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MULTI-STAGE VOTING, SEQUENTIAL ELIMINATION AND CONDORCET CONSISTENCY

by

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Multi-Stage Voting, Sequential Elimination and Condorcet Consistency*

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Abstract

A class of voting procedures based on repeated ballots and elimination of one candidate in each round is shown to always induce an outcome in the top cycle and is thus Condorcet consistent, when voters behave strategically. This is an important class as it covers multi-stage, sequential elimination extensions of all standard one-shot voting rules (with the exception of negative voting), the same one-shot rules that would fail Condorcet consistency. The necessity of repeated ballots and sequential elimination are demonstrated by further showing that Condorcet consistency would fail in all standard voting rules that violate one or both of these conditions.

JEL Classification: P16; D71; C72

Key Words: Multi-stage voting; sequential elimination; Condorcet consistency; top cycle; scoring rules; Markov equilibrium

1 Introduction

Any voting rule as a means of reaching collective decisions can be assessed by several alternative criteria. One such criterion is whether the voting rule can result in an outcome that is majority-preferred to any other candidate on binary comparisons – known as the *Condorcet winner*, henceforth *CW*. This property, called *Condorcet consistency*, is "widely regarded as a compelling democratic principle" (Moulin [18]; sect. 9.4); voting rules with this property will be described as *Condorcet consistent* (or, *CC*).

In this paper, we will argue that a large class of voting procedures based on repeated ballots and elimination of one candidate in each round, henceforth called *multi-stage voting with sequential elimination* (or simply referred as, *sequential elimination voting*), will lead uniquely to the *CW* being elected, if it exists, when voters behave strategically. Moreover, if there is no *CW*, the equilibrium in this class of voting will elect a candidate in the 'top cycle,' that is, on majority comparison the winning candidate would dominate any other candidate either directly or indirectly.

Top cycle property (and Condorcet consistency) have been obtained for the well-known class of *binary voting* (see McKelvey and Niemi [15]).[1] However, such results are not directly helpful as many multi-stage vote procedures are not binary. When voters vote over more than two candidates simultaneously in the stage games of any multi-stage voting, the usual miscoordination problem of simultaneous voting becomes even more pronounced. But based on an equilibrium refinement and by applying some carefully constructed induction arguments, the top cycle property can be established for a large class of sequential elimination voting that are not necessarily binary.

The broad principles underlying our multi-stage voting can be understood by considering one specific voting rule that we call the *weakest link voting*: Voting occurs in rounds with all the voters simultaneously casting their votes for one candidate in each successive round. In any round the candidate with minimal votes is eliminated, with any ties broken by a deterministic tie-breaking rule. Continue with this process to pick a winner.[2] The weakest link voting can be interpreted

[1]The definitions of binary voting and various other voting rules relevant for this paper are collated in a glossary at the end of the Appendix.

[2]The Conservative Party in Great Britain roughly follows this procedure to choose its leader: the party's parliamentary members vote in successive rounds to reduce first a small number

as a natural sequential elimination extension of plurality voting, with elimination of the worst plurality loser in each round. By carrying out similar eliminations in each round based on an appropriately defined elimination rule, one can extend any single-round voting to its sequential elimination equivalent.

For sequential elimination voting to be *CC*, or yield a top cycle outcome, it is sufficient (and may even be almost necessary) that any (group of) majority voters have some minimal collective influence: the voting rule must be such that by coordinating their votes in any round a majority can always ensure that any particular candidate who survived up to that round is not eliminated in that round; further, such vote coordinations by the majority must be "stable" in the sense that should the majority fail to choose some appropriate coordination of votes that may lead to the particular candidate's elimination, there will be at least one member of the majority group who will have an incentive, if his aim were to protect that candidate, to further deviate by changing his vote. We call these twin requirements, the *majority non-elimination* (**MNE**) property.

We show that the **MNE** property will be satisfied by multi-stage, sequential elimination versions of all familiar single-round voting procedures with one exception – the sequential elimination analogue of negative voting. To understand how majority influence works, consider for instance multi-stage analogue of scoring rules which eliminate, at any round, only one candidate with the lowest total score. Clearly, for any candidate and any majority, placing the candidate at the top by every member of a majority is stable; furthermore the candidate will have a total score that is strictly higher than the average score of the remaining candidates, even if every voter outside the majority places that candidate at the bottom, if the following property holds: the scores in any round for various ranks be such that the average of the two scores corresponding to the top and the bottom ranks weakly exceeds the average score for all the intermediate ranks combined (this property clearly holds for the weakest link and the sequential elimination analogue of Borda). Thus, if this property holds the majority is able to protect the candidate from being eliminated and hence satisfies the **MNE** property.

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of candidates to only two candidates, and eventually the party members vote to elect the final winner. See *http://politics.guardian.co.uk/Print/0,3858,4196604,00.html*. Also, the last contest in 2005 to select the host city for the 2012 olympic games had the characteristics of weakest link voting (London emerged the winner after four rounds of elimination). See *http://news.bbc.co.uk/sport1/hi/front_page/4655555.stm*.

Finally, we will also argue why in general *one-by-one elimination* and *repeated ballots* – the two characteristics of our multi-stage voting – are important for Condorcet consistency. All standard one-shot voting rules and several multi-stage voting lack one or both these characteristics and will fail to be *CC*.

As we mentioned earlier, binary voting rules are also *CC*. In the standard formulation of such rules, the winning alternative at each stage is matched against another alternative in the next stage. Our result clarifies that in multi-round voting what is important for Condorcet consistency (and top cycle) is not that each stage picks only "one winner," but rather that there is only "one loser." In fact, in binary voting there is also only "one loser" (as well as "one winner") at each stage.

More broadly, our multi-stage voting framework and results should be seen as a significant progress beyond the special class of binary voting (e.g., [15], [17], [2], [11], [12], [10], [4] and [5]). In contrast to only two choices from which to pick at each round in binary voting, in our multi-stage schemes there is virtually no exogenous restriction on how many choices might be considered. Thus, our multi-stage voting games are complementary to binary voting (sequential binary voting being the only common element). In our setup, the sets of candidates (i.e., choices) available at later rounds evolve endogenously (rather than determined by an exogenous ordering of agendas in binary voting) through equilibrium behavior at earlier rounds. Furthermore, in our multi-stage schemes the voting rule need not remain the same in every round. Of course, with the simple binary comparison lacking, our multi-stage voting poses a far greater challenge as the backwards induction arguments involving iterated deletion of dominated strategies do not necessarily generate a unique continuation outcome in each subgame (as in binary voting; see [15]). It is of course possible to generalize our multi-stage schemes to include all binary voting rules as special cases; in section 3.3 we describe one such generalization that preserves the top cycle property.

The analysis in this paper assumes complete information. One may ask why do we care about properties of voting rules with full information and where we know what should be selected (a CW)? First, even when the choice of an ideal social alternative is not an issue, the problem of vote coordinations, and as a result the potential multiplicity of voting outcomes, have been a major concern in the voting literature. Hence, any satisfactory resolution of this problem should be viewed as a positive contribution. Second, understanding how to deal with the complexity of coordination under complete information is a necessary step to deal with the more difficult challenges of incomplete information.[3] Third, studies of voting rules, because of their extensive use, are of considerable practical relevance. Finally, our analysis should also offer a useful normative guide as to how to design voting games, or modify some of the existing vote methods, to achieve Condorcet consistency as a social objective.

The next section presents the voting rules and related equilibrium solution concepts. Section 3 contains results on sequential elimination voting. In section 4, we analyze single-round voting and some multi-stage voting rules that do not involve one-by-one elimination. Positive results and general statements over broad classes of voting rules appear in theorems and results on particular voting rules appear in propositions. The proofs not contained here, in the text or the Appendix, can be found in supplementary materials.

2 The voting rules and equilibrium solutions

Voting games

The set of candidates is denoted as $\mathcal{K}$ with cardinality k, and the voter set is denoted as $\mathcal{N}$ with cardinality n, where both k and n are at least three. Throughout we assume n to be an odd number, but this can be relaxed (see footnote 6). Also for simplicity of exposition, $\mathcal{K} \cap \mathcal{N} = \emptyset$. Each voter $i \in \mathcal{N}$ has a strict, ordinal preference ordering over the candidates given by $\succ_i$. The voters have complete information about preferences.

The class of voting games we consider is quite general and is described as follows. Each voting rule consists of the voters/players voting in at most J rounds/stages, $J < k$. At each stage the voters simultaneously vote (i.e., take an action) and at least one candidate is removed. At the end of a maximum of J rounds of voting one candidate survives who is the winner. If C is the set of candidates left at any stage $j \leq J$ with $|C| \geq 2$ ($|.|$ denoting cardinality) then a choice for voter i at that stage consists of choosing an element from an arbitrary choice set $A_i(C, j)$. Moreover, if each i chooses $a_i \in A_i(C, j)$ at this stage then we shall denote the set of eliminated candidate(s) by $e(a^j, C) \subset C$ where $a^j = (a_1, ..., a_n)$ is the profile of votes at stage

[3]In voting with incomplete information, beside the coordination problem, there is also the additional issue of information aggregation. Recent literature (e.g., [8], [22], [3]) has mainly concerned with this latter issue.

j. So if the voting finishes in some $\mathcal{J} \leq J$ rounds and voters choose the sequence of votes $\{a^j\}_{j=1}^{\mathcal{J}}$, then the winning candidate is $w \notin \cup_{j=1}^{\mathcal{J}} e(a^j, C)$.

For any $j \leq J$ let $h^j = (a^1, ..., a^{j-1})$ be a complete history (description) of the *actual* voting decisions up to stage j. Define $\mathcal{H}^j$ to be the set of histories at round j and $\mathcal{H} = \bigcup_j \mathcal{H}^j$ be the set of all histories, with the convention that $\mathcal{H}^0$ refers to the initial null history. Also, let $C(h)$ be the set of remaining candidates at $h \in \mathcal{H}$.

Now a (pure) strategy for voter i is a function $s_i : \mathcal{H} \rightarrow \bigcup_{j,C} A_i(C, j)$ such that $s_i(h) \in A_i(C(h), j)$ if $h \in \mathcal{H}^j$. Also, denote the set of (pure) strategies of voter i by S_i and let $S = \times_i S_i$.

The above set of games clearly includes the weakest link voting, and more generally any multi-stage voting with sequential elimination, and any single-round voting. In the case of the weakest link, the number of voting rounds J is $k-1$, at each stage one candidate is eliminated so that $|e(a^j, C)| = 1$ and the set of choices $A_i(C, j) = C$. More generally, any voting game belongs to our class of multi-stage voting if and only if it has $k-1$ stages and at each stage one candidate is eliminated.

In the case of single-round voting, $J = 1$, all voters submit their strategies at the first stage. In some (such as plurality rule, approval voting, Borda rule and negative voting), all the candidates except one are eliminated simultaneously. In some others, such as *instant runoff voting*, the process of elimination is in one or more attempts following a single ballot.

Also, included in our voting games will be the class of repeated voting rules in which more than one candidate are eliminated in some round. This includes both the class of games in which the number of rounds J is fixed and less than $k-1$ (such as *plurality runoff* voting with $J = 2$), and the case in which the number of rounds J is endogenous (for example, repeated procedures, such as *exhaustive ballot*, have the following majority vote trigger property: if at any round a candidate receives majority votes then he is immediately declared the winner and voting stops).

The equilibrium

Since the voting games we consider may have a dynamic structure, we require our equilibrium concept to be subgame perfect. In addition, as is common in the literature on voting, we need to eliminate choices that are weakly dominated, otherwise there are a large number of trivial equilibria in which each voter's choice is immaterial. Therefore, an equilibrium in our setup is a strategy profile for the voters that is a subgame perfect equilibrium and is such that at each stage the votes of each player is not weakly dominated given the *equilibrium continuation strategies of others in future stages.*

In other words, any equilibrium strategy profile $s^* \in S$ in a voting game must have the following properties. In any subgame at the final stage J, s^* must be a weakly undominated Nash equilibrium in the subgame. In any subgame starting with stage $J-1$, the voters' strategies must be an undominated Nash equilibrium in the subgame given that the voters play the game according to s^* in the continuation game. This backward elimination procedure continues all the way to stage 1.

Formally, for any history $h \in \mathcal{H}$, let $\Gamma(h)$ be the subgame at h and $w(s,h)$ be the candidate elected in the subgame $\Gamma(h)$ if the voters follow strategy profile s in this subgame. Also, for any strategy profile $s \in S$ and any history $h \in \mathcal{H}$, define the set of strategies for all players other than i that are consistent with s in every subgame after h by

$$\widetilde{S}_{-i}(h,s) = \{s'_{-i} \in S_{-i} \mid s'_{-i}(h,h') = s_{-i}(h,h') \text{ for all non-empty } h' \text{ s.t. } (h,h') \in \mathcal{H}\}.$$

Definition 1. *A strategy profile s^* is an equilibrium if for any history $h \in \mathcal{H}$ it satisfies the following properties in the subgame $\Gamma(h)$:*

$$\text{(Nash)} \quad \text{For any } i, \quad w(s^*,h) \succeq_i w(s_i, s^*_{-i}, h) \quad \forall s_i \in S_i, \quad \text{where } \succeq_i \text{ means either } \succ_i \text{ or } =;$$

$$\left.\begin{array}{l} \text{(Weak non-domination)} \quad \text{For any } i, \quad \nexists s_i \in S_i \text{ s.t.} \\ \qquad w(s_i, s_{-i}, h) \succeq_i w(s^*_i, s_{-i}, h) \quad \forall s_{-i} \in \widetilde{S}_{-i}(h,s^*) \\ \text{and} \quad w(s_i, s_{-i}, h) \succ_i w(s^*_i, s_{-i}, h) \quad \text{for some } s_{-i} \in \widetilde{S}_{-i}(h,s^*). \end{array}\right\} \quad (1)$$

Notice that for any $s \in S$, at any $h \in \mathcal{H}^j$ we can define a one-shot reduced form voting game $\widehat{\Gamma}(h,s)$ in which voter i's strategy set is $A_i(C(h),j)$ and, given any profile $a^j \in A(C(h),j)$ $(= \prod_i A_i(C(h),j))$ of votes, the outcome of the game is given by $w(s,(h,a^j))$ being elected. Clearly, our definition of equilibrium strategy in Definition 1 is equivalent to showing that the choices that the equilibrium strategies prescribe at any history h constitute an undominated Nash equilibrium of the one-shot reduced voting game at h. Thus, s^* *is an equilibrium if and only if* $s^*(h)$ *is an undominated Nash equilibrium of* $\widehat{\Gamma}(h,s^*)$, for all h.

Remark 1. Our equilibrium concept is effectively a backward elimination procedure. However, note that it differs from the more familiar procedure of iterative elimination of (weakly) dominated strategies; while in the latter approach the weak-domination check is carried out in relation to the entire game, ours is only along the subgames.[4,5] Iterative elimination on its own is unlikely to solve the coordination problem that results in undesirable outcomes.

Remark 2. Note also that any trembling-hand perfect equilibrium in extensive form satisfies our definition of equilibrium. This is because any trembling-hand perfect equilibrium in extensive form is a subgame perfect equilibrium and excludes weakly dominated choices at different information sets. We could have alternatively started with trembling-hand perfect equilibrium in extensive form as our equilibrium concept (see also our remark following Theorem 3). However, for ease of exposition we adopt the above definition of equilibrium.

Remark 3. For single-round voting the standard equilibrium concept is *undominated Nash*. Our twin requirements of subgame perfection and non-domination reduce to this standard equilibrium definition for single-round voting rules.

Next we define Markov equilibrium.

Definition 2. *An equilibrium s^* is said to be Markov if for any i and any j,*

$$s_i^*(h) = s_i^*(h') \quad \forall h, h' \in \mathcal{H}^j \text{ such that } C(h) = C(h').$$

Markov equilibrium strategies are such that at any stage onwards the strategies depend only on the candidates who have survived up to that stage and *not* on the specific history leading up to it.

3 Multi-stage voting with sequential elimination

3.1 Condorcet consistency of the weakest link

Much of our insight about multi-stage voting with sequential elimination can be gained by studying the weakest link voting, so we start with this particular voting rule and then broaden our analysis to a very general class of sequential elimination voting.

First, some notation. Given the voters' strict preference ordering over candidates, a binary comparison operator T defines a candidate x to be *majority-preferred*

[4] Moulin [16] formally analyzed the iterative elimination procedure and applied it to a significant class of voting – voting by veto, kingmaker and voting by binary choices.

[5] In our setup the two definitions may differ because at each stage our voters vote simultaneously (the game is not one of perfect information) over more than two alternatives.

over another candidate y, written as xTy, if the number of voters preferring x over y exceeds the number of voters preferring y over x.[6]

Next, the *CW*, if it exists, is defined as a candidate $z \in \mathcal{K}$ such that zTz', for all $z' \in \mathcal{K}$. Similarly, for any set of remaining candidates $C \subseteq \mathcal{K}$ the *CW* with respect to C, if it exists, is a candidate $z \in C$ such that zTz' for all $z' \in C$.

We say that an equilibrium s^* of a voting rule is *CC* at every subgame if for every $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ has a *CW*, $z(h)$, the equilibrium strategy induces the *CW* with respect to $C(h)$ in the subgame defined by h (i.e. $w(s^*, h) = z(h)$ if $z(h)$ is defined for h).

Our first result is an equilibrium characterization of the weakest link game:

Theorem 1. *Any Markov equilibrium of the weakest link voting is* CC *at every subgame.*

Proof. We demonstrate this by (backward) induction on the number of remaining candidates in any subgame. First, consider any subgame at stage $k-1$ with only two candidates, z and z'. Because sincere voting is the only Nash equilibrium that is also undominated in this final stage subgame, the *CW* must be the winner.

Now suppose the following induction hypothesis is true: *For every history $h \in \mathcal{H}$ such that the set of remaining candidates $C(h)$ consists of j candidates, the following holds*: *if $C(h)$ has a CW, z, then z will become the ultimate winner in the subgame defined by h* (i.e., $w(s^*, h) = z$). We then prove that the same holds at any history/subgame with $j+1$ remaining candidates.

Suppose not; then there exists a subgame defined by some history $\tilde{h}$ such that the set of the remaining candidates $C(\tilde{h})$ has $j+1$ candidates, $C(\tilde{h})$ has a *CW*, z, and some other candidate $z' \neq z$ becomes the ultimate winner in this subgame. Now since z is the *CW* with respect to $C(\tilde{h})$, it follows by the induction hypothesis that z is eliminated immediately at $\tilde{h}$ at stage $k-j$ (since at $\tilde{h}$ there are $j+1$ candidates, the subgame defined by $\tilde{h}$ begins in round $k-j$). Otherwise, since z is also the *CW* with respect to the set of candidates in the next round, by the hypothesis z will become the ultimate winner.

Next, consider those voters who prefer z over z' and their immediate vote at $\tilde{h}$ in stage $k-j$. By definition of z, these voters will form a majority. Therefore, it

[6]To relax the assumption of *odd* number of voters, extend the definition of majority preference, whenever there is a tie, by applying a tie-breaker.

must be that at least one such voter, say voter i, who voted for some candidate z'' other than z. But then we establish a contradiction by showing that for i voting for z weakly dominates voting for z'' at this stage, given the equilibrium continuation strategies in the future stages.

To show this, first notice that if voter i chooses z'' there are two possible outcomes depending on the choices of others at this stage: either (i) z survives at this stage and, by the induction hypothesis, all the subsequent stages and becomes the ultimate winner; or (ii) z is eliminated and, by the Markov property of the equilibrium strategies, z' becomes the ultimate winner. Now if (i) is the case then if i switches his vote from z'' to z the outcome will be the same with z surviving all stages and becoming the winner. If (ii) is the case then if i switches his vote from z'' to z, either z is eliminated and the outcome will be the same with z' becoming the ultimate winner or z survives this stage, and by the induction hypothesis, all the subsequent stages and becomes the ultimate winner.

Finally, we need to show that there is a vote profile for all voters other than i such that if voter i votes for z'' then z would be eliminated and z' goes on to win whereas if he votes for z then z is not eliminated and z wins. To show this let $\overline{Z} \subset C(\tilde{h})$ be the set of remaining candidates other than z'' that are lower in the tie-breaker than z and let m be the cardinality of this set. Then, since voting at $\tilde{h}$ eliminated candidate z, it must be that $n - 1 \geq m$. Otherwise, it must be that some $x \in \overline{Z}$ receives zero vote at $\tilde{h}$ and therefore is eliminated (contradiction). Now consider a vote profile for all voters other than i such that every $x \in \overline{Z}$ receives at least one vote and no other candidate receive any vote; since $n - 1 \geq m$, this is feasible. Now, if i votes for z, he is not eliminated (z'' receives zero vote). On the other hand, if i votes for z'', candidate z is eliminated; this is because in this case z receives zero vote and any other candidate(s) with zero vote belong to the set $C(\tilde{h})\backslash\{\overline{Z} \cup z''\}$ and hence must be higher up in the tie-breaker than z in the case of a tie. This completes the claim that z weakly dominates z'' for i, contradicting the supposition that z is eliminated at this stage.

Since we already proved our hypothesis for subgames with two candidates, it follows by the induction step above that if there is a CW for the set C, he will be elected in any subgame with C. **Q.E.D.**

The above result is a characterization result for Markov equilibria of the weakest link voting when the set of (remaining) candidates has a CW. However, in order to ensure that the result is not vacuous one has to show that the weakest link game has a (Markov) equilibrium. This is particularly important because even if a set of candidates has a CW, there could be subgames off-the-equilibrium path without a CW among the remaining candidates and it is by no means clear that there is an equilibrium in such subgames. Thus, Theorem 1 should be viewed in combination with Theorem 2 below.[7]

Theorem 2. *Assume $n \geq 2k - 1$. Then in the weakest link game there exists a Markov equilibrium.*

The proof of this result can be found in supplementary materials. There are several further points to note concerning the characterization result in Theorem 1. First, notice that the arguments in the proof does not make any reference to the tie-breaking rule; thus the weakest link voting is CC for any arbitrary deterministic tie-breaking rule. Also, if the preferences of the voters can be represented using expected utility framework then by an analogous argument one can show that Theorem 1 holds for random tie-breaking rules.

Second, limiting the result to equilibria that are Markov could be considered a limitation of Theorem 1. However, there are two points that we like to make with respect to the Markov restriction. First, a weaker version of the Markov property would suffice for the proof of Theorem 1. All we require to obtain the result is that the equilibrium strategies do not depend on the history through the specific configuration of votes that lead to the particular candidates' eliminations. However, the strategies can still depend on the order in which the candidates are eliminated. In fact, if we assume that the votes are not revealed between stages but only the identity of the eliminated candidate at each stage is announced, then we do not need the Markov property. Second, it could be shown that if, in choosing the strategies, players have, at least at the margin (lexicographically), a preference for simplicity (aversion to complexity) then all equilibria are Markov.[8] The basic

[7] Since we wrote an earlier version of this paper (available under a different title: Bag et al. [1]), we came across Peress [19] who also examines the issue of Condorcet consistency using the weakest link (that he calls multistage runoff) but under a very restrictive assumption that every subset of candidates has a CW (all candidates can be majority ranked). In particular, he does not need to consider the possibility that off-equilibrium subgames may not have a CW. This makes the required analysis in Peress [19] much simpler.

[8] Properties of Markov equilibrium in general dynamic games have been studied by Chatterjee and Sabourian [6], Sabourian [20], and Gale and Sabourian [13].

reason is that in our multi-stage voting games at each round some candidate is eliminated, and therefore for any equilibrium strategy profile every set of remaining candidates occur on the equilibrium path at most once. If any player i's strategy is non-Markov, then i makes a different choice at two different subgames with the same set of remaining candidates C; but then since C occurs at most once on the equilibrium path, player i could economize on complexity by always making the same choice at every subgame with C without sacrificing payoffs. In supplementary materials, we provide a formal justification for this claim for any voting game such that at least one candidate is eliminated in each voting round.

Finally, as discussed after the equilibrium definition, since every trembling hand perfect equilibrium in extensive form satisfies our equilibrium concept, it follows that every Markov trembling hand perfect equilibrium in extensive form of the weakest link voting is CC at every subgame.

3.2 Sequential elimination voting and top cycle consistency

Next we extend our analysis in two ways: (1) consider general sequential elimination voting games; (2) allow arbitrary voter preferences that do not necessarily admit a CW. For the latter, we consider the broader concept of *'top cycle,'* which always exists and is same as the CW when the CW exists.

Arbitrary voter preferences including no $\boldsymbol{CW}$

Fix any set of candidates $C \subseteq \mathcal{K}$. Then candidate $x \in C$ is said to be directly or indirectly majority preferred to candidate $y \in C$, denoted by $xT^C y$, if either xTy or there exists a sequence of candidates $x^1, \ldots, x^\tau \in C$ such that $xTx^1T \ldots Tx^\tau Ty$, where, as before, T is the binary operator representing majority preference. Then the *top cycle with respect to* C is defined as $\mathcal{TC}(C) = \{x \in C : \forall y \in C, y \neq x,\ xT^C y\}$. We also refer to $\mathcal{TC}(\mathcal{K})$ simply by the *top cycle*.

Sequential elimination voting with majority property

A general sequential elimination voting is one where in each round only one candidate is eliminated. In these games, as mentioned before, players vote in $k-1$ rounds, the set of votes for voter i at round $j < k$ when C is the set of remaining candidates is $A_i(C, j)$, and one candidate $e(a^j, C)$ is eliminated at each round j with votes a^j.

An important aspect of this procedure would be the decisive role that any group of majority voters can play: at any round a majority of voters can ensure that any particular candidate is *not* eliminated. We now specify this important property for the set of sequential elimination voting games as follows.

Majority non-elimination* (MNE) *property: *For any stage $j < k$, any set of remaining candidates C, any $c \in C$, and any set of majority voters $\phi \subseteq \mathcal{N}$, there exists a set of strategy profiles $\mathcal{D}^c_\phi(C,j) \subseteq \Pi_{i\in\phi} A_i(C,j)$ for the majority ϕ such that the following two conditions hold:*

***[i]* (Majority protection)** *If all members of ϕ choose some profile $a_\phi \in \mathcal{D}^c_\phi(C,j)$ then c is not eliminated, i.e.,*

$$e(a_\phi, a_{-\phi}, C) \neq c, \ \forall a_{-\phi} \in \Pi_{\ell \notin \phi} A_\ell(C,j).$$

***[ii]* (Protection stability)** *For any profile $a_\phi \notin \mathcal{D}^c_\phi(C,j)$ such that $e(a_\phi, a_{-\phi}, C) = c$ for some $a_{-\phi} \in \Pi_{\ell\notin\phi} A_\ell(C,j)$, there exists some member of the majority $i \in \phi$ and an action $a^c_i \in A_i(C,j)$ such that*

$$\forall a'_{-i} \in A_{-i}(C,j) \ \text{ if } \ e(a_i, a'_{-i}, C) \neq c \ \text{ then } \ e(a^c_i, a'_{-i}, C) \neq c \tag{2}$$

$$\textit{and} \ \ \exists a'_{-i} \in A_{-i}(C,j) \ \text{ s.t. } \ e(a_i, a'_{-i}, C) = c \ \text{ and } \ e(a^c_i, a'_{-i}, C) \neq c. \tag{3}$$

That is, a_i is "inferior" to a^c_i in protecting c.

All sequential elimination voting rules satisfying [i] and [ii] above constitute the family $\mathcal{F}$. ||

Note that $\{\mathcal{D}^c_\phi(C,j)\}$ are sets of actions/votes for non-elimination of any candidate c. For instance, if each stage of the sequential elimination voting involves voters ranking the candidates, one can think of $\{\mathcal{D}^c_\phi(C,j)\}$ as all actions by the majority that place c at the top of their ranking; then the two conditions in the **MNE**-property require that [i] if a majority of voters place c at the top then c cannot be eliminated, and [ii] if a majority fails to place c at the top and c is eliminated then there is some voter from that majority who will have an action that is (weakly) better than his particular action in the 'failed majority action profile' in protecting c. Later we will verify that the MNE-property is a fairly mild condition and *multi-stage, sequential elimination extensions* of a very large class of one-shot voting rules fall under the family $\mathcal{F}$.

Theorem 3. *Fix any sequential elimination voting rule in the family $\mathcal{F}$. In all Markov equilibria, candidate w is the winner in any subgame with remaining candidates C only if $w \in \mathcal{TC}(C)$. Hence, all Markov equilibria are* CC *in every subgame.*

The proof appears in the Appendix. Before discussing the scope of the family $\mathcal{F}$, there are several points to note. First, the top cycle result in Theorem 3 does not require strategies to be Nash as part of the equilibrium definition; we impose the Nash requirement mainly to make the equilibrium definition consistent with the voting games of section 4 and a related negative result in Theorem 4. Second, as in Theorem 1, any Markov trembling hand perfect equilibrium in extensive form will be in the top cycle and is CC when a CW exists. Third, the previous justifications for the Markov restriction in the weakest link game extend to this setup as well (see the supplementary material).

The scope of $\mathcal{F}$

To fully appreciate Theorem 3, it is important that we elaborate the scope of the voting family $\mathcal{F}$. First consider *scoring rules.*

Definition 3. (***Scoring voting rules*** (***Moulin*** [***18***]***, ch. 9***)) *Fix a nondecreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \ldots \leq \varsigma_k$ with $\varsigma_1 < \varsigma_k$. Voters rank the candidates, giving ς_1 score to the one ranked last, ς_2 to the one ranked next to last, and so on. A candidate with a maximal total score is elected.*

Definition 4. (***Sequential elimination scoring rule***) *A sequential elimination scoring rule is the multi-stage, sequential elimination analogue of scoring rules: At any stage and for any set of remaining $J \leq k$ candidates, fix a non-decreasing sequence of real numbers $\varsigma_1 \leq \varsigma_2 \leq \cdots \leq \varsigma_J$ (with $\varsigma_1 < \varsigma_J$) to correspond to J ranks. Each voter ranks the candidates at the particular stage, thus assigning each candidate a score and the candidate receiving the lowest total score is eliminated at that stage.*

Proposition 1. *Any sequential elimination scoring rule belongs to the family $\mathcal{F}$, if at each stage the scores associated with different ranks are such that*

$$\frac{1}{2}(\varsigma_1 + \varsigma_J) \geq \frac{1}{(J-2)} \sum_{j=2}^{J-1} \varsigma_j. \qquad (4)$$

Condition (4) implies that if any majority voters place a candidate c at the top and the remaining voters place c at the bottom then the resulting total score of c can never be the lowest (exceeds the average score of the other candidates). Therefore, this condition ensures that c is not eliminated, irrespective of what others do, and thus the set of actions by a majority that place a candidate at the top satisfies majority protection and hence the **MNE**-property (protection stability is also satisfied by any strategy that does not place c at the top because it is then always possible to protect c better by improving its ranking). In fact, the **MNE**-property cannot be guaranteed in sequential elimination scoring if condition (4) fails.

Both plurality and Borda rules satisfy (4), so the corresponding sequential elimination extensions of these two rules satisfy the **MNE**-property. However, the negative voting with $\varsigma_1 = 0$ and $\varsigma_j = 1$ for all $j > 1$ would fail (4). Moreover, one can show that its sequential elimination extension (in each stage each voter vetoes one candidate and the one receiving the maximum number of vetoes is eliminated) fails the **MNE**-property. This is because a majority of voters may not always be able to guarantee non-elimination of a candidate c by giving it the maximum point, 1. The only way to ensure non-elimination of c is for the majority to coordinate to veto some other candidate(s) other than c; but this may violate *protection stability* because strategies that do not coordinate on vetoing some other candidate(s) need not be inferior in protecting the particular c.

There are some other one-shot voting rules – approval voting, Copeland rule and Simpson rule – that are not part of scoring rules even though each candidate receives a score. These have similar sequential elimination extensions (at any round the candidate receiving the lowest score is eliminated, applying a tie-breaker wherever necessary). The next result demonstrates that these sequential elimination extensions also satisfy the **MNE**-property, and hence are top cycle consistent.

Proposition 2. *The sequential elimination extensions of approval, Copeland and Simpson voting rules belong to the family $\mathcal{F}$.*

The proof of Proposition 1 appears in the Appendix. Proposition 2 proof is very similar and omitted (see also footnote 14).

Note that since our multi-stage voting is quite general – voters can submit a weak or strict ranking, or the preference submission may even be more abstract than a simple ranking of candidates – the scope of the family $\mathcal{F}$ goes well beyond sequential elimination extensions of one-shot voting games that are generally based on a ranking of *all* remaining candidates at every stage. For example, *sequential binary voting* also eliminates candidates one-by-one, even though at each stage the set of actions/votes is restricted to consist of only two candidates. Furthermore, this class trivially satisfies the **MNE**-property. Hence, this important class also comes under $\mathcal{F}$.

3.3 Family $\mathcal{F}$, binary voting and a generalization

McKelvey and Niemi [15] also obtain the top cycle result for general binary voting that satisfy a monotonicity property – a very mild requirement met, for instance, using majority rule. Our sequential elimination voting family is inherently different: in contrast to the binary voting games of McKelvey and Niemi, sequential elimination voting does not restrict the choices at each round to only two; binary voting, on the other hand, does not necessarily assume one-by-one elimination property (binary voting may involve no candidate or multiple candidates being eliminated in a single stage, including selecting a winner even in the first stage). The exception is the sequential binary voting which belongs to both setups (it satisfies one-by-one elimination and the choices at each round are binary).

In comparing our results with the earlier results on binary voting, there are several further points worth noting. First, to establish their top cycle result, McKelvey and Niemi define an equilibrium concept that involves solving *uniquely* the various constituent stage games backwards using elimination of weakly dominated strategies. This concept is well-defined in binary voting games in which every decision node involves two choices; as a result working backwards each voter has a unique dominant choice at each stage and the game can be solved *uniquely* at each stage. In our multi-stage voting framework with more than two remaining candidates voters usually have more than two choices, making the backwards induction type reasoning used in McKelvey and Niemi problematic because there is not necessarily a unique dominant choice and hence a unique continuation path.[9] To deal

[9] In binary voting, at the final decision nodes with only two choices (and each choice corresponds to a single candidate), sincere voting is the unique dominant choice and thus one can associate each final decision node with its "sophisticated equivalent" (Shepsle and Weingast [21]) – the candidate that wins conditional on reaching that particular subgame; iterating back up the tree, by the same reasoning, voters again have two choices over two sophisticated equivalents and voting sincerely over these choices is dominant. In our multi-stage voting scheme, working backwards and iteratively deleting dominated strategies does not typically yield a unique choice at each stage because the choice is not necessarily between two alternatives (sophisticated equivalents). In the case of the weakest link voting with three candidates, for instance, there are three final decision nodes, each involving a pairwise vote, so identifying the sophisticated equivalents is not a problem; but then in the previous (first) stage there is a three-way choice over the sophisticated equivalents and an individual voter's best vote choice at this stage depends on the choices of others.

with the non-uniqueness problem, our equilibrium concept solves recursively the different stages backwards by taking at each stage the continuation strategies as given and assuming Markov property.[10] Second, as a procedure the binary nature of choices is clearly very restrictive and makes the outcome of the binary voting game critically dependent on the exogenous order of choices. On the other hand, sequential elimination voting are not necessarily dependent on some exogenous ordering of candidates (or sets of candidates) with respect to which the voters must vote; rather, all remaining candidates can be simultaneously considered by the voters as voting moves from one round to another. Furthermore, sequential elimination voting extends the scope of decision making in obvious ways (allowing multi-stage, sequential elimination extensions of well-known one-shot voting rules). Third, our result together with the earlier literature establish that the top cycle result (and hence Condorcet consistency) follows from one-by-one elimination or from restricting choices to only two at every stage. It turns out that we can easily generalize our multi-stage voting games to also include binary voting as a special case and preserve the top cycle property.

Generalization: Consider the following three specific changes to the voting game in Section 2: (i) voting occurs in J stages, with J finite but not necessarily less than k (the number of candidates); (ii) at any stage $j \leq J$, the choice of a voter i is an element from an arbitrary choice set $A_i(h^j)$, rather than $A_i(C, j)$ (i.e., history may matter); and (iii) at any stage j with an action profile a^j, there is no restriction on the cardinality $|e(a^j, h^j)|$, i.e., the number of candidates eliminated in any round can be anything between zero and $|C| - 1$. ||

Next, consider an extended voting game with the above modifications such that at *every* stage, either (i) $|A_i(h)| = 2$ as in binary voting and the voting rule satisfies the monotonicity of McKelvey and Niemi [15], or (ii) exactly one candidate is eliminated according to a rule that satisfies the **MNE**-property as in sequential elimination voting. (Thus, in principle, it is possible that in some of the stages binary voting rule is in play while in other stages one-candidate elimination rule of ours applies.) Now applying our equilibrium solution to this extended voting game, it can be shown by a similar argument as in the proof of Theorem 3 that the

[10]The Markov restriction ensures that the equilibrium choices at each stage do not depend on payoff-irrelevant past history. McKelvey and Niemi do not require this restriction because in their setup, by the binary nature of choices at every decision node, the equilibrium in any continuation game is unique and hence is history-independent.

outcome always belongs to the top cycle.

4 Necessity of repeated ballots and one-by-one elimination

In this section we look at non-binary voting rules that differ from the family $\mathcal{F}$ in two important respects: either (1) the elimination of candidates is *not* sequential, or (2) the elimination which may even be sequential is through a *single ballot*, or both. This complementary class includes all single-round voting, a plurality runoff rule, the exhaustive ballot method, instant runoff voting, etc. We shall examine the Condorcet consistency property (or the lack of it) of this complementary class.

First define a general class of single-round voting rules. For any set of candidates $\mathcal{K}$ with cardinality k, the set of strategies for a voter is to rank the k candidates in J different categories for some J such that $1 < J \leq k$ subject to some bounds on the number of candidates in each category. Denote the minimum and the maximum number of candidates in each category $j \leq J$ by $m(j)$ and $M(j)$, respectively. Let Λ be the set of all such J rankings over $\mathcal{K}$. Thus, the strategy for voter i, denoted by $R_i \in \Lambda$, is a profile $(X_1, ..., X_J)$ with J components such that it partitions the set $\mathcal{K}$ into J non-empty cells $X_1, ..., X_J$ and $m(j) \leq |X_j| \leq M(j)$. Since it is a partition, it must be the case that $\sum_j m(j) \leq k$. From R_i we can also specify for each $x, y \in \mathcal{K}$ whether x is ranked strictly above y, denoted by $x \, P_i \, y$, or y is ranked strictly above x, denoted by $y \, P_i \, x$, or x is ranked the same as y (in the same category), denoted by $x \, I_i \, y$. For any set of n voters, the one-shot voting game also specifies the winning candidate as a function of the submitted strategies of the n voters given by an outcome function $f^n : \Lambda^n \rightarrow \mathcal{K}$. A voting rule with k candidates is then defined by the number of categories, the bounds on the size of each category and the outcome function. We refer to such a one-shot voting rule by $v(k) = (J, \{m(j)\}_{j \leq J}, \{M(j)\}_{j \leq J}, \{f^n\}_{n \in \mathbb{N}})$, where $\mathbb{N}$ is the set of odd numbers (as elsewhere, this restriction is made for simplicity).

Rankings Λ can accommodate all scoring voting rules as well as many others that do not fall under scoring rules category (e.g. approval voting, Copeland and Simpson rules). Thus, our one-shot voting game is the most comprehensive (one-shot) generalization of scoring rules.

We shall see that, for any fixed number of candidates k, all single-round voting games satisfying two intuitive properties, called scale invariance and responsiveness, are not *CC* in strategic voting. This will be a strong assertion because (i) all standard one-shot voting rules satisfy these two properties and (ii) the lack of Condorcet consistency is demonstrated for any arbitrary k. (The number of voters can of course vary.) The meaning of scale invariance is rather straightforward.

Definition 5. *A voting rule $v(k)$, for any k, is 'scale invariant' if replicating the set of voters with their submitted strategies by any multiple will not alter the winner.*

Before defining responsiveness, we need to define sincere behavior and Condorcet consistency (in sincere voting) in the above class of voting games. We say that a strategy $R_i = (X_1, ..., X_J) \in \Lambda$ submitted by voter i is *sincere* if $X_1 = \{c^1, ..., c^{m(1)}\}$, $X_2 = \{c^{m(1)+1}, ..., c^{m(1)+m(2)}\}$,...,$X_J = \{c^{\sum_{j<J} m(j)+1}, ..., c^k\}$, when the true preference ranking of voter i is $c^1 \succ_i \ldots \succ_i c^k$.[11] Then for any k, a voting rule $v(k)$ is said to be *CC* under sincere voting if for any number of voters and any preference profile over k candidates that admits a *CW*, the voting rule $v(k)$ selects the *CW* whenever the voters' strategies are sincere.

Responsiveness is about voter pivotalness. Roughly, it requires that for each voter, there is a scenario at which the voter is pivotal in determining the winner between any two candidates.

Definition 6. *A voting rule $v(k)$, for any k, is 'responsive' if it satisfies the following two conditions for each voter i:*

1. *For any pair of candidates x and y and any two strategies R_i and R'_i such that $x\, P_i\, y$ and $y\, P'_i\, x$ there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R'_i, R_{-i}) elects y as the winner.*

2. *If the voting rule is* CC *under sincere voting then (i) the submissions are strict ($J = k$),[12] and (ii) for any three candidates $X = \{x, y, z\}$, there exists a candidate z in X such that the following holds: for any pair of strategies $R_i = (X_1, X_2, X_3, ..., X_k)$ and $R'_i = (X_2, X_1, X_3, ..., X_k)$ such that $X_1 = x, X_2 = y$ and $X_3 = z$, there exists a profile of strategies R_{-i} by the remaining voters such that (R_i, R_{-i}) elects x as the winner, and (R'_i, R_{-i}) elects z as the winner.*

[11] This definition is a generalization of the standard definition of sincere behavior when $J = k$.

[12] The only two one-shot voting rules known to be *CC* under sincere voting (Copeland and Simpson; see Moulin [18]) are based on strict rankings.

Theorem 4. *Fix any k. For any single-round voting rule $v(k)$, if $v(k)$ satisfies responsiveness and scale invariance then $v(k)$ is not* CC.

Proposition 3. *Suppose $n \geq k-1$. Then all scoring rules (including plurality rule, negative voting, Borda rule), approval voting, the two variants of Instant runoff voting (with and without the majority top-rank trigger), Copeland rule and Simpson rule will all satisfy responsiveness and scale invariance conditions of Theorem 4. Hence none of these one-shot voting rules will be* CC.

For the proof of Theorem 4 see the Appendix. The proof of Proposition 3 appears in supplementary materials. Notice that the voting rules in Proposition 3, other than Copeland and Simpson, are not *CC* under sincere voting; therefore, condition 2 of Definition 6 is trivially satisfied for these other rules.

While failure of Condorcet consistency for specific one-shot voting rule(s) is not that surprising (see [9] and [7]), to our knowledge there is nothing to suggest that Condorcet consistency (under strategic voting) should fail for the entire class of one-shot voting. On the contrary, significant positive results in the implementation literature would have led one to believe otherwise. In this respect, failure of Condorcet consistency for the family of one-shot voting rules for *any* arbitrary number of candidates is an important result.[13]

So far in this section we have considered only single-round voting rules that rank candidates. Next we consider (non-binary) voting rules that do not belong to either the above class of single-round voting or the sequential elimination voting family of section 3. Obviously one can think of many voting rules that come under a third complementary group. We are not going to make any general observation here. Instead, we present some voting rules to indicate why both *one-by-one elimination* and *repeated ballots* are important for Condorcet consistency.

Proposition 4. *The plurality runoff rule, the exhaustive ballot method, and the one-shot weakest link voting (with voters submitting their entire weakest link strategies once-for-all in a single round followed by one-one-by elimination) are not* CC.

[13]Note that the Condorcet map is *Maskin monotonic* ([14]) on the restricted domain of preferences where *CW* exists (and will be Maskin monotonic even in unrestricted domains if one defines social choice rule to select all outcomes when *CW* fails to exist). Since the Condorcet map also satisfies 'no veto power,' it is Nash implementable if one considers arbitrary, rather than just one-shot voting, mechanisms.

The proof, based on counter-examples, appears in supplementary materials. Note that the plurality runoff rule and the exhaustive ballot method, used in various political appointments, share features of weakest link voting in that both use multiple ballots but fail *one-by-one elimination.* On the other hand, the one-shot version of the weakest link voting eliminates candidates sequentially but fails *repeated ballots* (and likewise for the instant runoff voting without the majority top-rank trigger, noted in Proposition 3).

In contrast to the results in Theorem 4 and Propositions 3 and 4, repeated ballots and one-by-one elimination allow the voters to coordinate their votes and ensure that the CW is never eliminated. This ability of the voters to coordinate derives from both one-by-one elimination as well as the power of the equilibrium refinement (based on backward inductions) associated with the dynamic (repeated) structure of the game. An intuition on why elimination of more than one candidate in some round may lead to a non-Condorcet outcome would be instructive. The basic idea is that with one-by-one elimination, when the CW is eliminated in some voting round the (off-equilibrium) outcome is unique in the induction argument. When more than one candidate are eliminated, following the CW's elimination the outcome is not necessarily unique – it depends on who else is being eliminated along with the CW; as a result, in this case, the voters may not vote for the CW in order to influence the final outcome in the case when the CW is eliminated.

Notice also the role of repeated ballots in obtaining Condorcet consistency. With one-shot voting models (including the one-shot version of weakest link described above), the game is treated as a normal form and Nash equilibrium is refined using weak domination (a normal form concept). Such a refinement is not sufficient to solve the coordination problem associated with selecting a CW, unless there are only two candidates. With repeated ballots, the equilibrium refinement uses a perfection type reasoning that requires the choices to be undominated in any subgame (given the continuation equilibrium path). This immediately implies that with repeated ballots the CW will be selected in any subgame with two remaining candidates. Then backward induction type reasoning, together with one-by-one elimination, does the rest by selecting the CW in every subgame.

The miscoordination problem (in terms of the lack of Condorcet consistency) we have identified above in most standard voting models can be worse when there is no CW, as these voting rules, in contrast to the family $\mathcal{F}$ (see Theorem 3), may not even select a member of the top cycle. We shall next provide an intuition for such possibilities by providing examples of winning candidate outside the top cycle set in the context of plurality rule and plurality runoff voting (plurality and plurality runoff are respectively typical examples of one-shot voting and multi-round voting without one-by-one elimination).

Consider the case of five voters and six alternatives with the following preferences: type 1: a, b, c, d, e, f (two voters); type 2: b, c, a, e, d, f (two voters) ; type 3: c, a, b, d, e, f (one voter). Assume further that the tie-breaker is such that e is eliminated last and d second last. Clearly, d is outside the top cycle. In the case of plurality rule, voting for d by each voter is an equilibrium outcome (that is, Nash and undominated) because d is not lowest in any one's ranking. In the case of the plurality runoff it can be checked that the following strategies will be an equilibrium: in the second stage voters vote sincerely; in the first stage two type 1 voters vote for d, two type 2 voters vote for e, and the type 3 voter votes for d. Thus, in both voting rules, the alternative d will be the winner in an equilibrium.

Appendix

Proof of Theorem 3. We use induction on the number of remaining candidates. Assume the following induction hypothesis: *Theorem 3 is true for any subgame with j candidates.*

We want to show that the result is also true for any subgame with $j+1$ remaining candidates. Suppose not. Then there is a subgame Γ at stage $k-j$ with remaining candidates C of cardinality $j+1$ such that w is the ultimate winner and $w \notin \mathcal{TC}(C)$. This implies there exists some $y \in C$ such that

$$\text{it is } \textit{not} \text{ the case that } \; w \; T^C \; y. \tag{5}$$

Next we establish two intermediate claims.

Claim 1: y must be the first eliminated candidate in the subgame Γ.

If not, let $y' \neq y$ be the candidate eliminated at this stage. Then in this subgame the remaining candidate set is $C \setminus y'$ and w wins, which implies by the induction hypothesis $w \in \mathcal{TC}(C \setminus y')$. But then $w \; T^{C \setminus y'} \; y$, contradicting (5). ||

Claim 2: For any $a \in C$ with $a \neq y$, y is the winner in any subgame with remaining candidates $C \backslash a$.

Suppose not: there is $a \in C$, $a \neq y$ and a subgame with remaining candidates $C \backslash a$ such that the winner is $\hat{w}$ and $\hat{w} \neq y$. Then by the induction hypothesis $\hat{w} \in \mathcal{TC}(C \setminus a)$. But this, together with $y \in C \setminus a$ and (5), imply that $\hat{w}\ T^{C\backslash a}\ y$ and $\hat{w} \neq w$. Also, by Claim 1 and the induction hypothesis $w \in \mathcal{TC}(C \setminus y)$; this together with $\hat{w} \neq y$ and $\hat{w} \neq w$, imply that $w\ T^{C\backslash y}\ \hat{w}$. Since $\hat{w}\ T^{C\backslash a}\ y$, we must then have $w\ T^{C}\ y$, contradicting (5). So Claim 2 must be true. ||

Now in the subgame Γ with remaining candidates C, consider any voter i such that $y \succ_i w$; there will be a majority of such voters because $y\,T\,w$. Denote these majority voters by ϕ. By condition *[i]* of the **MNE**-property, there exists a set $\mathcal{D}^y_\phi(C, k-j) \subseteq A_\phi(C, k-j)$ such that for any $a_\phi \in \mathcal{D}^y_\phi(C, k-j)$, such that $e(a_\phi, a_{-\phi}, C) \neq y$, $\forall a_{-\phi} \in \Pi_{\ell \notin \phi} A_\ell(C, k-j)$. Then since by Claim 1 y must be the first eliminated candidate in the subgame Γ, it must be that the majority ϕ chose some vote profile $\tilde{a}_\phi \notin \mathcal{D}^y_\phi(C, k-j)$. By condition *[ii]* of the **MNE**-property, this implies that there is some voter $i \in \phi$ whose vote choice $\tilde{a}_i$ (corresponding to the profile $\tilde{a}_\phi$) is "inferior" to some other vote choice a^y_i (as defined in condition *[ii]* of the **MNE**-property in section 3.2) in protecting y. But then we establish a contradiction by showing that for i voting for a^y_i weakly dominates voting for $\tilde{a}_i$ at this stage $k - j$, given the equilibrium continuation strategies in the future stages.

To show this, first note that if i votes for $\tilde{a}_i$, there are two possible outcomes depending on the choices of others at this stage: ***[1]*** y survives and becomes the ultimate winner, by Claim 2; ***[2]*** y is immediately eliminated in which case by Claim 1 and the Markov property of the equilibrium strategies, w becomes the ultimate winner. Now if ***[1]*** is the case and i switches his vote from $\tilde{a}_i$ to $\tilde{a}^y_i$ then y would still survive this stage (by (2) in condition *[ii]* of the **MNE**-property) and, by Claim 2, become the ultimate winner. If ***[2]*** is the case and i switches from $\tilde{a}_i$ to $\tilde{a}^y_i$ then either y is immediately eliminated that ensures, by Claim 1 and the Markov property of the equilibrium strategies, that w is the winner, or y survives and becomes the ultimate winner (by Claim 2). Finally, by (3) in condition *[ii]* of the **MNE**-property, there is some $a'_{-i} \in A_{-i}(C, k-j)$ such that $e(\tilde{a}_i, a'_{-i}, C) = y$ (and w wins by Claim 1 and the Markov property of the equilibrium strategies), and yet $e(\tilde{a}^y_i, a'_{-i}, C) \neq y$ that would result in y winning (by Claim 2).

This completes the claim that a^y_i weakly dominates voting for $\tilde{a}_i$, contradicting the supposition that $w \notin \mathcal{TC}(C)$ is the winner.

By a similar argument as above, it is easy to check that the hypothesis is true for $j = 2$, hence by induction Theorem 3 proof is now complete. **Q.E.D.**

Proof of Proposition 1. Fix a stage with the set of remaining candidates C having the cardinality J. Also, fix a candidate $c \in C$ and a majority ϕ.

For any voter i, let $\mathcal{D}_i^c(C, j)$ be the set of all strategies that place c at the top (with no other restriction on the positions of other candidates).[14] Also, let $\mathcal{D}_\phi^c(C, j) = \Pi_{i \in \phi} \mathcal{D}_i^c(C, j)$.

First we verify condition *[i]*. Fix any $a \in A(\mathcal{C}, j)$ such that $a_\phi \in \mathcal{D}_\phi^c(C, j)$. We need to show that $e(a, C) \neq c$.

For any $x \in C$ and any $a' \in A(\mathcal{C}, j)$, denote the total score of candidate x at this stage when action profile a' is chosen by $TS(x, a')$.

Next, define θ_{top} to be the total score of a candidate if he receives the highest score, ς_J, from a majority of $(n+1)/2$ voters and gets the lowest score, ς_1, from the remaining $n - (n+1)/2$ voters:

$$\theta_{\text{top}} = \frac{(n+1)}{2}\varsigma_J + [n - \frac{(n+1)}{2}]\varsigma_1.$$

Since a is such that the majority ϕ place c at the top, it follows that $TS(c, a) \geq \theta_{\text{top}}$. Therefore, the average score that the other candidates receive when a is chosen cannot exceed

$$\theta_{\text{rest}} = \frac{n[\varsigma_J + \ldots + \varsigma_1] - \theta_{\text{top}}}{J-1}.$$

But then there must exist a candidate $d \in C$ such that $TS(d, a) \leq \theta_{\text{rest}}$. Now to complete verification of condition *[i]*, it suffices to show that $\theta_{\text{top}} - \theta_{\text{rest}} > 0$. Write

$$(J-1)(\theta_{\text{top}} - \theta_{\text{rest}}) = J \cdot \theta_{\text{top}} - n \sum_{\ell=1}^{J} \varsigma_\ell$$

$$= \frac{(J-2)n + J}{2}\varsigma_J + \frac{(J-2)n - J}{2}\varsigma_1 - n \sum_{\ell=2}^{J-1} \varsigma_\ell.$$

Therefore, $\theta_{\text{top}} - \theta_{\text{rest}} > 0 \Leftrightarrow \frac{1}{2}(\varsigma_J + \varsigma_1) + \frac{J}{2n(J-2)}(\varsigma_J - \varsigma_1) > \frac{1}{(J-2)} \sum_{\ell=2}^{J-1} \varsigma_\ell.$

[14] We omit the proof of Proposition 2. It follows a similar argument as in the proof of Proposition 1. For sequential elimination extension of approval voting, $\mathcal{D}_i^c(C, j)$ will consist of the unique strategy of voter i approving only candidate c and disapproving all the remaining candidates. For sequential elimination extensions of Copeland and Simpson rules – given that these rules are based on strict order submissions – $\mathcal{D}_i^c(C, j)$ will place only candidate c at the top.

Since $\frac{1}{2}(\varsigma_J + \varsigma_1) \geq \frac{1}{(J-2)} \sum_{\ell=2}^{J-1} \varsigma_\ell$ and $\varsigma_J > \varsigma_1$, it follows that $\theta_{\text{top}} - \theta_{\text{rest}} > 0$. $\quad ||$

Next, we verify condition *[ii]*. Fix $a \in A(C,j)$ such that $a_\phi \notin \mathcal{D}^c_\phi(C,j)$ and $e(a,C) = c$. For any i, let m^i be a candidate to whom i attaches the highest score ς_J: $a_i(m^i) = \varsigma_J$. Also, without loss of generality, denote the set of voters in the ϕ-majority by $\{1, 2, \ldots, |\phi|\}$. Next, consider the sequence of vote profiles, $a^{(0)}, a^{(1)}, \ldots, a^{(|\phi|)}$, defined as follows: $a^{(0)} = a$ and

$$a^{(i)}_\ell(x) = \begin{cases} \varsigma_J & \text{if } x = c \text{ and } \ell \leq i \\ a_\ell(c) & \text{if } x = m^\ell \text{ and } \ell \leq i \\ a_\ell(x) & \text{otherwise,} \end{cases}$$

for any i and ℓ such that $1 \leq i, \ell \leq |\phi|$. Note that $a^{(|\phi|)}$ is such that $a^{(|\phi|)}_i(c) = \varsigma_J$ for all $i \in \phi$. Therefore, $a^{(|\phi|)}_\phi \in \mathcal{D}^c_\phi(C,j)$ and hence, by condition *[i]*, $e(a^{(|\phi|)}, C) \neq c$. Moreover, by assumption $e(a^{(0)}, C) = c$. Therefore, there exists some i, $1 \leq i \leq |\phi|$, such that $e(a^{(i-1)}, C) = c$ and $e(a^{(i)}, C) \neq c$. Furthermore, by the definition of the sequence $a^{(0)}, a^{(1)}, \ldots, a^{(|\phi|)}$ we have that $a^{(i-1)}_i = a_i$ and $a^{(i-1)}_{-i} = a^{(i)}_{-i}$. Therefore, we have $e(a_i, a^{(i-1)}_{-i}, C) = c$ and $e(a^{(i)}_i, a^{(i-1)}_{-i}, C) \neq c$ verifying (3) in condition *[ii]*.

To verify (2), for $a_i (= a^{(i-1)}_i)$ and $a^{(i)}_i$ note that $a^{(i)}_i(c) = \varsigma_J$, $a^{(i)}_i(m^i) = a_i(c)$ and $a^{(i)}_i(x) = a_i(x)$ for all $x \neq c$. Thus, for any $a_{-i} \in A_{-i}(C,j)$ we have $TS(c, a^{(i)}_i, a_{-i}) \geq TS(c, a_i, a_{-i})$, $TS(m^i, a^{(i)}_i, a_{-i}) \leq TS(m^i, a_i, a_{-i})$ and $TS(x, a^{(i)}_i, a_{-i}) = TS(x, a_i, a_{-i})$ for all $x \neq c$. But this implies that if $e(a_i, a_{-i}, C) \neq c$ then $e(a^{(i)}_i, a_{-i}, C) \neq c$ for all $a_{-i} \in A_{-i}(C,j)$, hence verifying (2). **Q.E.D.**

Proof of Theorem 4. First we show that *sincere* submission of one's ranking is never a weakly dominated strategy. Without loss of generality assume that voter i has the preference relation $c^1 \succ_i c^2 \succ_i \ldots \succ_i c^k$. Suppose $R_i = (X_1, \ldots, X_J)$ is sincere and R_i is dominated by $R'_i = (X'_1, \ldots, X'_J)$. Note that for any $\tau \leq J$, $|X_\tau| = m(\tau)$ and $|X'_\tau| \geq m(\tau)$. Let j be the first cell such that $X'_j \neq X_j$. If $|X_j| < |X'_j|$ then for some $r > j$ it must be that $|X_r| > |X'_r|$, but this is not possible. So $|X_j| = |X'_j|$, hence there exist some $x \in X_j$ and $y \in X'_j$ such that $x \in X'_\ell$ for some $\ell > j$ and $y \in X_r$ for some $r > j$. Hence $x\, P_i\, y$ and $y\, P'_i\, x$. But then by condition 1 in Definition 6 there exists R_{-i} such that (R_i, R_{-i}) results in x winning, and (R'_i, R_{-i}) results in y winning, thus contradicting that R_i is dominated by R'_i.

Now consider two separate cases.

Case A: The voting rule is not CC with respect to sincere voting.

Consider any specific preference profile $(\succ_1, ..., \succ_n)$ and the sincere strategy profile $R_{\mathcal{N}} \equiv (R_i)_{i \in \mathcal{N}}$ for which Condorcet consistency is violated in sincere voting. By the above argument, each i submitting R_i is weakly undominated. Replicate this voting game sufficiently large with every voter with preference ordering $\succ_i$ submitting R_i (so that the *scale invariance* of Definition 6 applies) such that unilateral deviation does not alter the non-Condorcet outcome and hence constitute a Nash equilibrium in undominated strategies.

Case B: The voting rule is CC in sincere voting.

Consider the first three candidates c^1, c^2 and c^3. Without any loss of generality assume that c^3 is the candidate among the first three candidates that satisfies the property in condition 2 in Definition 6 (i.e. c^3 is in the role of candidate z in condition 2). Next, let $\kappa = \max\{\kappa' \mid 3\kappa' \leq n\}$, where n is the number of voters. Suppose that the true preference profile of the voters is such that the set of voters can be partitioned into three sets $\mathcal{S}^1, \mathcal{S}^2$ and $\mathcal{S}^3$ as follows: The set $\mathcal{S}^1$ consists of $n - 2\kappa$ voters and each $i \in \mathcal{S}^1$ has preferences given by $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^2$ consists of κ voters and each $i \in \mathcal{S}^2$ has preferences given by $c^2 \succ_i c^1 \succ_i c^3 \succ_i c^4 \succ_i ... \succ_i c^k$; the set $\mathcal{S}^3$ consists of κ voters and each $i \in \mathcal{S}^3$ has preferences given by $c^3 \succ_i c^1 \succ_i c^2 \succ_i c^4 \succ_i ... \succ_i c^k$. Then note that c^1 is the *CW*.

Now since the voting rule is *CC* in sincere voting, by condition 2 in Definition 6, $J = k$. Next let $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$ and consider for any $i \in \mathcal{S}$ the strategy $R_i = (c^2, c^1, c^3, ..., c^k)$. First we show that for any $i \in \mathcal{S}$, R_i is not weakly dominated.

Suppose not; then for some $i \in \mathcal{S}$, R_i is weakly dominated by another strategy $R'_i = (X'_1, ..., X'_k)$. Now since R_i is sincere if $i \in \mathcal{S}^2$ and voting sincerely is not weakly dominated, it follows that $i \in \mathcal{S}^1$ and $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$. Using this, we next establish in several steps that $X'_\tau = c^\tau$ for all $\tau \leq k$.

Step 1: We claim that $X'_1 \neq c^\tau$ for any $\tau > 2$. Suppose not; then by condition 1 in Definition 6 there exists R_{-i} such that R_i results in c^2 winning, and R'_i results in c^τ for some $\tau > 2$ winning, thus contradicting that R_i is dominated by R'_i.

Step 2: We claim that $X'_1 = c^1$. Suppose not; then by the previous step $X'_1 = c^2$. But since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $c^1 \succ_i c^2 \succ_i c^3 \succ_i ... \succ_i c^k$, it must then be that $X'_2 = c^1$. Otherwise $X'_2 = c^\tau$ for some $\tau > 2$, and by condition 1 there exists some R_{-i} such that (R_i, R_{-i}) elects c^1 whereas (R'_i, R_{-i}) elects c^τ, contradicting that R_i is dominated by R'_i. That is, from $X'_1 = c^2$ follows $X'_2 = c^1$, and continuing with a similar reasoning using induction yields $R'_i = R_i$. But this is a contradiction.

Step 3: We claim that for $X'_j = c^j$ for all $j \leq J$. Since by the previous step the claim is true for $j = 1$, by induction, it suffices to show that for any $j \le J$, if $X'_{j'} = c^{j'}$ for all $j' < j$, then $X'_j = c^j$. To show this suppose contrary to the claim that $X'_{j'} = c^{j'}$ for all $j' < j$ and $X'_j \ne c^j$. Then $X'_j = c^\tau$ for some $\tau > j$. This implies, by condition 1 in Definition 6, that there exists R_{-i} such that R_i results in either c^1 (if $j = 2$) or c^j (if $j > 2$) winning, and R'_i results in c^τ. Since i prefers both c^1 and c^j to c^τ ($\tau > j$), this contradicts R_i being dominated by R'_i.

Now since $R_i = (c^2, c^1, c^3, ..., c^k)$ and $R'_i = (c^1, c^2, c^3, ..., c^k)$, by condition 2 in Definition 6, there exists a strategy profile R_{-i} such that (R'_i, R_{-i}) elects c^3 whereas (R_i, R_{-i}) elects c^2. Since c^3 is worse than c^2 in i's true ranking, R'_i cannot weakly dominate R_i. But this is a contradiction. Hence R_i is not weakly dominated.

Now consider the strategy profile $R_\mathcal{N}$ in which every $i \in \mathcal{S}$ submits the strategy R_i, and the rest of the voters vote sincerely by submitting $(c^3, c^1, c^2, c^4, ..., c^k)$. First, note that by the previous arguments $R_\mathcal{N}$ is undominated. Next, we show that such a profile results in c^2 being elected. Consider any preference profile $\succ' = (\succ'_1, ..., \succ'_n)$ such that $c^2 \succ'_i c^1 \succ'_i c^3 \succ'_i ... \succ'_i c^k$ for every $i \in \mathcal{S}$ and $R_{i'}$ is sincere with respect to $\succ'_{i'}$ for every $i' \in \mathcal{N} \backslash \mathcal{S}$. Clearly, $R_\mathcal{N}$ is sincere with respect to $\succ'$. Moreover, since $\succ'$ is such that c^2 is the most preferred for every $i \in \mathcal{S}$ and the set $\mathcal{S}$ constitutes a majority, it follows that c^2 is the *CW* with respect to $\succ'$. Hence, since the voting rule is, by assumption, *CC* in sincere voting and $R_\mathcal{N}$ is sincere with respect to $\succ'$, it follows that c^2 must be elected when the voters submit $R_\mathcal{N}$.

Now assume that $n > 5$ and $R_\mathcal{N}$ is chosen. Then no individual voter can affect the outcome because for any single deviation there are at least $n - 2\kappa + \kappa - 1 = n - \kappa - 1 \ge 2\kappa - 1$ voters (the numbers of $\mathcal{S}_1$ and $\mathcal{S}_2$ minus 1) who put c^2 first. Since $2\kappa - 1$ forms a majority if $n > 5$ and the voting rule is *CC* with respect to sincere voting, it follows that c^2 is still elected if any single voter deviates. Thus the strategy profile $R_\mathcal{N}$ is a Nash equilibrium with undominated strategies, yielding the candidate c^2. But c^1 is the *CW* with respect to the true preferences. **Q.E.D.**

A glossary of voting rules (Moulin [18])

Binary voting: In binary voting (described by a binary tree) there are many rounds of voting and in each round voters vote over only two choices. The voting proceeds by elimination of choices through the voting rounds (often using majority rule) until the voting reaches a final stage at which each choice corresponds to the selection of a candidate. A special case of binary voting is the ***sequential binary voting*** (also known as the amendment procedure) in which the candidates are ordered before the voting rounds begin, then in the first round the voters choose between the first two candidates, in the second round they choose between the winner of the first round and the third candidate, and so on until the last round in which the voting is between the last candidate and the winner of the penultimate round.

Approval voting: A voter may approve or disapprove any number of candidates (point 1 indicates approval of a candidate and point 0 denotes disapproval) except that the voter cannot approve all or disapprove all the candidates. The candidate with maximal votes wins.

Borda rule: Voters strictly rank the candidates and a candidate's total score is then calculated based on scores associated with each rank. The candidate with the highest total score wins.

Negative voting: Each voter is asked to name a candidate whom he least likes to win. The candidate with the least number of such votes wins.

Instant runoff voting: Instant runoff voting, also known as *single transferrable voting*, requires voters to submit a full ranking of candidates in a single ballot. If no candidate wins a majority of the top rank, the candidate with the fewest top-rank votes gets eliminated and a fresh count is taken with rankings rearranged. This process continues until some candidate secures a majority of the top rank. A second variant of this voting does not use the *majority top-rank trigger*, instead eliminates candidates sequentially: start with the candidate with the minimum top-rank votes, then after vote transfers again eliminate who has the least top-rank votes, and so on (with ties broken by a deterministic tie-breaking rule).

Plurality runoff rule: All but two candidates are eliminated in the first round using plurality rule and then the winner is selected in a second ballot from the remaining two using majority rule.

Exhaustive ballot: It is same as the weakest link voting (described in the Introduction) except for the *majority vote trigger*.

Copeland and Simpson rules: These two one-shot voting rules are based on voters submitting only strict order rankings (so that $J = k$). For Copeland rule, candidate a, compared with another candidate b, is assigned a score $+1$ if a majority prefers a to b, -1 if a majority prefers b to a, and 0 if it is a tie. Summing up the scores over all $b, b \neq a$, yields the Copeland score of a. A candidate with the highest such score, called a *Copeland winner*, is elected. For Simpson rule, for candidate a

denote by $N(a, b)$ the number of voters preferring a to another candidate b. The Simpson score of a is the minimum of $N(a, b)$ over all $b, b \neq a$. A candidate with the highest such score, called a *Simpson winner*, is elected.

Supplementary Material

("Multi-Stage Voting, Sequential Elimination and Condorcet Consistency" by P.K. Bag, H. Sabourian and E. Winter)

25 November, 2008

Proof of Theorem 2. (Existence of a Markov equilibrium for the weakest link voting when $n \geq 2k - 1$.)

To prove existence we need to show that there exists a Markov strategy profile s^* such that at each stage it is Nash and undominated assuming that all players play according to s^* in any later stages. This is done by defining s^* inductively in subgames with a given number of candidates as the inductive variable, as follows.

First, let $k(h)$ denote the number of candidates at h. Then at any h with $k(h) = 2$, assume that voter i chooses *sincerely*. Clearly such a strategy profile is an undominated Nash equilibrium in this last stage and is independent of h.

Induction hypothesis. *Now suppose for all h' such that $k(h') \leq J - 1$, $s^*(h')$ is defined, and is undominated Nash and Markov*[15] *from here onwards.*

We need to define a profile of choices for all voters $s^*(h)$, for all h such that $k(h) = J$, such that $s^*(h)$ is an undominated Nash equilibrium and Markov from here onwards, assuming that all follow $s^*(h')$ at all later stages h' s.t. $k(h') \leq J-1$.

Fix any h s.t. $k(h) = J$. Let $C = \{c^1, ..., c^J\}$ be the set of candidates at h. Without any loss of generality assume that $c^{j'}$ is higher in the tie-breaking rule than c^j (i.e., if at all, c^j is eliminated before eliminating $c^{j'}$) if and only if $j' < j$.

Also let $\sigma(c)$ be the winner if c is eliminated at the start of play of the subgame $\Gamma(h)$. Notice that $\sigma(c)$ is unique because by the induction hypothesis $s^*(h)$, when there are $J - 1$ candidates, is independent of the past history.

Next define, for any $i \in \mathcal{N}$,

$$M_i = \begin{cases} \Theta_i & \text{if } \exists c \text{ and } c' \in C \text{ s.t. } \sigma(c) \neq \sigma(c'); \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\Theta_i = \{c \in C \mid \nexists c' \in C \text{ s.t. } \sigma(c') \succ_i \sigma(c)\}$ consists of voter i's best elimination candidate(s) in this round of play. Note that M_i is empty-valued when the subgame

[15]That is, the strategies depend only on the candidates around and not on the precise history leading up to it.

is degenerate (the identity of the eventual winner is independent of who is eliminated at this round). Finally, let $M_i^c = C \backslash M_i$. Clearly, $M_i^c \neq \emptyset$.

Lemma 1. *In the subgame $\widehat{\Gamma}(h)$, any $c \in M_i^c$ is not weakly dominated for voter i.*

Proof of Lemma 1. Suppose $M_i \neq \emptyset$ (if $M_i = \emptyset$, Lemma 1 holds trivially). Fix $c \in M_i^c$ and any $c' \in C$, $c' \neq c$. We want to argue that switching his vote from c to c' would be worse for voter i for at least one profile of other voters' votes.

If the tie-breaker places *some* $\hat{c} \in M_i$ ahead of c and $\hat{c} \neq c'$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(\hat{c}) = \vartheta(c) = 0 \text{ and } \vartheta(\tilde{c}) > 0, \ \forall \tilde{c} \neq \hat{c}, c,$$

where $\vartheta(\cdot)$ denotes the number of votes in favor of a candidate by all voters other than i. Now if i votes for c then the distribution of votes as above leads to the elimination of $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, candidates $\hat{c}$ and c will be tied with minimal votes and by the tie-breaker c will be eliminated, which is worse for voter i. If $\hat{c} = c'$, the argument holds with even greater force as c would be eliminated (as i switches to c') without having to invoke the tie-breaker.

If the tie-breaker is such that c is placed ahead of *all* $\hat{c} \in M_i$, let the distribution of votes of all the voters other than i be as follows:

$$\vartheta(c) = 0, \ \vartheta(\hat{c}) = 1 \ \forall \ \hat{c} \in M_i, \ \text{ and } \vartheta(\tilde{c}) \geq 2 \ \forall \tilde{c} \neq c, \ \tilde{c} \in M_i^c.$$

Now if i votes for c then this leads to the elimination of some $\hat{c}$. However, if i switches to c' while the rest stay with their votes as above, c will be unique with minimal votes and therefore be eliminated, which is worse for voter i. This completes the proof of Lemma 1. ||

Next for any $r = 1, ..., J$ we define the following property.

Definition 7. *Any $r \in \{2, ..., J\}$ satisfies property α if there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1})$ consisting of $2(r-1)$ different voters such that*

$$c^j \in M_i^c \ \text{ for } \ i = u_j, v_j \text{ for all } \ j < r. \tag{6}$$

Lemma 2. *Suppose that the following two conditions hold for some $1 \leq r < J$: (i) either $r = 1$ or r satisfies property α; and (ii) $r + 1$ does not satisfy property α. Then there exist a choice profile $s^*(h)$ that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 2. Given that r satisfies (i) and (ii) above, there exists a set of voters Ω (that is empty if $r=1$) consisting of $2(r-1)$ different voters $(u_1, v_1, u_2, v_2, ..., u_{r-1}, v_{r-1}) \subset \mathcal{N}$ such that

$$c^j \in M_i^c \text{ for } i = u_j, v_j \text{ for all } j < r \tag{7}$$

and there exists a set of voters $V \subset \mathcal{N} \backslash \Omega$ such that

$$|\, V \,| = n - 2(r-1) - 1 \tag{8}$$

and

$$c^r \in M_v \text{ for any } v \in V. \tag{9}$$

Let

$$C^r = \{c \in C \,|\sigma(c) = \sigma(c^r)\} \quad \text{and} \quad \overline{C}^r = \{C \backslash C^r\} \cap \{c^{r+1}, ..., c^J\}.$$

Since the preferences of each voter is strict, it follows that

$$\overline{C}^r \subset M_v^c \text{ for any } v \in V. \tag{10}$$

Also since $|\, V \,| = n - 2(r-1) - 1$, $|\, \overline{C}^r \,| \leq J - r$ and by assumption $n \geq 2k - 1 \geq 2J - 1$ and $r < J$, it follows that the number of voters in V is at least twice the number of candidates in $\overline{C}^r$. But this implies that there exists a choice profile $\{s_v^*(h)\}_{v \in V}$ such that

$$s_v^*(h) \in \overline{C}^r \text{ for each } v \in V, \tag{11}$$

$$|\{v \in V\,| s_v^*(h) = c\}| \geq 2 \text{ for each } c \in \overline{C}^r. \tag{12}$$

(The second condition says that each candidate $c \in \overline{C}^r$ receive at least two votes). Next set the choice $s_i^*(h)$ of each $i \in \Omega$ to be such that

$$s_i^*(h) = c^j \text{ for } i = u_j, v_j. \tag{13}$$

Finally, denote the remaining voter $\mathcal{N} \backslash \{V \cup \Omega\}$ by x and set the choice of voter x to be such that

$$\begin{array}{ll} s_x^*(h) \in M_x^c \backslash c^r & \text{if } M_x^c \backslash c^r \text{ is not empty;} \\ s_x^*(h) = c^r & \text{otherwise.} \end{array} \tag{14}$$

Now by Lemma 1 and conditions (7), (10), (11), (13) and (14), the choice $s_\ell^*(h)$ is undominated in this round for any voter $\ell \in \mathcal{N}$ and is Markov.[16] Next we show that $s^*(h) = \{s_\ell^*(h)\}_{\ell \in \mathcal{N}}$ is Nash. There are two possible cases.

[16]If $M_x^c \backslash c^r$ is an empty set then c^r must be an element of M_x^c because $M_x^c \neq \emptyset$. Thus, $s_x^*(h)$ is undominated.

Case A. $M_x^c \neq c^r$**.** First, note that by (12) and (13), in this round each candidate $c \in \overline{C}^r \cup \{c^1, ..., c^{r-1}\}$ receives at least two votes, c^r receives zero vote (follows from (14), given the fact that $M_x^c \neq \emptyset$ and $M_x^c \neq c^r$), and any other $c' \in C^r \cap \{c^{r+1}, \ldots, c^J\}$ receives at most one vote. This means that some candidate $c \in C^r$ is eliminated and $\sigma(c^r)$ will be the final winner. Moreover, since c^r receives zero vote, it must be that the eliminated candidate $c^e \in C^r$ receives zero vote and $e \geq r$.

Since, by (9), $\sigma(c^r)$ is a best outcome for each $v \in V$, it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Moreover, no voter $i \in \Omega$ can change the final outcome $\sigma(c^r)$ by changing its action because the choice $s_i^*(h) \in \{c^1, ..., c^{r-1}\}$ receives at least two votes, the eliminated candidate c^e has zero vote and $e \geq r$. Finally, voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because either the choice $s_x^*(h) \in \{c^1, ..., c^{r-1}\} \cup \overline{C}^r$, in which case $s_x^*(h)$ receives at least three votes and as before c^e has zero vote, or $s_x^*(h) \in C^r \cap \{c^{r+1}, \ldots, c^J\}$ in which case $s_x^*(h)$ receives one vote and any deviation results in some candidate in the set C^r to be eliminated.

Case B. $M_x^c = c^r$**.** Then for each $c' \neq c^r$, $c' \in M_x$. Therefore

$$\forall c', c'' \neq c^r, \;\; \sigma(c') = \sigma(c''). \tag{15}$$

This implies that $\overline{C}^r = \{c^{r+1}, ..., c^J\}$. But then, by (12) and (13), in this round each candidate $c \neq c^r$ receives at least two votes, c^r receives one vote (the vote of x), c^r is eliminated and $\sigma(c^r)$ will be the final winner. As in the previous case, since this is a best outcome for each $v \in V$ it follows that $s_v^*(h)$ is a best choice for any $v \in V$. Next note that for each voter $i = u_j, v_j$ for $j < r$ we have $s_i^*(h) = c^j \in M_i^c$ and thus $c^r \in M_i$. Therefore, eliminating c^r is also the best outcome for any $i \in \Omega$. Finally, note that voter x cannot change the final outcome $\sigma(c^r)$ by changing its action because every $c \neq c^r$ receives two votes. $||$

Lemma 3. *Suppose that J satisfies property α. Then there exist a choice profile $s^*(h)$ that is Nash, is not weakly dominated, and is Markov.*

Proof of Lemma 3. Given that J satisfies property α, there exists a set of voters $\Omega = (u_1, v_1, u_2, v_2, ..., u_{J-1}, v_{J-1})$ consisting of $2(J-1)$ different voters such that

$$c^j \in M_i^c \;\text{ for }\; i = u_j, v_j \text{ for all }\; j < J. \tag{16}$$

Set the choice profile $\{s_i^*(h)\}_{i\in\Omega}$ to be such that

$$s_i^*(h) = c^j \text{ if } i = u_j, v_j. \tag{17}$$

Also partition the remaining voters as follows:

$$\begin{aligned}\Gamma^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c = c^J\right\}\\ \overline{\Gamma}^J &= \left\{v \in \mathcal{N}\backslash\Omega | M_v^c \neq c^J\right\}.\end{aligned}$$

Let the choice profile $\{s_v^*(h)\}_{v\in\mathcal{N}\backslash\Omega}$ be such that

(i)
$$s_v^*(h) \in \begin{cases} c^J & \text{if } v \in \Gamma^J \\ M_v^c\backslash c^J & \text{if } v \in \overline{\Gamma}^J; \end{cases}$$

(ii) if Γ^J is non-empty,

$$|n(c) - n(c')| \leq 1 \;\; \forall c, c' \neq c^J \text{ s.t. } \sigma(c) = \sigma(c'), \tag{18}$$

where

$$n(c) = |\{v \in \mathcal{N}\, | s_v^*(h) = c\}| \quad \text{for any } c.$$

(Note that $M_v^c\backslash c^J \neq \emptyset$ for $v \in \overline{\Gamma}^J$.) Notice that if Γ^J is non-empty, (18) is possible for the following reasons. First, since Γ^j is non-empty,

$$\forall c', c'' \neq c^J, \;\; \sigma(c') = \sigma(c'') \neq \sigma(c^J). \tag{19}$$

Next note that each $c^j,\ j < J$ receives two votes from the set of voters Ω. The only other voters that vote for the candidates $c^j,\ j < J$ are from the set $\overline{\Gamma}^J$. Because Γ^J is non-empty it follows from (19) that for each $v \in \Gamma^J$, $M_v = \{c^1, \ldots, c^{J-1}\}$; therefore votes by the members of $\overline{\Gamma}^J$ can be arranged so that (18) is satisfied: the first member of $\overline{\Gamma}^J$ votes for c^1, the second for c^2 etc. until the $(J-1)$st member votes for c^{J-1}, the J-th member for c^1, $(J+1)$st for c^2 etc.

By Lemma 1, $s^*(h)$ is not weakly dominated. Also by definition, $s^*(h)$ is Markov. Next we show that it is a Nash equilibrium.

Case A. Γ^J is empty. Then every $c \neq c^J$ receives at least two votes, c^J receives no votes and is eliminated. This together with c^J having the lowest rank in the tie-breaking rule imply that no player can change the final outcome by changing their choices and thus $s^*(h)$ constitutes a Nash equilibrium.

Case B. Γ^J **is non-empty.** By (19), since for each $v \notin \Gamma^J$ there exists a $c \neq c^J$ such that $c \in M_v^c$, it follows that

$$\forall v \notin \Gamma^J, \;\; c^J \in M_v. \tag{20}$$

Now there are two possibilities.

Subcase 1: Candidate c^J is eliminated. Then, by (20), this is the best outcome for any $v \notin \Gamma^J$ and therefore, each such v is choosing his optimal action. Moreover, each $v \in \Gamma^J$ cannot change the outcome by deviating from $s_v^*(h)$ because $s_v^*(h) = c^J$ and c^J is the candidate that is eliminated.

Subcase 2: Some $c \neq c^J$ is eliminated. Then, by the tie-breaking rule

$$n(c) < n(c^J). \tag{21}$$

Next note that by (19) and the definition of Γ^J, this is the best outcome for any $v \in \Gamma^J$ and therefore, each such v is choosing his optimal action. Next we show that no voter $v \notin \Gamma^J$ can change the outcome by deviating. Suppose not; then some voter $v \notin \Gamma^J$ can deviate from $s_v^*(h) = c^j (\neq c)$ for some $j < J$ and change the final outcome $\sigma(c)$ by voting for another candidate. Since the outcome is changed, by (19), it must be that c^J is eliminated. This implies that

$$n(c^j) - 1 \geq n(c^J).$$

But this together with (21) imply that

$$n(c^j) > n(c) + 1$$

But this contradicts condition (18). Therefore no $v \notin \Gamma^J$ can change the final outcome by deviating. ||

The last two lemmas together establish that there exists a choice profile $s^*(h)$ that is Nash, not weakly dominated, and is Markov. **Q.E.D.**

Justifying the use of Markov strategies

Consider any voting game described in section 2, where at each round at least one candidate is eliminated. Recall, S_i is the strategy set of voter i with $s_i : \mathcal{H} \rightarrow \cup_{C,j} A_i(C,j)$ s.t. $s_i(h) \in A_i(C,j) \;\; \forall h \in \mathcal{H}_C^j$, where $\mathcal{H}_C^j = \mathcal{H}_C \cap \mathcal{H}^j$. Also, let $S = \Pi_i S_i$.

Definition 8. *A strategy $s_i \in S_i$ is more complex than another strategy $s'_i \in S_i$ if $\exists C$ and j s.t.*
(i) $s_i(h) = s'_i(h) \quad \forall h \notin \mathcal{H}_C^j$;
(ii) $s'_i(h) = s'_i(h') \quad \forall h, h' \in \mathcal{H}_C^j$;
(iii) $s_i(h) \neq s_i(h')$ for some $h, h' \in \mathcal{H}_C^j$.

The above ordering of complexity is only a partial ordering. Nevertheless, it will prove a powerful one for our purpose. Based on this ordering, let us refine our original definition of *equilibrium* as follows.

Definition 9. *A equilibrium strategy profile $s^* \in S$ will be called a simple equilibrium if for any $i \in \mathcal{N}$*

$$\nexists s_i \in S_i \text{ s.t. } w(s_i, s^*_{-i}) = w(s^*_i, s^*_{-i}) \text{ and } s^*_i \text{ is more complex than } s_i, \qquad (22)$$

where $w(s)$ is the winner if profile s is adopted.

Note that while the definition of *simple* equilibrium allows history-dependent (i.e., non-Markov) strategies, the condition in (22) reflects the implicit assumption that the voters are averse to complexity *unless* it helps to change the final outcome. Thus, simplicity of the simple equilibrium is a very weak, and in our view plausible, requirement for any descriptive analysis. We can therefore use the simplicity criterion for equilibrium selection.

Theorem. *Any simple equilibrium is also a Markov equilibrium.*

Proof. Suppose $s^* \in S$ is a simple equilibrium but not a Markov equilibrium. Then there exists some i, C, j and $h, h' \in \mathcal{H}_C^j$ s.t. $s^*_i(h) \neq s^*_i(h')$. Clearly, if $\mathcal{H}_C^j \cap E \neq \emptyset$ where E is the equilibrium path corresponding to the simple equilibrium s^*, then $\mathcal{H}_C^j \cap E$ is *unique*; that is, C happens on the equilibrium path at stage j at most once. Now consider another strategy $s_i \in S_i$ s.t.

$$\begin{aligned} s_i(h) &= s^*_i(h) \quad \forall h \notin \mathcal{H}_C^j; \\ \forall h \in \mathcal{H}_C, \quad s_i(h) &= \begin{cases} s^*_i(\mathcal{H}_C^j \cap E) & \text{if } \mathcal{H}_C^j \cap E \neq \emptyset, \\ a_i \in C & \text{if } \mathcal{H}_C^j \cap E = \emptyset, \end{cases} \end{aligned}$$

where a_i denotes any arbitrary element of C (note that s_i is well-defined because $\mathcal{H}_C^j \cap E$ is unique when defined).

It is easy to see that s_i is simpler than s^*_i. Moreover, because s_i differs from s^*_i only possibly for histories in $\mathcal{H}_C^j$ that are off-the-equilibrium path, (s_i, s^*_{-i}) will result in the same winner as the equilibrium s^*, so that $w_i(s_i, s^*_{-i}) = w_i(s^*_i, s^*_{-i})$. Hence, s^* cannot be a simple equilibrium – a contradiction. **Q.E.D.**

Proof of Proposition 3. We prove that the voting rules considered satisfy the two conditions in Definition 6 separately.

Condition 1

To show this, fix any x and y and any two strategies $R_i = (X_1, ..., X_J)$ and $R'_i = (X'_1, ..., X'_J)$ such that $x \in X_\tau$, $y \in X_{\tau'}$ and $\tau < \tau'$, so that $x\, P_i\, y$. Suppose also $x \in X'_\nu$, $y \in X'_{\nu'}$ and $\nu' < \nu$, so that $y\, P'_i\, x$. Also, let $m = (n-1)/2$ and consider the set of voters other than i. Enumerate this set (we are assuming an odd number of voters) and denote the enumeration by $\{\alpha_1, ..., \alpha_{2m}\}$, with a typical voter denoted as α_ℓ. Also enumerate the candidates other than x, y as $\{c^1, c^2, \ldots, c^{k-2}\}$.

Next, for any voter α_ℓ consider any strategy $R_{\alpha_\ell} = (\hat{X}_1, ..., \hat{X}_J)$ satisfying[17]

$$
\begin{array}{rcl}
x & \in & \begin{cases} \hat{X}_1 & \text{if either } \ \ell \leq m \quad \text{or} \quad M(1) > 1 \\ \hat{X}_2 & \text{if } \ \ell > m \quad \text{and} \quad M(1) = 1 \end{cases} \\
y & \in & \begin{cases} \hat{X}_1 & \text{if either } \ \ell > m \quad \text{or} \quad M(1) > 1 \\ \hat{X}_2 & \text{if } \ \ell \leq m \quad \text{and} \quad M(1) = 1 \end{cases} \\
\text{and} \qquad c^r & \in & \hat{X}_J \qquad \text{for voter } \ \alpha_r, \quad 1 \leq r \leq k-2.
\end{array}
$$

Thus, each of the candidates other than x and y is placed in at least one voter's lowest-ranked cell. This is possible because there are $k-2$ such candidates and $k - 2 \leq 2m$ (by assumption $k - 1 \leq n$).

Next, consider the different voting rules under consideration.

Scoring rules and Approval Voting: Denote the score attached to the j-th cell in either of the two voting rules by ς_{J-j+1}. Also, denote respectively the total score that any candidate c receives for strategy profile (R_i, R_{-i}) and (R'_i, R_{-i}) by $TS(c, R_i, R_{-i})$ and $TS(c, R'_i, R_{-i})$. Then it follows from the definition of R_{-i} above that:

$$
\left.
\begin{aligned}
TS(x, R_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\tau+1} \\
TS(y, R_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\tau'+1} \\
TS(c^r, R_i, R_{-i}) &\leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2,
\end{aligned}
\right\} \tag{23}
$$

$$
\text{where} \qquad \gamma = \begin{cases} \varsigma_J & \text{if } \ M(1) > 1 \\ \varsigma_{J-1} & \text{if } \ M(1) = 1. \end{cases}
$$

Therefore, it follows from (23) and $\varsigma_{J-\tau+1} > \varsigma_{J-\tau'+1} \geq \varsigma_1$ that $TS(x, R_i, R_{-i}) - TS(y, R_i, R_{-i}) > 0$, and $TS(x, R_i, R_{-i}) - TS(c^r, R_i, R_{-i}) > 0$. Therefore, (R_i, R_{-i}) results in x being elected.

[17] One should index the cells to reflect individualistic voting, but we keep to minimal notations.

Also, it follows from the definition of R_{-i} above that:

$$\begin{aligned} TS(x, R'_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu+1} \\ TS(y, R'_i, R_{-i}) &= m\varsigma_J + m\gamma + \varsigma_{J-\nu'+1} \\ TS(c^r, R'_i, R_{-i}) &\leq \varsigma_J + (2m-1)\gamma + \varsigma_1, \qquad 1 \leq r \leq k-2. \end{aligned}$$

But this together with $\varsigma_{J-\nu'+1} > \varsigma_{J-\nu+1} \geq \varsigma_1$ imply that $TS(y, R'_i, R_{-i}) - TS(x, R'_i, R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_{J-\nu+1} > 0$, and $TS(y, R'_i, R_{-i}) - TS(c^r, R'_i, R_{-i}) \geq \varsigma_{J-\nu'+1} - \varsigma_1 > 0$. Thus, (R'_i, R_{-i}) results in y being elected.

Instant runoff voting (with and without the majority top-rank trigger). For both variants of instant runoff voting, the strategy profile (R_i, R_{-i}) described above results in candidate x having the highest number of votes at each round and therefore in x being elected, and the strategy profile (R'_i, R_{-i}) described above results in candidate y having the highest number of votes at each round and therefore in y being elected (this follows from the same reasoning as in the previous case with scoring rules and approval voting).

Copeland rule. To calculate Copeland scores for (R_i, R_{-i}) submissions, let us do binary comparisons: comparing x against any other candidate yields x each time a score of $+1$, thus the Copeland score of x is $k-1$. Since $k-1$ is the maximum possible Copeland score, it follows that (R_i, R_{-i}) results in x being the winner. By the same reasoning, the Copeland score of y when (R'_i, R_{-i}) is chosen is $k-1$; therefore in this case the Copeland winner is y. Thus, condition 1 is satisfied.

Simpson rule. The strategy profile (R_i, R_{-i}) described above results in candidate x having the highest Simpson score and therefore being elected. This is because the Simpson score of x in this case is $m+1$ ($N(x,a) = 2m$ for all $a \neq x, y$ and $N(x,y) = m+1$), whereas the Simpson score of y is m and that of any other candidate $a \neq x, y$ is no greater than 1. By the same reasoning, it follows that the strategy profile (R'_i, R_{-i}) described above results in candidate y having the highest Simpson score and therefore being elected. ||

Condition 2

Scoring rules, approval voting and instant runoff voting (with and without the majority top-rank trigger). For these voting rules condition 2 holds vacuously because these voting rules are not CC with respect to sincere voting. To see this, consider each of the voting rules under consideration.

For scoring rules, the assertion follows from a three candidates, seventeen voters example due to Fishburn (1973) with a CW that fails to be elected under sincere voting (see also Theorem 9.1 in Moulin, 1988). To show that the same holds for arbitrary number of candidates k, consider Fishburn's example and add $k-3$ more candidates below the three candidates for all voters.

For approval voting and instant runoff voting (with and without the majority top-rank trigger), consider the following example of five voters and k candidates with strict preferences over candidates in a descending order:

$$\begin{aligned} 1,4 &: y,\ x,\ z,\ w, X_5, \ldots, X_k \\ 2,5 &: z,\ x,\ y,\ w, X_5, \ldots, X_k \\ 3 &: w,\ x,\ z,\ y, X_5, \ldots, X_k. \end{aligned}$$

While x is the CW, sincere voting (for approval voting, 'sincere' in the sense defined in our paper) will eliminate x under these voting rules and thus will not be CC.

Copeland rule and Simpson rule. Consider any three candidates $X = \{x, y, z\}$. Suppose that the tie-breaker places z above x and y. Fix any two strategies $R_i = (x, y, X_3, X_4, \ldots, X_k)$ and $R_i' = (y, x, X_3, X_4, \ldots, X_k)$, for any $(X_3, X_4, \ldots, X_k)$ with $X_\tau = z$ for some specific $\tau > 2$. Next, specify R_{-i} as follows: $(n-1)/2$ voters submit $(x, z, \underbrace{\ldots, X_{\tau-1}, y, X_{\tau+1}, \ldots})$ and $(n-1)/2$ other voters submit $(z, y, \underbrace{\ldots,\ X_{\tau-1}, x, X_{\tau+1} \ldots})$, where the two underbraced $(\underbrace{\ldots})$ rankings by the two groups of $(n-1)/2$ voters are otherwise the same as the ranking $(X_3, X_4, \ldots, X_k)$ except that X_τ's slot is filled in respectively by y and x.

Let us now calculate Copeland scores first. For R_i submission by i, comparing x against any other candidate yields x each time a score of $+1$, so candidate x's Copeland score $CSc(x) = k-1$, and thus x is the Copeland winner. On the other hand if i submits R_i' instead, the Copeland scores are calculated as follows. Candidate x: comparison x, y yields x the score -1 and comparison of x against any other candidate yields each time x the score $+1$, so $CSc(x) = k-3$. Candidate y: comparison y, z yields y the score -1 and comparison of y against any other candidate yields each time y the score $+1$, so $CSc(y) = k-3$. Candidate z: comparison z, x yields z the score -1 and comparison of z against any other candidate yields each time z the score $+1$, so $CSc(z) = k-3$. Since z is ahead of x and y in the tie-breaker, it follows that if R_i' is chosen z will be the Copeland winner (for any other candidate w, $CSc(w) \leq k-7$).

Next, consider the Simpson rule. For R_i submission by i, the Simpson scores are $SSc(x) = (n-1)/2 + 1,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) = 0$ for any other w; thus the Simpson winner is x. On the other hand, for R'_i submission the Simpson scores are $SSc(x) = (n-1)/2,\ SSc(y) = 1,\ SSc(z) = (n-1)/2$ and $SSc(w) \leq 1$ for any other w. With a tie-breaker placing z ahead of x and y, the Simpson winner is x. ||

Our required verifications for the specific one-shot voting rules are now complete. Thus, by Theorem 4, none of the voting rules considered in Proposition 3 are CC under strategic voting. **Q.E.D.**

Proof of Proposition 4.

[**Plurality runoff**] Suppose there are three voters and four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. The voters' ranking over candidates are as follows:

$$1: \quad x, w, y, z$$
$$2: \quad x, y, z, w$$
$$3: \quad z, w, y, x.$$

The CW is x. Also, $y\,T\,z\,T\,w\,T\,y$.

Under plurality-runoff rule, an equilibrium strategy profile is

$$1:\ y\ ; \qquad 2:\ z\ ; \qquad 3:\ \ w$$

in stage 1, followed by sincere voting in stage 2. In stage 1, x and w are eliminated, so that y is picked as the ultimate winner.

Given that sincere voting in stage 2 constitute a Nash equilibrium that is also weakly undominated, we only need to check that the proposed stage 1 strategies will be Nash equilibrium and weakly undominated. Clearly, the strategies are best responses to each other and therefore Nash in stage 1. So we will only verify that for any voter no other strategy weakly dominates his proposed equilibrium strategy.

The votes by voters 1 and 2 are the unique best responses, thus also weakly undominated. So let us consider voter 3's strategy. Let voters 1 and 2 choose in stage 1 respectively x and z. If voter 3 chooses w the outcome is z; on the other hand, if voter 3 chooses x or z the outcome is x, and if he chooses y the outcome is y, and both are worse compared to z.

Thus, plurality-runoff rule is not CC.[18]

[18]Note that given a counter example for a particular tie-breaking rule, one can construct similar

[Exhaustive ballot] There are three types of voters, with three voters of each type. There are three alternatives with the following preferences:

type 1 : x, y, z

type 2 : y, z, x

type 3 : x, z, y.

The CW is x.

Consider the following strategy profile: In round 1, type 1 voters vote for x and types 2 and 3 vote for z. In round 2, if reached, each type vote for the alternative (from the remaining two) which he prefers most.

The above strategy induces z as the winner. We claim that this strategy will be an equilibrium. That round 2 voting satisfies the equilibrium conditions is trivial. So consider round 1 voting. First, note that no player can gain by deviating unilaterally in this round (this is because each type has three voters). It thus remains to argue that no weakly dominated strategies are used in round 1. As type 1 voters vote for their top-ranked candidate, clearly the strategy is undominated. So we need to argue that voter types 2 and 3 are not using weakly dominated strategies in round 1. First consider type 3 voters. Let the strategy combination in round 1 be as follows: all type 1 vote for y, all type 2 vote for z, two type 3 vote for z and one type 3 votes for x. This leads to z being elected. If on the other hand one of the two type 3 voters who voted for z now switches to either y or x, then x will be eliminated and y is the ultimate outcome, which is worse for a type 3 voter. Consider now type 2 voters. Let the strategy combination in round 1 be as follows: all type 1 vote for x, all type 3 vote for z, one of type 2 votes for x and the other two vote for z. For this profile the outcome is z. If, however, one of the voters who earlier voted for z now switches to either y or x, then y will be eliminated and x is the ultimate outcome, which is worse for a type 2 voter.

[One-shot version of the weakest link voting] Consider the one-shot version of the weakest link game. Suppose there are four candidates, w, x, y, z. Fix a tie-breaking rule y, z, w, x. Below we specify voter preferences for which CW will not be elected. For any other tie-breaking rule, a counter example can be constructed by permuting voter preferences appropriately (see footnote 4).

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counter examples for other tie-breakers by simply permuting voter preferences in the same way the alternatives are permuted to obtain the new tie-breaking rule.

Consider three types of voters with two voters of each type with the following preferences:

$$\begin{aligned} \text{type 1} &: \; y, \; x, \; z, \; w \\ \text{type 2} &: \; z, \; x, \; y, \; w \\ \text{type 3} &: \; w, \; x, \; z, \; y. \end{aligned}$$

The CW is x.

For only two candidates sincere submission (i.e., voting for one's favorite candidate) is clearly the only undominated strategy for any voter. We now specify the strategies for each type of voter for three and four candidates:

$$\begin{array}{lllll} s_1(w,y,z)=y, & \mathbf{s_1(x,y,z)=x}, & s_1(w,x,z)=x, & s_1(w,x,y)=y, & s_1(w,x,y,z)=y \\ s_2(w,y,z)=z, & s_2(x,y,z)=z, & s_2(w,x,z)=z, & s_2(w,x,y)=x, & s_2(w,x,y,z)=z \\ s_3(w,y,z)=w, & s_3(x,y,z)=x, & \mathbf{s_3(w,x,z)=x}, & \mathbf{s_3(w,x,y)=x}, & s_3(w,x,y,z)=w. \end{array}$$

We verify that the strategies constitute a Nash equilibrium. For candidates $\{w, y, z\}$, the proposed strategies lead to z as the ultimate winner (given that for any two candidates under consideration the voters would vote sincerely). For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or z, still result in z as the winner; for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{x, y, z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to y or z, for a type 2 voter deviation to y or x, and for a type 3 voter deviation to y or z – all leave the voting outcome unchanged (i.e., x is the winner).

For candidates $\{w, x, z\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or z, and for a type 3 voter deviation to y or w, still result in x as the winner; for a type 2 voter deviation to w or x also result in x as the winner (in the first deviation, w gets eliminated using the tie-breaker).

For candidates $\{w, x, y\}$, the proposed strategies lead to x as the ultimate winner. For a type 1 voter deviation to w or x, and for a type 3 voter deviation to w or y, leave the winner x unchanged; and for a type 2 voter clearly deviation cannot be optimal.

For candidates $\{w, x, y, z\}$, the proposed strategies lead to x's elimination in the first round and ultimately lead to z being the ultimate winner. Given that x is placed below other alternatives in the tie-breaker and there are two voters of each type with the same types choosing the same strategy in the proposed equilibrium, no individual voter can prevent x's elimination by altering his vote. Thus, deviation by any of the voters is never optimal.

We next argue why the proposed strategies are undominated as well. We will make our assertions for only the three vote choices indicated in bold; for the remainder, to verify that the choices are undominated is easy because in each case the voter votes for his top-ranked candidate.

First, consider a type 1 voter's decision $s_1(x, y, z) = x$. To show that for this voter voting for y cannot (weakly) dominate, suppose all other five voters vote for z, and for only two candidates remaining all vote sincerely. Then the type 1 voter by voting for y will induce z as the winner, whereas if he votes for x the winner is x; and he prefers x over y. To show that voting for z cannot dominate either, suppose the other type 1 voter votes for x and the remaining four voters all vote for y. Then if the type 1 voter votes for z the winner is z, whereas if he votes for x the winner is x; and he prefers x over z.

Next consider a type 3 voter's decision $s_3(w, x, z) = x$. To show that voting for w cannot dominate, suppose the other type 3 voter votes for x and the remaining four voters all vote for z. Then if the type 3 voter votes for w, the tie-breaker would eliminate x and sincere voting in the next round of elimination would elect z; on the other hand, if the type 3 voter voted for x then in the next round of elimination (with x and z as the candidates) sincere voting would elect x, which the type 3 voter prefers over z. To show that voting for z cannot dominate either, suppose the other type 3 voter votes for x, one from the remaining four votes for w and three others vote for z. Then if the type 3 voter votes for z, the tie-breaker eliminates x and sincere voting (with two candidates remaining) elects z as the winner. On the other hand, if the type 3 voter votes for x, alternative w will be eliminated and sincere voting with two candidates remaining would elect x, which the type 3 voter prefers.

Finally, by an argument similar to the one just given, one can show that a type 3 voter's decision $s_3(w, x, y) = x$ is undominated.

Thus, we have constructed a Nash equilibrium with undominated strategies in which the *CW*, x, is eliminated and z gets elected. **Q.E.D.**

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