i(j) + \sum_{i>j} a_i(j)\right)
\end{aligned}
$$

Note that $s_i(j) = s_j(i)$ and $a_i(j) = -a_j(i)$. In addition $a_i(j) > 0$ when $i > j$ due to the acyclic tournament and the consistent ranking. Therefore, $V = n \cdot c - \frac{1}{2}\left(\sum_{i<j} s_i(j) - \sum_{i<j} |a_i(j)|\right) = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym})$. ■

Proof of Corollary 3.2 The result follows immediately from Proposition 3, where we show that $V = n \cdot c - \frac{1}{2}\sum_{i\ j} w_i(j) - \frac{1}{2}\sum_{\ i<j} |w_i(j) - w_j(i)|$, and from $V_{multiple} = n \cdot c - \sum_{i\ j} w_i(j)$. Taken together, the two yield $V - V_{multiple} = \frac{1}{2}\sum_{i\ j} w_i(j) - \frac{1}{2}\sum_{i<j} |w_i(j) - w_j(i)| = \frac{1}{2}(K_{agg} - K_{asym})$. ■

Proof of Proposition 4 Consider a subset of arcs S where G_{-S} is acyclic, and assume that the tournament ranking of G_{-S} is $\varphi = (j_1, j_2, ..., j_n)$. The payment of the principal V under the DAC mechanism with respect to φ is $V = n \cdot c - \left\{\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k)\right\}$. Note that each $(i,j) \in S$ satisfies an inconsistency in tournament ranking φ. More specifically, if $(i,j) \in S$, then i beats j, agent j is positioned above agent i.Note that in this case $w_i(j) = w_j(i) - t(i,j)$, where $w_i(j) < w_j(i)$ and $t(i,j) > 0$. Consider the following substitution: If $(i,j) \in S$ then $w_i(j) = \hat{w}_j(i) - t(i,j)$; otherwise $w_i(j) = \hat{w}_i(j)$ and rewrite the principal's payment as $V = n \cdot c - \left\{\sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k)\right\} + t(S)$. Note that $\hat{w}_i(j) = \max(w_i(j), w_j(i))$; therefore rankings differ only in the level of $t(S)$. Hence, the subset S with the lowest $t(S)$ brings V to a minimum. Hence, the optimal mechanism is the DAC mechanism with respect to the tournament ranking of G_{-S^*}. ■

Proof of proposition 5 As demonstrated in proposition 4, the payment of the principal can be written as $V = n \cdot c - \left\{\sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k)\right\} + t(S)$ when $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Following the argument of proposition 3, denote $s_i(j) = [\hat{w}_i(j) + \hat{w}_j(i)]$ and $a_i(j) = [\hat{w}_i(j) - \hat{w}_j(i)]$ and the principal's payment is $V = n \cdot c - \frac{1}{2}\left(\sum_{i<j} s_i(j) + \sum_{i<j} |a_i(j)|\right) + t(S) = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$. ■

Proof of Proposition 6 Given mechanism v, participation is a dominant strategy for all agents, under the worst-case scenario in which all other players participate since $u_i = \sum_{i=1}^{n} w_i(j) + v_i = c$ for every $i \in N$ [16]. To show that v is optimal, consider a mechanism m for which $m_i < v_i$ for some agents and $m_i = v_i$ for the rest. By contradiction, assume full participation equilibrium holds under mechanism m. Consider an agent i for which $m_i < v_i$. If all other players are participating, then player i's best response is to default since $u_i = \sum_{i=1}^{n} w_i(j) + m_i < c$. Hence, v is a unique and optimal mechanism. ■

Proof of Proposition 7 Assume by contradiction that there exists a mechanism $\tilde{v}$ that sustains full participation unique equilibrium in the participation problem (N, w, c) such that $\sum_i \tilde{v}_i < \sum_i v_i$. Lets adjust payoffs of v and $\tilde{v}$ by substracting the compensations for the negative externalities, hence $\tilde{u}_i = \tilde{v}_i - \sum_{D_i} |w_i(j)|$ and $u_i = v_i - \sum_{D_i} |w_i(j)|$ when $D_i = \{j \mid w_i(j) < 0 \ s.t. \ i,j \in N\}$. Mechanisms u and $\tilde{u}$ provide incentives for the participation problem (N, q, c) where $q_i(j) = w_i(j) \ \textit{if} \ w_i(j) > 0 \ \textit{and} \ q_i(j) = 0 \ \textit{if} \ w_i(j) < 0$. But then, $\sum_i \tilde{u}_i < \sum_i u_i$ in contradiction to the optimality of u. ■

[16] As mentioned earlier, since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate.

Proof of Proposition 9 In both segregated and desegregated environments the externality structure is symmetric and, following Corollary 5.1, all rankings are optimal. Let's consider first the segregated environment. Since all rankings are optimal, a possible optimal mechanism is $v = (c, ..., c - (n_1 - 1), c, ..., c - (n_2 - 1))$. Hence, the optimal payment for the principal is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - \sum_{k=1}^{n_2-1} k = n \cdot c - \frac{n_1(n_1-1)}{2} - \frac{(n-n_1)(n-n_1-1)}{2}$. Assuming that $v(n_1, n_2)$ is continuous with n_1 then $\frac{\partial v(n_1,n_2)}{\partial n_1} = n - 2n_1$ and maximum is achieved at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is declining with $\mid n_1 - n_2 \mid$. In desegregated example, a possible optimal mechanism is $v = (c, ..., c, c - n_1, ..., c - n_1)$. Therefore, the principal's payment is $v(n_1, n_2) = n \cdot c - (n - n_1) \cdot n_1$. Again, let's assume that $v(n_1, n_2)$ is continuous with n_1, in which case solving $\frac{\partial v(n_1,n_2)}{\partial n_1} = 2n_1 - n = 0$ results that the minimum payment for the principal in the desegregated environment is received at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is increasing with $|n_1 - n_2|$. In a status environment, since group B_1 is the more esteemed group, all agents from B_1 beat all agents from B_2; therefore agents from B_1 should precede the agents from B_2 in the optimal ranking. A possible optimal ranking is $\varphi = \{i_1, ., ., , i_{n_1}, j_1, ..., j_{n_2}\}$ when $i_k \in B_1$, $j_m \in B_2$ and $1 \leq k \leq n_1$, $1 \leq m \leq n_2$. Therefore, a possible optimal mechanism is $v = (c, c-1..., c-(n_1-1), c-n_1, ..., c-n_1)$. The principal's payment is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - n_2 \cdot n_1 = n \cdot c - \frac{n_1(n_1-1)}{2} - (n - n_1)n_1 = \frac{1}{2}n_1 - nn_1 + \frac{1}{2}n_1^2 + cn$. Again, assuming that $v(n_1, n_2)$ is continuous with n_1, $\frac{\partial v(n_1,n_2)}{\partial n_1} = n_1 + \frac{1}{2} - n = 0$ and the minimal payment is achieved at $n_1^* = n - \frac{1}{2}$. Note that $V(n_1 = n) = V(n_1 = n - 1)$. Therefore, the best scenario for the principal is when $n_1 = n$. Alternatively, the cost of incentivizing is decreasing with n_1. ■

Proof of Proposition 10 Since the optimal mechanism is a DAC mechanism, it is a result of the following optimization problem:

$$\max_{(j_1,j_2,...,j_n)} [u_{j_2}(j_1) + u_{j_3}(j_1, j_2) + ... + u_{j_n}(j_1, ..., j_{n-1})]$$

Assume, without loss of generality, that the tournament ranking φ is the identity permutation, hence $\varphi(i) = i$, and $W_\varphi = u_2(1) + u_3(1, 2) + ... + u_n(1, ..., n-1)$. W_φ is the principal's revenue extraction. By contradiction let's assume that there exists a different ranking denoted by σ such that $W_\varphi \leq W_\sigma$. First, assume σ is obtained from having two *adjacent* agents i and j ($j = i+1$) in φ trade places such that i precedes j in φ (hence i beats j) and j precedes i in σ. Therefore, $\sigma = \{1, ..., i-1, j, i, ..., n\}$.First note that all the players that appear after j in order φ earn the same payoff in the DAC mechanism of both φ and σ. The same holds also for all the agents who appear before i in the order φ. So the cost of the DAC mechanisms with respect to φ and σ

differs only in terms of the payoff of players i and j. and we get that

$$W_\sigma = W_\varphi + A$$

When $A = [u_i(1, ..., i-1, j) - u_i(1, ..., i-1)] - [u_j(1, ..., i-1, i) - u_j(1, ..., i-1)]$. The term A compares the marginal contribution of i relative to the marginal contribution of j, given a subset $S = \{1, ..., i-1\}$. Therefore, $A < 0$, which entails $W_\sigma < W_\varphi$. Consider now the case in which i and j are not adjacent. Since any substitution is a result of a series of adjacent substitutions, using the previous argument iteratively results with $W_\sigma < W_\varphi$, and the rent extraction from σ is lower . Consider now any arbitrary ranking σ different from φ. Since we can move from σ to φ by a finite number of swaps of the sort described above we get again the result that $W_\sigma < W_\varphi$. Therefore the DAC mechanism with respect to the tournament ranking is unique and optimal. ■