Contracting with Heterogeneous Externalities[*]

Shai Bernstein[†]   Eyal Winter[‡]

August 29, 2011

Abstract

We model situations in which a principal offers contracts to a group of agents to participate in a project. Agents' benefits from participation depend on the identity of other participating agents. We assume heterogeneous externalities and characterize the optimal contracting scheme. We show that the optimal contracts' payoff relies on a ranking of the agents, which arise from a tournament among the agents. The optimal ranking cannot be achieved by a simple measure of popularity. Using the structure of the optimal contracts we derive results on the principal's revenue extraction and the role of the level of externalities' asymmetry.

Keywords: Bilateral contracting, heterogeneous externalities, mechanism design

[†]Harvard University, e-mail: sbernstein@hbs.edu
[‡]Hebrew University, Center for the Study of Rationality, e-mail: mseyal@mscc.huji.ac.il

1   Introduction

What is the optimal structure of contracts to induce a group of agents to participate in a joint activity? How should these contracts take into account the complex externalities that prevail among the agents? These questions arise in various settings. Governments around the world seek to foster growth and innovation by emulating the success of Silicon Valley and creating planned science parks.[1] To attract companies, policy makers devote substantial resources.[2] Can governments lower the costs of establishing science parks by exploiting the heterogeneous externalities that arise between companies? Mall owners use such a strategy when leasing stores. Gould et al. (2005) show that national brand stores (which attract the most consumer traffic to malls) are being used to attract leases of smaller stores. These smaller stores generate most of the mall owners' leasing revenue.

In many situations a group member decision to participate depends on the choices of others. These relations are hardly symmetrical; in particular, participation choices may depend not only on how many members decide to participate, but also on the identity of the other participating agents. In a mall, a small store substantially gains from the presence of national brand stores, which attract a high volume of buyer traffic. The opposite externality, induced by the small store, has hardly any effect. The recruitment of a senior star to an academic department can easily attract a junior researcher to apply to that department. Invited party guests base their participation decisions on the participation of their close friends. In all of these examples the relations between the agents should be taken into account when structuring incentives.

In this paper we analyze a principal's problem of coordinating participation given heterogeneous externalities between group members. We explore a project initiated by a principal, when its success depends on the participation of a group. The principal structures a set of incentive contracts to coordinate the group members' participation. Such incentives can be tax credits, discounts, gifts, celebrities' participation, or any other benefits that are conditional on an agent's participation. We characterize the optimal, i.e., the least expensive, contracts that induce the participation of the group members.

---

[1] The International Association of Science Parks (IASP) currently has members in 49 countries outside the United States. According to the IASP, the number of science parks in the U.S. alone has increased from 16 in 1980 to 170 in 2010.

[2] For example, Hong Kong spent more than $2 billion to develop a planned research and development park (Cheng 1999).

In our model, the heterogeneous externalities are additive and described by a matrix whose entry $w_i(j)$ represents the extent to which agent i benefits from joint participation with agent j.[3] Following Segal (2003) we focus on situations in which the principal cannot coordinate agents to his preferred equilibrium in a multiple-equilibria setting. That is, we mainly focus on contracts that sustain agents' participation in a unique Nash equilibrium.[4] This set of contracts is of the form of *divide and conquer*.[5] For any given ranking of the agents, divide-and-conquer contracts are structured in the following way: offer each agent a reward that would convince him to participate in the belief that the agents who precede him in the ranking participate, and all subsequent agents abstain. Thus, the optimal contract is achieved by the ranking (henceforth, optimal ranking) that produces the least expensive divide-and-conquer incentive scheme.

Given the complex relations between the agents due to heterogeneity we ask: (1) Who should be getting a higher-powered incentive for participation? In other words, how should we determine the optimal ranking of the agents? (2) How do changes in the structure of externalities affect the principal's cost of sustaining the group's participation?

We show that the optimal ranking can be constructed using a virtual popularity tournament between the agents. In this tournament, we say that agent i beats agent j if agent j's benefit from i's participation is greater than i's benefit from j's participation. This binary relation is described by a directed graph. We use basic graph theory arguments to characterize the optimal ranking which depends on the number of winnings in the virtual tournament.[6] Hence, the agents' payoffs are determined with respect to their success in the tournament. This idea that agents who induce higher externalities receive higher-powered incentive rewards is supported by an empirical paper by Gould et al. (2005) who demonstrate that while national brand stores occupy over 58% of the total leasable space in shopping malls they pay only 10% of the total rent collected by the mall owners.

---

[3]We consider non-additive externalities in Section 4 of the paper

[4]Recent experimental papers (see Brandt and Cooper 2005) indicate that in an environment of positive externalities agents typically are trapped in the bad equilibrium of no-participation.

[5]Segal (2003) uses a similar structure to characterize a setting of homogeneous externalities. Che and Yoo (2001) show that a similar structure arises as an optimal mechanism in a moral hazard in team setups.

[6]The ranking is directly determined by the number of winnings if the directed graph is acyclic. If the graph is cyclic, ranking depends on the number of winnings as well as on the differences between agents' externalities.

A key characteristic of group externalities is the level of asymmetry[7] between the pairs of agents, which we show to reduce the principal's cost. Greater asymmetry offers the principal more leverage in exploiting the externalities to lower costs. This result has a significant implication on the principal's choice of group for the initiative in the selection stage.

Our problem surprisingly connects to two quite distinct topics: (1) ranking sport teams based on tournament results, which has been discussed in the Operations Research literature, and (2) ranking candidates in a voting problem based on the outcomes of pairwise elections, which was suggested by Condorcet (1785). Condorcet's solution uses a similar approach to ours, where candidates are the nodes in the graph, and the arcs' directions are the election results of pair-wise voting.

This work is part of an extensive body of literature on multi-agent contracting in which externalities arise between the agents.[8] Our general approach is closely related to the seminal papers by Segal (1999, 2003) on contracting with externalities. Segal (2003) introduced a general model of trade contracts that admit externalities among agents. He shows that increasing externalities implies that the principal gains from using a divide-and-conquer mechanism, when he cannot coordinate players to play his most-preferred equilibrium. Segal's model is sufficiently general to fit nicely into a variety of IO applications (like takeovers, vertical contracting, exclusive dealing, and network externalities). While Segal (2003) defines the divide-and-conquer mechanism in a general contracting setup that allows for heterogeneity, he doesn't solve for the optimal mechanism except for special cases such as the symmetric case (although he is able to obtain some comparative static results without deriving the optimal mechanism explicitly). Our objective here is to solve for the optimal mechanism for any matrix of externalities. While our environment is more restrictive than Segal's in the sense that agents' choices are binary (participate or not), we develop a sharper characterization by allowing externalities to be heterogeneous

---

[7]By asymmetry we refer to the sum of differences in bilateral externalities.

[8]To give a few examples, these applications include vertical contracting models (Katz and Shapiro 1986a; Kamien, Oren, and Tauman 1992) in which the principal supplies an intermediate good to N identical downstream firms (agents), which then produce substitute consumer goods; employment models (Levin 2002) in which a principal provides wages to induce effort in a joint production of a group of workers; exclusive dealing models (Rasmusen, Ramseyer, and Wiley 1991; Segal and Whinston 2000) in which the principal is an incumbent monopolist who offers exclusive dealing contracts to N identical buyers (agents) in order to deter the entry of a rival; acquisition for monopoly models (Lewis 1983; Kamien and Zang 1990; Krishna 1993) in which the principal makes acquisition offers to N capacity owners (agents); and network externalities models (Katz and Shapiro 1986b).

and thus capture the contracting implications of complex relations between the agents.

In fact, most of the literature assumes that externalities are homogeneous (in our setting, such an assumption implies that the benefit from joint participation depends on the number of participants), and not on the identity of the agents. Exceptions to this assumption are Jehiel and Moldovanu (1996) and Jehiel, Moldovanu, and Stachetti (1996), who consider an auction in which a single indivisible object is sold to multiple heterogeneous agents. Jehiel and Moldovanu (1999) introduce resale markets and consider the implications of the identity of the initial owner of the good to the initial consumer. Our paper is also related to Milgrom and Roberts (1990) who pointed out that a principal can gain from collusion or coordination among his agents in an interaction that gives rise to strategic complementarity.

We consider several extensions to verify the robustness of our assumptions. First, we study situations in which agents' choices are sequential and we show that our solution applies when the principal is interested in implementing effort via a stronger solution concept that admits a dominant strategy for each player at his relevant subgame. We show that the analysis remains valid when we allow the externalities to affect agents' outside options, as well as for more complicated contingent contracts. We consider more general externalities structures. In particular, we allow externalities to be both negative and positive, and provide the conditions under which the solution for the mixed externalities participation problem can be derived by decomposing the mixed problem to two problems one of which is positive and the other negative. Finally, we consider the case of a non-additive externalities structure.

The rest of the paper is organized as follows. In Section 2 we introduce the general model. Section 3 provides the solution to a participation problem with positive externalities between the agents. In Section 4 we consider several extensions of the model, by which we demonstrate that our results apply in more general settings. Section 5 demonstrates how the model can be used to solve selection problems and Section 6 concludes. Proofs are presented in the Appendix.

2   The Model

A participation problem is given by a triple (N, w, c) where N is a set of n agents. The agents' decision is binary: participate in the initiative or not. The structure of externalities w is an $n \times n$ matrix specifying the bilateral externalities between the agents. An entry $w_i(j)$ represents agent i's added value from participation in the initiative jointly with agent j. Agents gain no additional benefit from their own participation, i.e., $w_i(i) = 0$. Agents' preferences are additively separable; i.e., agent i's utility from participating jointly with a group of agents M is $\sum_{j \in M} w_i(j)$ for every $M \subseteq N$. In the extensions section we consider a model in which agents' preferences are non-additive; i.e., externalities are defined over all subsets of agents in group N.

We assume that the externalities structure w is fixed and exogenous. Also, c is the vector of the outside options of the agents. For simplicity, and with a slight abuse of notation, we assume that every outside option is constant and equals c for all agents. In the extensions section we demonstrate that our results hold also when the outside options are affected by the participation choices of the agents.

We assume that contracts offered by the principal are simple and descriptive in the sense that the principal cannot provide payoffs that are contingent on the participation behavior of other agents. Many of the examples discussed above seem to share this feature. Based on the data used by Gould et al. (2005) which includes contractual provisions of over 2,500 stores in 35 large shopping malls in the U.S., there is no evidence that contracts make use of such contingencies. The theoretical foundation for the absence of such contracts is beyond the scope of this paper. One possible explanation is the complexity of such contracts. In Section 5 we demonstrate that our analysis remains valid even if we allow contingencies to be added to the contracts.

The set of contracts offered by the principal can be described as an incentives vector $v = (v_1, v_2, ..., v_n)$ in which agent i receives a payoff of v_i if he decides to participate and zero otherwise. v_i is not constrained in sign and the principal can either pay or charge the agents but he cannot punish them for not participating (limited liability). Given

a contracting scheme v, agents face a normal-form game $G(v)$.[9],[10] Each agent has two strategies in the game: participation or abstention. For a given set M of participating agents, each agent $i \in M$ earns $\sum_{j \in M} w_i(j) + v_i$ and each agent $j \notin M$ earns c, his outside option. We define *full implementation* contracts to be contracts that induce group participation as a unique Nash equilibrium. Alternatively, *partial implementation* contracts induce the group to participate in a Nash equilibrium, which is not necessarily unique.

3   Contracting with Positive Externalities

Positive externalities are likely to arise in many contracting situations. Network goods, opening stores in a mall and attracting customers, and contributing to public goods are a few such examples. In this section we consider situations in which agents benefit from the participation of the other agents in the group. Suppose that $w_i(j) > 0$ for all $i, j \in N$, such that $i \neq j$. In this case, agents are more attracted to the initiative as the set of participants grows.

   As a first step toward characterizing optimal full implementation contracts, we show in Proposition 1 that an optimal contracting scheme is part of a general set of contracts characterized by the *divide-and-conquer*[11] property. This set of contracts is constructed by ranking agents in an arbitrary fashion, and by offering each agent a reward that would induce him to participate in the belief that all the agents who precede him in the ranking participate and all subsequent agents abstain. Due to positive externalities, "later" agents are induced to participate (implicitly) by the participation of others and thus can be offered smaller (explicit) incentives. More formally, the *divide-and-conquer*

---

[9]We view the participation problem as a reduced form of the global optimization problem faced by the principal, which involves both the selection of the optimal group for the initiative and the design of incentives. Specifically, let U be a (finite) universe of potential participants. For each $N \subseteq U$ let $v^*(N)$ be the total payment made in an optimal mechanism that sustains the participation of the set of agents N. The principal will maximize the level of net benefit he can guarantee himself, which is given by the following optimization problem: $\max_{N \subseteq U}[u(N) - v^*(N)]$, where $u(N)$ is the principal's gross benefit from the participation of the set N of agents and is assumed to be strictly monotonic with respect to inclusion; i.e., if $T \subsetneq S$, then $u(T) < u(S)$.

[10]In the extensions section we also consider the case of a sequential offers game.

[11]See Segal (2003) and Winter (2004) for a similarly structured optimal incentive mechanism in a setting of homogeneous externalities.

(*DAC*) contracts have the following structure:

$$v = (c, c - w_{i_2}(i_1), c - w_{i_3}(i_1) - w_{i_3}(i_2), ..., c - \sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary order of agents. We say that v is a DAC contracting scheme with respect to the ranking φ. The following proposition, which is similar to the analysis in Segal (2003, subsection 4.1.1) provides a necessary condition for optimal contracts.

Proposition 1 *If v is an optimal full implementation contracting scheme then it is a divide-and-conquer contracting scheme.*

Note that given contracting scheme v, agent i_1 has a dominant strategy in the game $G(v)$ to participate.[12] Given the strategy of agent i_1, agent i_2 has a dominant strategy to participate as well. Agent i_k has a dominant strategy to participate provided that agents i_1 to i_{k-1} participate as well. Therefore, contracting scheme v sustains full participation through an iterative elimination of dominated strategies.

3.1 Optimal Ranking

The optimal contracting scheme satisfies the divide-and-conquer property with the ranking that minimizes the principal's payment. The optimal ranking is determined by a virtual popularity tournament among the agents, in which each agent is "challenged" by all the other agents. The results of the matches between all pairs of agents are described by a *simple* and *complete*[13] directed graph $G(N, A)$, where N is the set of nodes and A is the set of arcs. N represents the agents, and $A \subset N \times N$ represents the results of the matches, which is a binary relation on N. We refer to such graphs as *tournaments*.[14] More precisely, the set of arcs in tournament $G(N, A)$ is as follows:

---

[12]Since rewards take continuous values we assume that if an agent is indifferent then he chooses to participate. Alternatively, we can define an optimal contract to be a vector of payments v such that any arbitrary increase of the payoffs of all players will result with a unique equilibrium in which all agents participate. Note also that while the indifferences may generate multiple Nash equilibria, full participation is a unique Nash equilibrium with iterative elimination of weakly dominated strategies (without assuming that agents participate whenever they are indifferent).

[13]A directed graph $G(N, A)$ is *simple* if $(i, i) \notin A$ for every $i \in N$ and *complete* if for every $i, j \in N$ at least $(i, j) \in A$ or $(j, i) \in A$.

[14]We allow that (i, j) and (j, i) are both in A.

(1) $w_i(j) < w_j(i) \iff (i,j) \in A$

(2) $w_i(j) = w_j(i) \iff (i,j) \in A$ *and* $(j,i) \in A$

The interpretation of a directed arc (i,j) in the tournament G is that agent j values mutual participation with agent i more than agent i values mutual participation with agent j. We simply say that agent i *beats* agent j whenever $w_i(j) < w_j(i)$. In the case of a two-sided arc, i.e., $w_i(j) = w_j(i)$, we say that agent i is *even* with agent j and the match ends in a tie.

In characterizing the optimal contracts we distinguish between cyclic and acyclic tournaments. We say that a tournament is *cyclic* if there exists at least one node v for which there is a directed path starting and ending at v, and *acyclic* if no such path exists for all nodes.[15] The solution for cyclic tournaments relies on the acyclic solution, and therefore the acyclic tournament is a natural first step.

3.2 Optimal Ranking for Acyclic Tournaments

A ranking φ is said to be *consistent* with tournament $G(N, A)$ if for every pair $i, j \in N$, if i is ranked before j in φ, then i beats j. In other words, if agent i is ranked higher than agent j in a consistent ranking, then agent j values agent i more than agent i values j. We start with the following graph theory lemma:

Lemma 1 *If tournament* $G(N, A)$ *is acyclic, then there exists a unique ranking that is consistent with* $G(N, A)$.

We refer to the unique consistent ranking proposed in Lemma 1 as the *tournament ranking*.[16] In the tournament ranking, each agent's location in the tournament ranking is determined by the number of his wins. Hence, the agent ranked first is the agent who won all matches and the agent ranked last lost all matches. As we demonstrate later, there may be multiple solutions when tournament $G(N, A)$ is cyclic. Proposition 2 provides the solution for participation problems with acyclic tournaments, and shows that the solution is unique.

Proposition 2 *Let* (N, w, c) *be a participation problem for which the corresponding tournament* $G(N, A)$ *is acyclic. Let* φ *be the tournament ranking of* $G(N, A)$. *The*

---

[15]By definition, if $(i,j) \in A$ and $(j,i) \in A$, then the tournament is cyclic.

[16]The tournament ranking is actually the ordering of the nodes in the unique Hamiltonian path of tournament $G(N, A)$.

optimal full implementation contracting scheme is given by the DAC with respect to φ.

The intuition behind Proposition 2 is based on the notion that if agents $i, j \in N$ satisfy $w_i(j) < w_j(i)$ then the principal is able to reduce the cost of incentives by $w_j(i)$, rather than by only $w_i(j)$, by giving preferential treatment to i and placing him higher in the ranking. Applying this notion to all pairs of agents minimizes the principal's total payment to the agents, since it maximizes the inherent value of the participants from the participation of the other agents.

The optimal contracting scheme can be viewed as follows. First the principal pays the outside option c for each one of his agents. The winner of each match in the virtual tournament is the agent who imposes a higher externality on his competitor. The loser of each match pays the principal an amount equal to the benefit that he gets from mutually participating with his competitor. The total amount paid depends on the size of bilateral externalities and not merely on the number of winning matches. However, the higher agent i is located in the tournament, the lower is the total amount paid to the principal.

An intuitive solution for the participation problem is to reward agents according to their level of popularity in the group, such that the most popular agents would be the most rewarded. A possible interpretation of popularity in our context would be the sum of externalities imposed on others by participation, i.e., $\sum_{j=1}^{n} w_j(i)$. However, as we have seen, agents' ranking in the optimal contracting scheme is determined by something more refined than this standard definition of popularity. Agent i's position in the ranking depends on the set of peers that value agent i's participation more than i values theirs. This two-way comparison may result in a different ranking than the one imposed by a standard definition of popularity. This can be illustrated in the following example in which agent 3 is ranked first in the optimal contracting scheme despite being less "popular" than agent 1.

Example 1 *Consider a group of four agents with an identical outside option $c = 20$. The externalities structure of the agents is given by matrix w, as shown in Figure 1. The tournament G is acyclic and the tournament ranking is $\varphi = (3, 1, 2, 4)$. Consequently, the set of optimal contracts is $v = (20, 17, 14, 10)$, which is the divide-and-conquer contracting scheme with respect to the tournament ranking. Note that agent 3 who is ranked first is not the agent who has the maximal $\sum_{j=1}^{n} w_j(i)$.*

$$w = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 4 & 0 & 2 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 5 & 2 & 0 \end{pmatrix} \implies$$
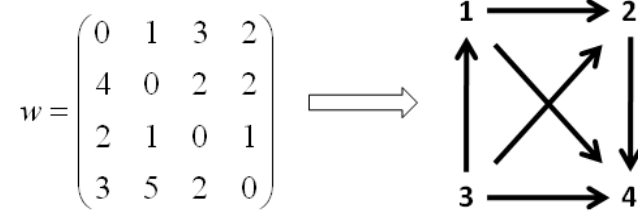

Figure 1

The derivation of the optimal contracting scheme requires the rather elaborate step of constructing the virtual tournament. However, it turns out that a substantially simpler formula can derive the cost of the optimal contracts. Two terms play a role in this formula: the first measures the aggregate level of externalities, i.e., $K_{agg} = \sum_{i\, j} w_i(j)$; the second measures the bilateral asymmetry between the agents, i.e., $K_{asym} = \sum_{i<j} |w_i(j) - w_j(i)|$. Hence, K_{asym} stands for the extent to which agents induce mutual externalities on each other. The smaller the value of K_{asym} the higher the degree of mutuality of the agents. Proposition 3 shows that the cost of the optimal contracting scheme is additive and declining in these two measures.

Proposition 3 *Let (N, w, c) be a participation problem and V_{full} be the principal's cost of the optimal full implementation contracts. If the corresponding tournament $G(N, A)$ is acyclic then $V_{full} = n \cdot c - \frac{1}{2}\left(K_{agg} + K_{asym}\right).$*

An interesting consequence of Proposition 3 is that for a given level of aggregate externalities, the principal's payment is decreasing with a greater level of asymmetry among the agents, as stated in Corollary 3.1.

Corollary 3.1 *Let (N, w, c) be a participation problem with an acyclic tournament. Let V_{full} be the principal's cost of the optimal full implementation contracts. For a given level of aggregate externalities, V_{full} is strictly decreasing with the asymmetry level of the externalities within the group of agents.*

The intuition behind this result is related to the virtual tournament discussed above. In each match the principal extracts "fines" from the losing agents. It is clear that these fines are increasing with the level of asymmetry (assuming $w_i(j) + w_j(i)$ is kept constant). Hence, a higher level of asymmetry allows the principal more leverage in exploiting the

externalities. This observation has important implications for the principal's selection stage.

Consider the comparison between the optimal full and partial implementation contracts, where in the latter the principal suffices with the existence of a full participation equilibrium, not necessarily unique. With partial implementation, the cost for the principal in the optimal contracting scheme is substantially lower. More specifically, in the least costly contracting scheme that induces full participation, each agent i receives $v_i = c - \sum_j w_i(j)$. However, these contracts entail a no-participation equilibrium as well; hence coordination is required. The total cost of the partial implementation contracts is $V_{partial} = n \cdot c - \sum_{i \ j} w_i(j)$ and the principal can extract the full revenue generated by the externalities.[17]

It is worth mentioning that for a fixed level of aggregate externalities, the difference between full and partial implementation contracts, $V_{full} - V_{partial}$, is stricty decreasing with the level of asymmetry of the externalities within the group. In the extreme case where $K_{asym} = 0$ ($w_i(j) = w_j(i)$ for all pairs), then the cost of moving from partial to full implementation is the most expensive. On the other hand, when externalities are always one-sided; i.e., for each pair of agents $i, j \in N$ satisfies that either[18] $w_i(j) = 0$ or $w_j(i) = 0$, then the additional cost is zero and full implementation is as expensive as partial implementation.

Note that increasing the aggregate level of externalities will not necessarily increase the principal extraction of revenue in the optimal contracting scheme. For example, in an asymmetric two-person problem raising slightly the externality that the less attractive agent induces on the other one will not change the principal's revenue.[19] From the perspective of the agents, their reward is *not* a continuous increasing function of the externalities they impose on the others. However, it is possible that a slight change in these externalities may increase rewards significantly, since a minor change in externalities

---

[17]Our emphasis on full implementation is motivated by the fact that under most circumstances the principal cannot coordinate the agent to play his most-preferred equilibrium. Brandts and Cooper (2005) report experimental results that speak to this effect. Agents' skepticism about the prospects of the participation of others trap the group in the worst possible equilibrium even when the group is small. Nevertheless, one might be interested in evaluating the cost of moving from partial to full implementation.

[18]Since this section deals with positive externalities, assume that $w_i(j) = \varepsilon$ or $w_j(i) = \varepsilon$ when ε is very small.

[19]It can be shown that in an n-person asymmetric problem one can raise the externalities in half of the matrix's entries (excluding the diagonal) without affecting the principal's surplus extraction.

may change the optimal ranking and thus affect agents' payoffs.

The extreme asymmetric case nicely contrasts with the completely symmetric case, where the principal's surplus increases with any slight increase of the externalities. With partial implementation, which allows the principal full extraction of surplus, the principal's revenue is sensitive to the values of externalities whether the problem is symmetric or asymmetric.

3.3   Optimal Ranking of Cyclic Tournaments

In the previous section we demonstrated that optimal full implementation contracts are derived from a virtual tournament among the agents in which agent i beats agent j if $w_i(j) < w_j(i)$. However, the discussion was based on the tournament being acyclic. If the tournament is cyclic, the choice of the optimal DAC contracting scheme (i.e., the optimal ranking) is more delicate since Lemma 1 does not hold. Any ranking is prone to inconsistencies in the sense that there must be a pair i, j such that i is ranked above j although j beats i in the tournament. To illustrate this point, consider a three-agent example where agent i beats j, agent j beats k, and agent k beats i. The tournament is cyclic and any ranking of these agents necessarily yields inconsistencies. For example, take the ranking $\{i, j, k\}$, which yields an inconsistency involving the pair (k, i) since k beats i and i is ranked above agent k. This applies to all possible rankings of the three agents.

The inconsistent ranking problem is similar to problems in sports tournaments, which involve bilateral matches that may turn out to yield cyclic outcomes. Various sports organizations (such as the National Collegiate Athletic Association - NCAA) nevertheless provide rankings of teams/players based on the cyclic tournament outcome. Extensive literature in Operations Research suggests solution procedures for determining the "minimum violation ranking" (e.g., Kendall 1955, Ali et al. 1986, Cook and Kress 1990, and Coleman 2005) that selects the ranking for which the number of inconsistencies is minimized. It can be shown that this ranking is obtained as follows. Take the cyclic (directed) graph obtained by the tournament and find the smallest set of arcs such that reversing the direction of these arcs results in an acyclic graph. The desired ranking is taken to be the consistent ranking (per Lemma 1) with respect to the resulting acyclic graph.[20]

---

[20] Multiple rankings may result from this method.

One may argue that this procedure can be improved by assigning weights to arcs in the tournament depending on the score by which team i beats team j and then look for the acyclic graph that minimizes the total weighted inconsistencies. In fact this approach goes back to Condorcet's (1785) classical voting paper in which he proposed a method for ranking multiple candidates. In the voting game, the set of nodes is the group of candidates, the arcs' directions are the results of pairwise voting, and the weights are the plurality in the voting. The solution to our problem follows the same path. In our framework arcs are not homogeneous and so they will be assigned weights determined by the difference in the bilateral externalities. As in Condorcet's voting paper, we will look for the set of arcs such that their reversal turns the graph into an acyclic one. While Young (1988) characterized Condorcet's method axiomatically, our solution results from a completely different approach, i.e., the design of optimal incentives to maximize revenues.

Formally, we define the weight of each arc $(i, j) \in A$ by $t(i, j) = w_j(i) - w_i(j)$. Note that weights are always non-negative as an arc (i, j) refers to a situation in which j favors i more than i favors j. Hence $t(i, j)$ refers to the extent of the one-sidedness of the externalities between the pairs of agents. If an inconsistency in the ranking arises due to an arc (i, j), then this implies that agent j precedes agent i despite the fact that i beats j. Relative to consistent rankings, inconsistencies generate additional costs for the principal. More precisely, the principal has to pay an additional $t(i, j)$ when inconsistency is due to arc $(i, j) \in A$.

For each subset of arcs $S = \{(i_1, j_1), (i_2, j_2), ..., (i_k, j_k)\}$ we define $t(S) = \sum_{(i,j) \in S} t(i, j)$, which is the total weight of the arcs in S. For each graph G and subset of arcs S we denote by G_{-S} the graph obtained from G by reversing the arcs in the subset S. Consider a cyclic graph G and let S^* be a subset of arcs that satisfies the following:

(1) G_{-S^*} is acyclic.

(2) $t(S^*) \leq t(S)$ for all S such that G_{-S^*} is acyclic.

Then, G_{-S^*} is the acyclic graph obtained from G by reversing the set of arcs with the minimal total weight, and S^* is the set of pairs of agents that satisfies inconsistencies in the tournament ranking of G_{-S^*}. Proposition 4 shows that the optimal ranking of G is the tournament ranking of G_{-S^*} since the additional cost from inconsistencies, $t(S^*)$, is the lowest.

Proposition 4 *Let (N, w, c) be a participation problem with a cyclic tournament G. Let φ be the tournament ranking of G_{-S^*}. Then, the optimal full implementation contracts are the DAC with respect to φ.*

In the symmetric case, the principal cannot exploit the externalities among the agents, as $K_{asym} = 0$, and the total payment made by the principal is identical for all rankings. This can be seen to follow from Proposition 4 as well by noting that the tournament has two-way arcs connecting all pairs of agents, and $t(i, j) = 0$ for all i, j and $t(S)$ is uniformly zero. An intriguing feature of the symmetric case is that all optimal contracting schemes are discriminative in spite of the fact that all agents are identical.

Corollary 4.1 *When the externalities structure w is symmetric then all DAC contracts are optimal.*

We can now provide the analogue version of Proposition 3 for the cyclic case. In this case, the optimal ranking has an additional term $K_{cyclic} = t(S^*)$ representing the cost of making the tournament acyclic, i.e., the cost borne by the principal due to inconsistencies.

Proposition 5 *Let (N, w, c) be a participation problem. Let V_{full} be the principal's optimal cost of a full implementation contract. Then $V_{full} = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym}) + K_{cyclic}$.*

Corollary 3.1 still holds for pairs of agents that are not in S^*. More specifically, if we increase the level of asymmetry between pairs of agents that are outside S^*, we reduce the total expenses that the principal incurs in the optimal contracting scheme.

4 Extensions

In this section we discuss the implications of the assumptions we made so far. We demonstrate that the optimal contracts remain optimal if we assume sequential participation choices when the principal desires to implement participation in a subgame perfect equilibrium with the property that each player has a dominant strategy on the subgame that he plays. In addition, we show that even when the outside option is affected by the agents' participation choices, the construction of the optimal contracts remains unchanged. We

demonstrate that when contracts can be contingent on the participation of a subset of the agents, then the optimal contracts are closely related to the analysis above. Our analysis is valid in more general setups in which externalities can be either negative or positive. Moreover, the solution is also relevant to non-additive externalities structures.

4.1 Sequential Participation Decisions

We first point out that our analysis applies to any sequential game except for one of perfect information, i.e., when each player is fully informed about all the participation decisions of his predecessors. Indeed, this extreme case of perfect information is a strong assumption as agents rarely possess the participation decisions of *all* their predecessors. Any partial information environment implies that some actions are taken simultaneously, and therefore the divide-and-conquer contracting scheme and the virtual tournament apply.

Nevertheless, it is interesting to point out that our analysis is also relevant to the extreme case of perfect information. Consider a game in which players have to decide sequentially about their participation based on a given order. Suppose that the principal wishes to implement the full participation in a subgame perfect equilibrium with the additional requirement that each player has a dominant strategy on the subgame in which he has to play.[21] It is easily verified that the optimal contracting scheme in this framework is the DAC applied to the order of moves; i.e., the first moving player is paid c and the last player is paid $c - \sum_{j \in N} w_i(j)$. Under this contracting scheme each player has a dominant strategy on each subgame. Assume now that the principal can control the order of moves (which he can do by making the offers sequentially and setting a deadline on agents' decisions). Then the optimal sequential contracting scheme is exactly identical to the one discussed in previous sections for the simultaneous case. If the principal suffices with a standard subgame perfect equilibrium (without the strategy dominance condition), then the optimal contracting scheme will allow him to extract more and he will pay $c - \sum_{j \in N} w_i(j)$ to all agents.

---

[21]Such a requirement may reflect the principal's concern that a player will fail to apply complex backward induction reasoning.

4.2   Participation-dependent Outside Options

In many situations non-participating agents are affected by the participation choices of other agents. Consider the case of a corporate raider who needs to acquire the shares of N identical shareholders to gain control (similar to Grossman and Hart 1980). If the raider is enhancing the value of the firm when he holds a larger stake in the firm, then selling shareholders impose positive externalities on non-participating agents. If the raider gains private benefits from the firm which will decrease its value, then selling shareholders induce negative externalities on the non-participating agents.

In this section we consider the case in which the agents' outside option is partly determined by the agents who choose to participate. For a given group of agents $P \subseteq N$ who participate, we define the outside option of non-participants as $c + \eta \sum_{j \in P} w_i(j)$. In the former analysis we assumed $\eta = 0$.[22] Segal (2003) defines externalities as increasing (decreasing) when an agent is more (less) eager to participate when more agents participate. In our setup, eagerness to participate is identity-dependent. When $\eta \leq 1$, we say that agents are more eager to participate when highly valued agents choose to participate. If $\eta > 1$, the benefits of non-participation outweigh the benefits of participation when highly valued agents choose to participate; hence agents are less eager to participate. In Segal's terminology, the former case is equivalent to *increasing* externalities, while the latter is equivalent to *decreasing* externalities.

Following the analysis of Proposition 1, if v is an optimal full implementation contracting scheme then it is easy to verify that under the current setup, v is a DAC of the form:

$$v = (c, c - (1 - \eta)w_{i_2}(i_1), ..., c - (1 - \eta) \sum_k w_{i_n}(i_k))$$

where $\varphi = (i_1, i_2, ..., i_n)$ is an arbitrary ranking. Instead if $\eta > 1$, the participation problem is identical to a standard participation problem (with fixed outside option) where externalities are $(1 - \eta)w_i(j) < 0$. In these negative externalities problems the DAC mechanism does not apply and the optimal scheme requires that the principal reimburse the agents for their total burden, i.e., $c - (1 - \eta) \sum_j w_i(j)$, which is a positive number whenever the outside option and $w_i(j)$ are positive. Finally, the case of $\eta = 1$ corresponds to an

---

[22]The following analysis can be generalized by specifying an externalities matrix q that distinguishes between the benefits of participating and non-participating agents. It can be shown that in such a case our analysis remains unchanged. However, we choose to use the simpler and more intuitive outside option form of $c + \eta \sum_{j \in C} w_i(j)$.

environment of no externalities at all and the optimal scheme requires simply to reimburse agents for their outside option. We can summarize with the following proposition:

Proposition 6 *Let (N, w, c^*) be a participation problem where $c_i^* = c + \eta \sum_{j \in P} w_i(j)$ and $P \subseteq N$ is a group of participating agents. Let $G(N, A)$ be the equivalent tournament. The optimal full implementation contracts are given as follows:*

(1) for $\eta < 1$, DAC contracts with respect to the optimal ranking;[23]

(2) for $\eta \geq 1$, the optimal mechanism pays agent i the payoff $c - (1 - \eta) \sum_j w_i(j))$, which is exectly c, whenever $\eta = 1$.

4.3   Contingent Contracts

Our model assumes that the principal cannot write contracts that make a payoff to an agent contingent on the participation of other agents. Under such contracts the principal could extract the total surplus from positive externalities among the agents.[24] We find such contracts not very descriptive. Based on the data used by Gould et al. (2005) which consists of contractual provisions of over 2,500 stores in 35 large shopping malls in the U.S., there is no evidence that contracts make use of such contingencies. Shopping malls are a natural environment for contingent contracting; the fact that these contracts are still not used makes it likely that in other, more complicated settings such contracts are exceptional as well. The theoretical foundation for the absence of such contracts is beyond the scope of this paper. However, one possible reason for their absence is the complexity of such contracts, especially in environments where participation involves long-term engagement and may be carried out by different agents at different points in time. We point out that if partial contingencies are used, i.e., participation is contingent on a subset of the group, our model and its analysis remain valid. Specifically, for each player i, let $T_i \subseteq N$ be the contingency set, i.e., the set of agents whose participation choice can appear in the contract with agent i. Let $T = (T_1, T_2, ..., T_n)$ summarize the contingency sets in the contracts. The optimal contracts under the contingency sets are closely related to the original optimal contract (when contingencies are not allowed). More precisely, Let w be the original matrix of externalities. Denote by w^T the matrix

---

[23] As described in Section 4.

[24] One possible contracting scheme is to offer agent i a participation reward of $v_i = c - \sum_{j \in N} w_i(j)$ if each of the other agents participates, and a reward of $v_i = c$ if any of the contingencies is violated. Such contracts will sustain full participation as a unique Nash equilibrium, and the principal extracts the entire surplus.

of externalities obtained from w by replacing $w_i(j)$ with zero whenever $j \in T_i$. Lemma 6.1 in the Appendix shows that the optimal full implementation contracting scheme is as follows: agent i gets c if one of the agents $j \in T_i$ does not participate; i.e., the contingency requirement is violated.[25] If all agents in T_i participate, then agent i gets the payoff $v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$, where $v_i(N, w^T, c)$ is the payoff for agent i for the participation problem (N, w^T, c) under no-contingencies (as developed in Section 4).

4.4 Mixed Externalities Structure

So far we have limited our discussion to environments in which agents' participation positively affects the willingness of other agents to participate. However, in many situations this is not the case, such as in environments of congestion. Traffic, market entry, and competition among applicants all share the property that the larger the number of agents who participate, the lower the utility of each participant. The heterogeneous property in our framework seems quite descriptive in some of these examples. In the context of competition it is clear that a more qualified candidate/firm induces a larger negative externality. It is also reasonable to assume, at least for some of these environments, that the principal desires a large number of participants in spite of the negative externalities that they induce on each other.

In Proposition 7 we demonstrate that in order to sustain full participation as a unique Nash equilibrium under negative externalities the principal has to fully compensate all agents for the participation of the others.

Proposition 7 *Let (N, w, c) be a participation problem with negative externalities. Then optimal full implementation contracts v are given by $v_i = c + \sum_{i \neq j} |w_i(j)|$, and v is unique.*

Naturally, real-world multi-agent contracting problems may capture both positive and negative types of externalities. In social events, individuals may greatly benefit from some of the invited guests, while preferring to avoid others. In a mall, the entry of a new store may benefit some stores by attracting more customers, but impose negative externalities on its competitors.

---

[25] In fact, the principal can offer lower payments to the agents in case of contingencies' violations, by exploiting the participation of other agents. However, these off-equilibrium payments do not affect the principal's payment in the full participation equilibrium.

Our analysis of the mixed externalities case is based on the following binary relation. We say that an agent i is *non-averse* to agent j if $w_i(j) \geq 0$, and we write it as $i \succeq j$. We will assume that $\succeq$ is symmetric and transitive, i.e., $i \succeq j \implies j \succeq i$ and if $i \succeq j$ and $j \succeq k$ then $i \succeq k$. Note that this assumption does not imply any constraint on the magnitude of the externalities, but just on their sign. While the symmetry and transitivity of the *non-averse* relation seem rather intuitive assumptions, not all strategic environments satisfy them. These assumptions are particularly relevant to environments where the selected population is partitioned into social, ethnic, or political groups with animosity potentially occurring only between groups but not within groups. We analyze a specific example of this sort of environment in Section 6.

It turns out that the optimal solution of participation problems with symmetry and transitivity of the *non-averse* relation is derived by a decomposition of the participation problem into two separate participation problems: one that involves only positive externalities, and the other that involves only negative externalities. This is done by simply decomposing the externalities matrix into a negative and a positive matrix. In the following proposition we show that the *decomposition contracting scheme,* a contract set that is the sum of the two optimal contracts of the two decomposed participation problems, is the optimal contracting scheme for the mixed externalities participation problem.

Proposition 8 *Consider a participation problem* (N, w, c). *Let* (N, w^+, c) *be a participation problem such that* $w_i^+(j) = w_i(j)$ *if* $w_i(j) \geq 0$ *and* $w_i^+(j) = 0$ *if* $w_i(j) < 0$, *and let* u^+ *be the optimal full implementation contracts of* (N, w^+, c). *Let* $(N, w^-, 0)$ *be a participation problem such that* $w_i^-(j) = w_i(j)$ *if* $w_i(j) < 0$ *and* $w_i^-(j) = 0$ *if* $w_i(j) \geq 0$, *and let* u^- *be the optimal full implementation contracts of* $(N, w^-, 0)$. *Then, the decomposition contracting scheme* $v = u^+ + u^-$ *induces a unique full participation equilibrium. Moreover, if agents satisfy symmetry and transitivity with respect to the non-averse relation,* v *is the optimal contracting scheme.*

Proposition 8 shows that the virtual popularity tournament discussed in earlier sections plays a central role also in the mixed externalities case as it determines payoffs for the positive component of the problem. When symmetry and transitivity hold, the principal can exploit the positive externalities to reduce payments. In this tournament i beats j whenever (1) $w_j(i) \geq 0$ and $w_j(i) \geq 0$, and (2) $w_j(i) > w_i(j)$. Note that under

the *non-averse* assumptions, the principal provides complete compensation for the agents who suffer from negative externalities, as in the negative externalities case. Finally, it is easy to show that equivalently to Proposition 5, the principal's cost of achieving full implementation in a mixed externalities setting is equivalent to the positive externalities setup, except that now the principal has to add the compensation for the negative externalities.

4.5   Non-additive Preferences

We propose here an extension of the model in which we impose no restrictions on agents' preferences; i.e., preferences are no longer assumed to be separably additive. Using an iterative procedure that makes use of the solution for the additive case allows us to narrow down the set of potential optimal incentive contracts, even when no structure is assumed.

A participation problem is described by a group of agents N and their outside option is equal to c, as noted previously. We assume a general externalities structure, which is composed of the non-additive preferences of the agents over all subsets of agents in the group N. More specifically, for each i, $v_i : 2^{N \setminus \{i\}} \longrightarrow R$. The function $v_i(S)$ stands for the benefit of agent i from the participation with the subset $S \subseteq N$. We normalize $v(\emptyset) = 0$. The condition of positive externalities now reads: for each i and subsets S, T such that $T \subset S$ we have $v_i(S) \geq v_i(T)$.

Arguments similar to those used in Proposition 1 show that the optimal contracting scheme that sustains full participation as a unique equilibrium also satisfies the divide-and-conquer property. Hence, the optimal contracts rely on the optimal ranking of the agents.

We leave the detailed description of the procedure to the proof of Proposition 9. Instead, we provide an example to illustrate the basic ideas.

4.5.1   A Simple Example

Consider a four-agent example. Given that the optimal solution is DAC for any given ranking of agents $\varphi = \{i_1, i_2, i_3, i_4\}$, the DAC contracts with respect to ranking φ are $(c, c - v_{i_2}(i_1), c - v_{i_3}(i_1, i_2), c - v_{i_4}(i_1, i_2, i_3))$. Instead of identifying the optimal ranking, we apply an iterative procedure of $N - 1$ steps to eliminate rankings that we infer cannot be optimal. Our starting point is the set of all possible rankings of the agents; in this

example there are 24 such rankings.

STEP 1. Let's assume that the bilateral externalities $v_i(j)$ between the agents result in the corresponding acyclic graph described below. Therefore the tournament yields the unique consistent ranking for step one when $\phi_1 = (3, 1, 2, 4)$.
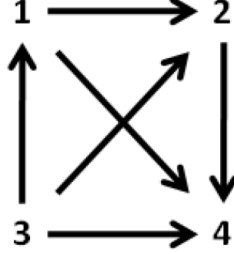

Figure 2

We argue that any ranking that orders the first two agents in a way that contradicts their relative ranking in ϕ_1 cannot be the optimal. To see that, consider the ranking $(4, 2, 1, 3)$ which is inconsistent with ϕ_1 with respect to the relative ranking of agents 4 and 2. We can immediately construct a cheaper ranking by reversing the position of the first two agents, and keeping the position of the remaining agents ranked lower in the same order. Hence, we can eliminate $(4, 2, 1, 3)$ from the set of potential optimal rankings. Applying this logic to the entire set of potential rankings we are left with 12 potential rankings; i.e., the optimal ranking of the original problem must start with any of the following pairs: $(3, 1), (3, 2), (3, 4), (1, 2), (1, 4), (2, 4)$.

STEP 2. We now proceed to the second iteration in which for each agent located in the first position we construct a graph that is based on the bilateral relations conditional on the participation of the first agent. In particular, we consider the case in which agent 1 is ranked first and build the graph based on agents' preferences conditional on the participation of agent 1; i.e., the externalities matrix is given by $(w_i(j) = v_i(j, 1)|j \in \{2, 3, 4\})$.

Let's assume that preferences take the following form:

$$v_2(3, 1) > v_3(2, 1)$$
$$v_2(4, 1) > v_4(2, 1)$$
$$v_3(4, 1) > v_4(3, 1)$$

Since the graph is acyclic the unique consistent ranking of the second iteration, conditional

on agent 1 being first, is $\phi_{2|1} = (4, 3, 2)$. Again, we require rankings to be consistent with $\phi_{2|1}$. For example, ranking $(1, 2, 4, 3)$ cannot be optimal since $(2, 4, 3)$ is not consistent with $\phi_{2|1}$ and transposing the order of 2 and 4 we get ranking $(1, 4, 2, 3)$, which is cheaper. While there are six rankings in which agent 1 is ranked first, we can immediately eliminate three that do not agree with $\phi_{2|1}$ and we are left with $\{(1, 4, 2, 3), (1, 4, 3, 2), (1, 3, 2, 4)\}$. However, these rankings must agree with the constraints from the previous step. This is not the case for ranking $(1, 3, 2, 4)$, as we can transpose the order of 1 and 3 and get a cheaper mechanism; thus we can eliminate it as well.[26] Hence, if the optimal ranking starts with agent 1 it must be followed by agent 4 ranked second. Rather than discussing the construction of cases where agents 2 and 3 are ranked first, we continue to explore the case where agent 1 is ranked first and proceed to step 3.

STEP 3. In this iteration we repeat and construct the graph based on agents 2 and 3's preferences, conditional on the participation of agents 1 and 4. Let's assume that $v_2(3, 1, 4) < v_3(2, 1, 4)$; hence $\phi_{3|1,4} = \{2, 3\}$. Thus, the only ranking that can be optimal in the original problem conditional on agent 1 being first is $(1, 4, 2, 3)$.

4.5.2 General Result

The example above illustrates our procedure for generating the optimal incentive contracts can also be used iteratively to eliminate non-optimal rankings, when we impose no structure on agents' preferences.

The starting point is the set of all agents' rankings. We proceed with an iterative procedure of $N - 1$ stages, in each we rule out possible rankings by constructing a graph that is based on the bilateral preferences of the agents conditional on the participation of the agents ranked above them. We assume that in each step the resulting graph is acyclic and thus generates a unique consistent ranking. We eliminate rankings that are inconsistent with the step's consistent ranking or with the constraints imposed in the previous step. The formal description of this iterative procedure is provided in the proof of Proposition 9.

Proposition 9 *Let (N, c) be a participation problem with non-additive preferences, for which all tournaments in the iterative procedure are acyclic. Then, the set of surviving rankings is non-empty and includes the optimal ranking.*

---

[26]We refer to this check as the *interface condition* and discuss it more fully in the proof of Proposition 9.

Proposition 9 demonstrates that the fundamental logic underlying our analysis of additive externalities also underlies our construction of optimal contracts while taking into account the complex structure of externalities among agents.

5 Group Identity and Selection

In this section we consider special externalities structures to demonstrate how the selection stage can be incorporated once we have solved the participation problem. Assume that the externalities take values of 0 or 1. In this environment an agent either benefits from the participation of his peer or gains no benefit. We provide three examples of group identities in which the society is partitioned into two groups and agents have hedonic preferences for members in these groups. We demonstrate how the optimal contracting scheme proposed in previous sections may affect the selection of the agents in the planning of the initiative.

(1) **Segregation** - agents benefit from participating with their own group's members and enjoy no benefit from participating with members from the other group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 1$. Otherwise, $w_i(j) = 0$.

(2) **Desegregation**[27] - agents benefit from participating with the other group's members and enjoy no benefit from participating with members of their own group. More specifically, consider the two groups B_1 and B_2 such that for each $i, j \in B_k$, $k = 1, 2$, we have $w_i(j) = 0$. Otherwise, $w_i(j) = 1$.

(3) **Status** - the society is partitioned into two status groups, high and low. Each member of the society benefits from participating with each member of the high-status group and enjoys no benefit from participating with members of the low-status group. Formally, let B_1 be the high status group and set $w_i(j) = 1$ if and only if $j \in B_1$. Otherwise $w_i(j) = 0$.

Proposition 10 *Let (N, w, c) be a participation problem. Let n_1 and n_2 be the number of agents selected from groups B_1 and B_2, respectively, such that $n_1 + n_2 = n$. Denote by*

---

[27] An example could be a singles party.

$v(n_1, n_2)$ *the principal cost of incentivizing agents under the optimal contracts given that the group composition is n_1 and n_2. The following holds:*

    1) under Segregation $v(n_1, n_2)$ is decreasing with $\mid n_1 - n_2 \mid$;

    2) under Desegregation $v(n_1, n_2)$ is increasing with $\mid n_1 - n_2 \mid$;

    3) under Status $v(n_1, n_2)$ is decreasing with n_1.

In the case of Segregation, the principal's cost of incentives is increasing with the mixture of groups; hence in the selection stage the principal would prefer to give precedence to one group over the other. In the Desegregation case the principal's cost is declining with mixture; hence in the selection stage the principal would like to balance between members of the groups. In the Status case the cost is declining with the number of agents recruited from B_1, who will be strongly preferred to the members of B_2.

6   Conclusion

In this paper we analyzed a contracting framework with heterogeneous externalities. Introducing a complicated structure of externalities allowed us to explore several aspects of the multi-agent contracting environments that are not apparent in the homogeneous case. These include the impact of externalities asymmetry on payments, the implications of externalities structure on the hierarchy of incentives, and the effect of variations in the externalities structures on both the principal's payments and the agents' rewards.

Exploring the role of heterogeneous externalities reveals the importance of externalities asymmetry within the group. More specifically, greater asymmetry between the agents' benefits reduced the principal's payment in the full implementation problem. In addition, externalities asymmetry turns out to play a role also in the selection between partial and full implementation, as it affects the premium required to sustain full participation as a unique equilibrium. Greater asymmetry decreases this premium, and thus makes full implementation more profitable.

The hierarchy of incentives is determined by a ranking that results from a virtual popularity tournament. In the simplest case, an agent i is ranked above agent j if agent i benefits less from joint participation than agen j does from it. We demonstrated that this ranking of incentives is different from the standard ranking that is based on agents'

popularity.

The implications of externalities on the rankings of agents may suggest a preliminary game in which agents invest effort to increase the positive externalities that they induce on others which can ultimately increase the rewards from the principal. For example, agents can invest in their social efforts to make themselves more attractive guests at social events. A firm may invest to increase its market share in order to improve its ranking in an acquisition game. Under certain circumstances such an investment may turn out to be quite attractive as we have seen that a slight change in externalities may result in a substantial gain, due to a change in the ranking. Our analysis contributes to the understanding of how to make such a strategic investment profitable. The new game will now have two stages. In the first stage agents' investment efforts change the exterrnalities they induce on other members of the group. In the second stage, the externalities determine the network formation and the consequent incentives provided by the principal. The analysis of such a game is beyond the scope of this paper but seems to be a natural next step.

References

Brandts, J. and D. Cooper. 2006. "A Change Would Do You Good... An Experimental Study on How to Overcome Coordination Failure in Organizations," *American Economic Review* 96, 669-693.

Calvò, A. and M. Jackson. 2004. "The Effect of Social Networks on Employment and Inequality," *American Economic Review* 94, 426-454.

Calvò, E., J. Lasaga, and A. van den Nouweland. 1999. "Values of Games with Probabilistic Graphs," *Mathematical Social Sciences* 37, 79-95.

Che, Y. K. and S. Yoo. 2001. "Optimal Incentives for Teams," *American Economic Review* 91, 525-541.

Cheng, M. "Hong Kong's Silicon Dreams." San Francisco Chronicle. October 12, 1999.

Condorcet, Marquis de. 1785. *Essai sur l'application de l'analyse a la probabilite des decisions rendues a la probabilite des voix,* Paris: De l'imprimerie royale.

Gould, E., P. Pashigian, and C. Prendergast. 2005. "Contracts, Externalities, and Incentives in Shopping Malls," *Review of Economics and Statistics* 87, 411-422.

Goyal, S. and J. L. Moraga. 2001. "R & D Networks," *RAND Journal of Economics* 32, 686-707.

Grossman, S. J. and O. D. Hart. 1980. "Takeover Bids, the Free-rider Problem, and the Theory of the Corporation," *The Bell Journal of Economics* 11, 42-64.

Jackson, M.. 2004. "A Survey of Models of Network Formation: Stability and Efficiency," in *Group Formation in Economics; Networks, Clubs and Coalitions*, edited by G. Demange and M. Wooders, Cambridge University Press.

Jehiel, P., B. Moldovanu, and E. Stachetti. 1996. "How (Not) to Sell Nuclear Weapons," *American Economic Review* 86, 814-829.

Jehiel, P. and B. Moldovanu. 1996. "Strategic Nonparticipation," *RAND Journal of Economics* 27, 84-98.

Jehiel, P. and B. Moldovanu. 1999. "Resale markets and the Assignment of Property Rights," *The Review of Economic Studies* 66, 971-991.

Kamien, M., S. Oren, and Y. Tauman. 1992. "Optimal Licensing of Cost-Reducing Innovation," *Journal of Mathematical Economics* 31, 483-508.

Kamien, M. and I. Zang. 1990. "The Limits of Monopolization through Acquisition," *Quarterly Journal of Economics* 105, 465-499.

Katz, M. and C. Shapiro. 1986a. "How to License Intangible Property," *Quarterly Journal of Economics* 101, 567-589.

Katz, M. and C. Shapiro. 1986b. "Technology Adoption in the Presence of Network Externalities," *Journal of Political Economy* 94, 822-841.

Kendall, M.. 1962. *Rank Correlation Methods,* 3rd ed., Hafner: New York.

Krishna, K.. 1993. "Auctions with Endogenous Valuations: The Persistence of Monopoly Revisited," *American Economic Review* 83, 147-160.

Levin, J.. 2002. "Multilateral Contracting and the Employment Relationship," *Quarterly Journal of Economics* 117, 1075-1103.

Lewis, T.. 1983. "Preemption, Divestiture, and Forward Contracting in a Market Dominated by a Single Firm," *American Economic Review* 73, 1092-1101.

Milgrom, P., and J. Roberts. 1990. "Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities," *Econometrica* 58, 1255-1277.

Mutuswami, S. and E. Winter. 2002. "Subscription Mechanisms for Network Formation," *Journal of Economic Theory* 106, 242-264.

Page, F., M. Wooders, and S. Kamat. 2001. "Networks and Farsighted Stability," Discussion Paper, University of Warwick.

Rasmusen, E., M. Ramseyer, and J. Wiley. 1991. "Naked Exclusion," *American Economic Review* 81, 1137-1145.

Segal, I.. 1999. "Contracting with Externalities," *Quarterly Journal of Economics* 114, 337-388.

Segal, I. and Whinston M.. 2000. "Naked Exclusion: Comment," *American Economic Review* 90, 296-309.

Segal, I.. 2003. "Coordination and Discrimination in Contracting with Externalities: Divide and Conquer?" *Journal of Economic Theory* 113, 147-327.

Winter, E.. 2004. "Incentives and Discrimination," *American Economic Review* 94, 764-773.

Winter, E.. 2006. "Optimal Incentives for Sequential Production," *RAND Journal of Economics* 37, 376-390.

Young, H. P.. 1988. "Condorcet's Theory of Voting," *The American Political Science Review* 82, 1231-1244.

Appendix

Proof of Proposition 1 Let $v = (v_{i_1}, v_{i_2}, ..., v_{i_n})$ be an optimal full implementation contracting scheme of the participation problem (N, w, c). Hence, v generates full participation as a unique Nash equilibrium. Since no-participation is not an equilibrium, at least a single agent, say i_1, who reward is at least as high as his outside option c. Otherwise, a no-participation equilibrium exists. Due to the optimality of v his payoff would be exactly c. Agent i_1 chooses to participate under any profile of other agents' decisions. Given that agent i_1 participates and an equilibrium of a single participation is not feasible, at least one other agent, say i_2, receives a reward at least as high as $c - w_{i_2}(i_1)$. Since v is the optimal contracting scheme, i_2's reward equals $c - w_{i_2}(i_1)$, and under any profile of decisions i_2 will participate. Applying this argument iteratively on the first $k - 1$ agents, at least one other agent, henceforth i_k, must get a payoff at least as high as $c - \sum_{j=1}^{k-1} w_{i_k}(j)$, but again, since v is optimal, the payoff for agent k must be equal to $c - \sum_{j=1}^{k-1} w_{i_k}(j)$. Hence, the optimal contracting scheme v must satisfy the *divide-and-conquer* property with respect to a ranking φ. ∎

Proof of Lemma 1 We will demonstrate that there is a single node with $n - 1$ outgoing arcs. Since the tournament is a complete, directed, and acyclic graph there cannot be two such nodes. If such a node does not exist, then all nodes in G have both incoming and outgoing arcs. Since the number of nodes is finite, we get a contradiction to G being acyclic. We denote this node as i_1 and place its corresponding agent first in the ranking (hence this agent beats all other agents). Now let us consider a subgraph $G(N^1, A^1)$ that results from the removal of node i_1 and its corresponding arcs. Graph $G(N^1, A^1)$ is directed, acyclic, and complete and, therefore, following the previous argument, has a single node that has exactly $n - 2$ outgoing arcs. We denote this node as i_2, and place its corresponding agent at the second place in the ranking. Note that agent i_1 beats agent i_2 and therefore the ranking is consistent so far. After the removal of node i_2 and its arcs we get subgraph $G(N^2, A^2)$ and consequently node i_3 is the single node that has $n - 3$ outgoing arcs in subgraph $G(N^2, A^2)$. Following this construction, we can easily observe that the ranking $\varphi = (i_1, i_2, ..., i_n)$ is consistent among all pairs of agents and due to its construction is also unique. ∎

Proof of Proposition 2 According to Proposition 1 the optimal contracting scheme satisfies the DAC property. Hence the optimal contracting scheme is derived from con-

structing the optimal ranking and is equivalent to minimizing the sum of incentives, V_{full} :

$$
\begin{aligned}
V_{full} &= \min_{(j_1, j_2, ..., j_n)} \left[n \cdot c - \left\{ \sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right\} \right] \\
&= \max_{(j_1, j_2, ..., j_n)} \left[\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k) \right]
\end{aligned}
$$

Since no externalities are imposed on nonparticipants, the outside options of the agents have no role in the determination of the optimal contracting scheme. We will show that the ranking that solves the maximization problem of the principal is the tournament ranking. Let us assume, without loss of generality, that the tournament ranking φ is the identity permutation: hence $\varphi(i) = i$, and $W_\varphi = \sum_{k=1}^{2} w_2(k) + ... + \sum_{k=1}^{n} w_n(k)$, where W_φ is the principal's revenue extraction. By way of contradiction, assume that there exists $\varphi \neq \sigma$ such that $W_\varphi \leq W_\sigma$. First, assume that σ is obtained from having two *adjacent* agents i and j in φ trade places such that i precedes j in φ and j precedes i in σ. By Lemma 1, agent i beats agent j; thus $W_\sigma = W_\varphi - w_j(i) + w_i(j)$ and $W_\sigma < W_\varphi$.

Note that since φ is the tournament ranking, agent 1 beats all agents, agent 2 beats all agents but agent 1, and so on. Now consider unconstrained $\sigma = \{i_1, ..., i_n\}$ such that $\varphi \neq \sigma$. If agent 1 is not located first, by a sequence of adjacent swaps $(1, i_j)$, we move agent 1 to the top of the ranking. In each of the substitutions agent 1 beats i_j. Next, if agent 2 is not located at the second place, by a sequence of adjacent substitutions $(2, i_j)$, we move agent 2 to the second place. Again, agent 2 beats all agents i_j. The process ends in at most n stages and produces the desired order φ. As demonstrated, any adjacent substitution results in a higher extraction, and so $W_\sigma < W_\varphi$. Therefore, the DAC contracting scheme with respect to the tournament ranking is unique and optimal. ∎

Proof of Proposition 3 Without loss of generality, assume that the tournament ranking φ is the identity permutation. Hence, under the optimal contracting scheme, the principal's payment is $V_{full} = n \cdot c - \left[\sum_{j=1}^{1} w_1(j) + ... + \sum_{j=1}^{n} w_n(j) \right]$. Denote $s_i(j) = [w_i(j) + w_j(i)]$ and $a_i(j) = [w_i(j) - w_j(i)]$. We can represent K_{agg} and K_{asym} in the following manner: $K_{agg} = \sum_{i\,j} w_i(j) = \sum_{i<j} (w_i(j) + w_j(i)) = \sum_{i<j} s_i(j)$ and $K_{asym} = \sum_{i<j} |a_i(j)|$.

Since $w_i(j) = \frac{1}{2}(s_i(j) + a_i(j))$ we can rewrite the principal's payment as

$$
\begin{aligned}
V_{full} &= n \cdot c - \frac{1}{2}\left[\sum_{j=1}^{1}\{s_1(j) + a_1(j)\} + ... + \sum_{j=1}^{n}\{s_n(j) + a_n(j)\}\right] \\
&= n \cdot c - \frac{1}{2}\left(\sum_{i>j} s_i(j) + \sum_{i>j} a_i(j)\right)
\end{aligned}
$$

Note that $s_i(j) = s_j(i)$ and $a_i(j) = -a_j(i)$. In addition $a_i(j) > 0$ when $i > j$ as the tournament is acyclic and ranking is consistent. Therefore,

$$
V_{full} = n \cdot c - \frac{1}{2}\left(\sum_{i<j} s_i(j) - \sum_{i<j} |a_i(j)|\right) = n \cdot c - \frac{1}{2}(K_{agg} + K_{asym})
$$

∎

Proof of Corollary 3.2 The result follows immediately from Proposition 3, where we show that $V_{full} = n \cdot c - \frac{1}{2}\sum_{i\,j} w_i(j) - \frac{1}{2}\sum_{i<j}|w_i(j) - w_j(i)|$, and from $V_{partial} = n \cdot c - \sum_{i\,j} w_i(j)$. Taken together, the two yield

$$
V_{full} - V_{partial} = \frac{1}{2}\sum_{i\,j} w_i(j) - \frac{1}{2}\sum_{i<j}|w_i(j) - w_j(i)| = \frac{1}{2}(K_{agg} - K_{asym})
$$

∎

Proof of Proposition 4 Let $G(N, A)$ be a cyclic graph. Consider a subset of arcs S such that G_{-S} is acyclic, and the tournament ranking of G_{-S} is $\varphi = (j_1, j_2, ..., j_n)$. The payment of the principal V_{full} under the DAC contracting scheme with respect to φ is

$$
V_{full} = n \cdot c - \left\{\sum_{k=1}^{1} w_{j_1}(j_k) + \sum_{k=1}^{2} w_{j_2}(j_k) + ... + \sum_{k=1}^{n} w_{j_n}(j_k)\right\}
$$

Note that each $(i, j) \in S$ satisfies an inconsistency in tournament ranking φ. More specifically, if $(i, j) \in S$, then i beats j, and agent j is positioned above agent i. In addition, $w_i(j) = w_j(i) - t(i, j)$, where $w_i(j) < w_j(i)$ and $t(i, j) > 0$. Consider the following substitution: if $(i, j) \in S$ then $w_i(j) = \hat{w}_j(i) - t(i, j)$; otherwise $w_i(j) = \hat{w}_i(j)$. This allows us to rewrite the principal's payment as $V_{full} = n \cdot c - \left\{\sum_{k=1}^{1} \hat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \hat{w}_{j_n}(j_k)\right\} + t(S)$. Note that $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Therefore, different rankings affect only the level of $t(S)$, as the first two terms in V_{full} remain indifferent to variations in the ranking.

This implies that the subset S with the lowest $t(S)$ brings V_{full} to a minimum. Hence, the optimal contracting scheme is the DAC with respect to the tournament ranking of G_{-S^*}. ■

Proof of Proposition 5 As demonstrated in Proposition 4, the optimal payment of the principal is the DAC contracting scheme with respect to the tournament ranking of G_{-S^*}. According to Proposition 4, this can be written as

$$V_{full} = n \cdot c - \left\{ \sum_{k=1}^{1} \widehat{w}_{j_1}(j_k) + ... + \sum_{k=1}^{n} \widehat{w}_{j_n}(j_k) \right\} + t(S)$$

when $\hat{w}_i(j) = \max(w_i(j), w_j(i))$. Following the argument of Proposition 3, denote $s_i(j) = [\hat{w}_i(j) + \hat{w}_j(i)]$ and $a_i(j) = [\hat{w}_i(j) - \hat{w}_j(i)]$ and the principal's payment is $V_{full} = n \cdot c - \frac{1}{2} \left(\sum_{i<j} s_i(j) + \sum_{i<j} |a_i(j)| \right) + t(S) = n \cdot c - \frac{1}{2} (K_{agg} + K_{asym}) + K_{cyclic}$. ■

Proof of Proposition 6 The cost of a full implementation contracting scheme is simply $V_{full} = nc - (1 - \eta) \sum_i \sum_{j<i} w_i(j)$. If $\eta = 1$, then the cost does not depend on the externalities. If $\eta < 1$, the minimal cost is obtained by selecting a ranking that maximizes $\sum_i \sum_{j<i} w_i(j)$. This is equivalent to the tournament ranking outlined in Proposition 4. If $\eta > 1$, the participation problem is identical to a standard participation problem (with fixed outside option) where externalities are $(1 - \eta)w_i(j) < 0$. In these negative externalities problems the DAC mechanism does not apply and the optimal scheme requires that the principal reimburse the agents for their total burden, i.e., $c - (1 - \eta) \sum_j w_i(j))$, which is a positive number whenever the outside option and $w_i(j)$ are positive. ■

Lemma 6.1 *Let (N, w, c) be a participation problem and $T = (T_1, ..., T_n)$ define the contingency sets. Define w^T to be such that $w_i^T(j) = w_i(j)$ if $j \notin T_i$ and $w_i^T(j) = 0$ otherwise. Let φ be the optimal ranking of the participation problem (N, w^T, c), and $v(N, w^T, c)$ the corresponding DAC payment vector. The optimal full implementation contracts set of (N, w, c) is such that it provides c for agent i if contingencies T_i are violated, and $v_i = v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$ otherwise.*

Proof of Lemma 6.1 Since externalities are positive, contingencies allow the principal to reduce payments. In particular, when exploiting all contingencies allowed in T, the contracting scheme that sustains a unique full participation Nash equilibrium offers each agent i a reward $v_i = c - \sum_{j \in T_i} w_i(j)$ if contingencies are met, and c if they are violated.

If for all agents $T_i = N/\{i\}$, then full extraction of surplus is possible as a unique equilibrium. However, if only partial contingencies are allowed, i.e., for some agents $T_i \subset N$ $/\{i\}$ then the principal can perform even better than in the contracts outlined above.

Let's define $\hat{w}_i(j) = w_i(j)$ if $j \notin T_i$ and $\hat{w}_i(j) = 0$ otherwise. Consider an arbitrary ranking of agents $\varphi = \{1, 2, ..., n\}$ in which the first agent is offered $v_1 = c - \sum_{j \in T_1} w_1(j)$ if contingencies are met, and c otherwise. Agent 1 will choose to participate. Given the participation of agent 1, we can offer agent 2 the following payment: $v_2 = c - \hat{w}_2(1) - \sum_{j \in T_2} w_2(j)$ if contingencies are met, and c otherwise. Hence, agent 2 will agree to participate given the participation of agent 1. Following the same argument, we could offer the last agent in the ranking $v_n = c - \sum_{i=1}^{n-1} \hat{w}_n(i) - \sum_{j \in T_n} w_n(j)$. This set of contracts will sustain full participation as a unique Nash equilibrium.

The optimal full implementation contracting scheme is thus achieved by obtaining the ranking of agents that will maximize $\sum_i \sum_{j>i} \hat{w}_i(j)$. Given our definition of $\hat{w}_i(j)$, this is equivalent to finding the optimal ranking of agents in the problem (N, w^T, c) when $w_i^T(j) = w_i(j)$ if $\notin T_i$ and $w_i^T(j) = 0$ otherwise. In other words, in the optimal full implementation contracting scheme, the payment for participation for each agent will be $v_i = v_i(N, w^T, c) - \sum_{j \in T_i} w_i(j)$ if contingencies are met, and c otherwise. ∎

Proof of Proposition 7 Given contracting scheme v, participation is a dominant strategy for all agents under the worst-case scenario in which all other agents participate, since $u_i = \sum_{i=1}^{n} w_i(j) + v_i = c$ for every $i \in N$. To show that v is optimal, consider a contracting scheme m for which $m_i < v_i$ for some agents and $m_i = v_i$ for the rest. By way of contradiction, assume full participation equilibrium holds under contracting scheme m. Consider an agent i for which $m_i < v_i$. If all other agents are participating, then agent i's best response is to abstain, since $u_i = \sum_{i=1}^{n} w_i(j) + m_i < c$. Hence, v is a unique and optimal contracting scheme. ∎

Proof of Proposition 8 See Complementary Note.

Proof of Proposition 9 We proceed in three steps. First we describe the iterative procedure formally. Then we show that the iterative procedure has a finite number of steps with a non-empty set of outcomes. Finally, we demonstrate that the optimal ranking of the original problem is among those orders that survive the procedure.

Formal Description of the Iterative Procedure. The starting point is the set of all possible rankings of the N agents. We start with the first two positions in the ranking.

We construct a tournament ranking in which $w_{ij} = v_i(j)$ and eliminate all the rankings in which the first two agents are ordered in a manner that is inconsistent with this ranking. All rankings that survived the elimination provide a possible assignment for the first position in the order. Suppose we have implemented the procedure k-1 steps and obtained a subset of the assignment to the first $k-1$ positions of the order. Let W_{k-1} be the set of sub-orders for the first $k-1$ agents obtained in step $k-1$. For each $w \in W_{k-1}$ we denote by S_w the set of agents assigned to one of the first $k-1$ slots in the assignment w. We now define a tournament T_w on the set of agents $N \backslash S_w$ in such a manner that $w_{ij} = v_i(S_w \cup j)$ defines the externalities matrix. Assuming the graph is acyclic we denote by π_w the uniquely consistent order of the agents in N/S_w.

Next, we construct a subset of assignments to the first k slots based on the set W_{k-1} and the orders π_w for each $w \in W_{k-1}$. We first look at the set of all orders of $N \backslash S_w$ and eliminate all orders in which the first two agents are ordered in a manner that is inconsistent with π_w. We denote by P_w the set of all ordered pairs that survive this elimination.

We now do the following check which we refer to as the *interface condition*: Take a pair $p_w \in Pw$. Suppose this pair is i_w, j_w and let k_w be the last player in w. If k_w beats i_w then the suborder w, i_w is a permissible suborder in step k and is added to W_k; otherwise it is excluded. We now proceed in a similar way for every $w \in W_{k-1}$ and the set of permissible suborders of length k which defines W_k.

Claim 1: The process ends in $N-1$ steps and results in a non-empty set of permissible orders.

Proof: We define inductively an order that survives all the steps of this procedure. The first agent in the order is the one ranked first under the tournament $w_{ij} = v_i(j)$ of the first step; call him i_1. The second agent is the one ranked first under the tournament T_w where $w = i_1$.

Suppose that the first $k-1$ slots of the order have been defined. The k-th agent in the order is the one ranked first under the tournament T_w , where $w = i_1, ... i_{k-1}$

Clearly the interface condition that we defined earlier will never be violated since at each step the agent who is added beats all the agents who are not yet ordered. This implies that the process yields a non-empty set of orders and the number of steps is finite.

Claim 2: The optimal order is among those orders that survive the procedure.

Proof: Suppose w.l.o.g. that the optimal order is the identity i.e., $1, 2, 3, ...n$. We denote this order by π. Suppose by way of contradiction that the optimal order is eliminated at some step k. This means that the interface condition between k-1 and k is violated or that the order was eliminated because the next two agents (say $i, i+1$) in the order π are not consistent with their order in the tournament ranking of $N \backslash S_w$ of the current stage. This means that $i + 1$ beats i in this tournament on $N \backslash S_w$. This implies that reversing their order will increase the principal's revenue in the divide-and-conquer scheme. Consider the tournament in stage k which is on the set of agents $k, k + 1, ..., n$ and $w_{ij} = v_i(\{1, ...k - 1\} \cup j)$. Let S be the subset of agents in $k, k + 1..., n$ such that for each $j \in S$ the agent j is not placed last under the consistent order of the tournament $w_{ij} = v_i(\{1, ...k - 1\} \cup j)$. Any such agent can be the next to be ordered and appear immediately after player k-1. By our assumption any such agent will violate the interface condition. This means that all players in S win against k-1 in the tournament defined in step k-1. We now distinguish between two cases. Case 1: $k \in S$. In this case consider the order obtained by switching the positions of k-1 and k in the original identity order. We denote this order by π'. We note that payments in the divide-and-conquer mechanism for the orders π and π' differ only in terms of players k-1 and k. Furthermore, since k is in S, k beats k-1 in the tournament defined in step k-1. Hence the total payment under π' is less than that under π, which contradicts the optimality of π. We now move to Case 2: k is not in S. In this case k must be ranked last under the tournament $w_{ij} = v_i(\{1, ...k-1\} \cup j)$. In particular $k + 1$ beats k in this tournament. Consider now the order π' which is identical to π except that k+1 appears before k. As in the argument made earlier, payments to all players but k and k+1 are identical in π and π' and because $k + 1$ bits k under the tournament $w_{ij} = v_i(\{1, ...k-1\} \cup j)$, the order π' corresponds to lower total payments to k and $k + 1$, again in contradiction to the optimality of π. ∎

Proof of Proposition 10 In both segregated and desegregated environments the externalities structure is symmetric and, following Corollary 5.1, all rankings are optimal. Consider first the segregated environment. Since all rankings are optimal, a possible optimal contracting scheme is $v = (c, ..., c - (n_1 - 1), c, ..., c - (n_2 - 1))$. Hence, the optimal payment for the principal is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1-1} l - \sum_{k=1}^{n_2-1} k = n \cdot c - \frac{n_1(n_1-1)}{2} - \frac{(n-n_1)(n-n_1-1)}{2}$. Assuming that $v(n_1, n_2)$ is continuous with n_1, it follows that $\frac{\partial v(n_1,n_2)}{\partial n_1} = n - 2n_1$, the maximal payment is achieved at $n_1^* = n_2^* = \frac{n}{2}$, and the cost

of incentivizing is declining with $|n_1 - n_2|$. In the desegregated example, a possible optimal contracting scheme is $v = (c, ..., c, c - n_1, ..., c - n_1)$. Therefore, the principal's payment is $v(n_1, n_2) = n \cdot c - (n - n_1) \cdot n_1$. Again, let us assume that $v(n_1, n_2)$ is continuous with n_1, in which case solving $\frac{\partial v(n_1, n_2)}{\partial n_1} = 2n_1 - n = 0$ results in the minimum payment for the principal in the desegregated environment being received at $n_1^* = n_2^* = \frac{n}{2}$, and the cost of incentivizing is increasing with $|n_1 - n_2|$. In a status environment, since group B_1 is the more esteemed group, all agents from B_1 beat all agents from B_2; therefore agents from B_1 should precede the agents from B_2 in the optimal ranking. A possible optimal ranking is $\varphi = \{i_1, ., , , i_{n_1}, j_1, ..., j_{n_2}\}$ when $i_k \in B_1$, $j_m \in B_2$, and $1 \leq k \leq n_1$, $1 \leq m \leq n_2$. Therefore, a possible optimal contracting scheme is $v = (c, c - 1..., c - (n_1 - 1), c - n_1, ..., c - n_1)$. The principal's payment is $v(n_1, n_2) = n \cdot c - \sum_{l=1}^{n_1 - 1} l - n_2 \cdot n_1 = n \cdot c - \frac{n_1(n_1 - 1)}{2} - (n - n_1)n_1 = \frac{1}{2}n_1 - nn_1 + \frac{1}{2}n_1^2 + cn$. Again, assuming that $v(n_1, n_2)$ is continuous with n_1, $\frac{\partial v(n_1, n_2)}{\partial n_1} = n_1 + \frac{1}{2} - n = 0$ and the minimal payment is achieved at $n_1^* = n - \frac{1}{2}$. Note that $V(n_1 = n) = V(n_1 = n - 1)$. Therefore, the best scenario for the principal is when $n_1 = n$. Alternatively, the cost of incentivizing is decreasing with n_1. ■

Complementary Appendix

Proposition 8 *Consider a participation problem* (N, w, c). *Let* (N, w^+, c) *be a participation problem such that* $w_i^+(j) = w_i(j)$ *if* $w_i(j) \geq 0$ *and* $w_i^+(j) = 0$ *if* $w_i(j) < 0$, *and let* u^+ *be the optimal full implementation contracts of* (N, w^+, c). *Let* $(N, w^-, 0)$ *be a participation problem such that* $w_i^-(j) = w_i(j)$ *if* $w_i(j) < 0$ *and* $w_i^-(j) = 0$ *if* $w_i(j) \geq 0$, *and let* u^- *be the optimal full implementation contracts of* $(N, w^-, 0)$. *Then, the decomposition contracting scheme* $v = u^+ + u^-$ *induces a unique full participation equilibrium. Moreover, if agents satisfy symmetry and transitivity with respect to the non-averse relation, v is the optimal contracting scheme.*

Proof of Proposition 8 To prove the proposition we use the following lemma.

Lemma 8.1: Consider a participation problem (N, w, c) with mixed externalities. Then the decomposition contracting scheme sustains full participation as a unique equilibrium.

Proof: Consider a decomposition contracting scheme v of the participation problem (N, w, c), when w includes mixed externalities. By definition $v_i = u_i^+ + u_i^-$, where u^+ is the optimal full implementation contracting scheme for the positive participation problem, and u^- is the optimal contracting scheme for the negative participation problem. Let $\phi = \{i_1, ..., i_N\}$ be the optimal ranking in the positive participation problem (N, w^+, c). As demonstrated in Proposition 6, the optimal contracting scheme for the negative participation problem is full compensation of agents for the negative externalities, i.e., $u_i^- = \sum_{j \in D_i} |w_i(j)|$ where $D_i = \{j \mid w_i(j) < 0 \ s.t. \ j \in N\}$. We will demonstrate that v sustains full participation as a unique equilibrium. To deter no-participation equilibrium, at least a single agent has to participate, regardless of the participation choices of the other agents. Agent i_1 gets $u_{i_1}^+ = c$, and $u_{i_1}^- = \sum_{j \in D_{i_1}} |w_{i_1}(j)|$; therefore, regardless of the others' choices, agent i_1 will participate in the initiative. Next, it is necessary to prevent an equilibrium in which only a single agent participates, and note that given agent $i_1's$ participation, agent i_2 will choose to participate, regardless of the choices of agents $i_3, ..., i_N$. This is true since $u_{i_2}^+ = c - w_{i_2}^+(i_1)$ and $u_{i_1}^- = \sum_{j \in D_{i_2}} |w_{i_2}(j)|$, and, given the participation of agent i_1 and the compensation for negative externalities, agent i_2 will participate. Applying the same logic for all subsequent agents, we get that the only

possible equilibrium is full participation, since all agents are willing to participate given the participation of the agents who preceded them in the optimal ranking of the positive participation problem, and since they are being compensated for negative externalities. **End of Lemma.**

Now we need to show that if the matrix of externalities satisfies symmetry and transitivity with respect to the *non-averse* relation, then the decomposition contracting scheme generates the optimal set of contracts for full participation. Note that since we have defined $w_{ii} = 0$ for each agent i, the *non-averse* relation $\succeq$ is reflexive (i.e., $i \succeq i$) and together with the symmetry and transitivity assumptions it is an equivalence relation. Therefore the set of agents can be partitioned into equivalence classes $S_1, ..., S_m$ such that $\cup_j S_j = N$ and $S_j \cap S_i = \oslash$ for all i, j, and for each $k, l \in S_i$ we have that k and l induce non-negative externalities on each other. Let us now define m new participation problems by restricting the matrix of externalities to the set S_j where $1 \leq j \leq m$. All these problems are non-negative and their optimal contracting schemes are all DAC. Let u_j^+ be an optimal contracting scheme for the problem defined on the set S_j, which is a vector of size $\#S_j$. Write $u^+ = (u_1^+, ..., u_m^+)$. For each agent $i \in N$ denote by d_i the total of negative externalities imposed on agent i, i.e., $d_i = \Sigma_{\{j; w_{ij} < 0\}} w_{ij}$, and set $u^- = (d_1, ..., d_n)$. We will show that $u = u^+ + u^-$ is an optimal contracting scheme for the mixed problem.

Let w be the original matrix of externalities and let $w = w^+ + w^-$ be the decomposition of w, where w^+ is a non-negative matrix and w^- is a non-positive matrix. Clearly, u^+ is an optimal contracting scheme for w^+. This is true since under w^+ the externalities between agents of the same class are non-negative and between agents of different classes they are zero. By a similar argument, u^- is the optimal contracting scheme for w^-. Hence $u = u^+ + u^-$ is a decomposition contracting scheme and as such by Lemma 1 it sustains full participation as a unique equilibrium.

To show that $u = u^+ + u^-$ is optimal for the participation problem (N, w, c) we have to show that the principal cannot extract more from the agents in the full participation unique equilibrium. Let ij denote the agent at the $i-$th location in the optimal order of the DAC contracting scheme within class S_j. Consider first an alternative contracting scheme u', which is identical to u in terms of incentives for all agents except for a single agent, who is placed first in the optimal order of some class j. This agent is getting less in u', i.e., $u'_{1j} < u_{1j}$. Under the contracting scheme u' there exists a Nash equilibrium

in which the set of participating agents is $N \backslash S_j$. This is true since $1j$ does not have a dominant strategy to participate and therefore all agents in S_j fail to participate in a Nash equilibrium in the game restricted to S_j, and hence also in the entire game. Furthermore, any agent in $N \backslash S_j$ chooses to participate when S_j participates, and even more so if the S_j class stays out (given the negative externalities between S_j and the rest). Hence u' does not qualify as an optimal contracting scheme. A similar argument holds whenever we reduce the payoff of more than one agent while limiting ourselves to one class only, say S_j. As we proved in Propositions 2 and 4, this induces an equilibrium on the game restricted to S_j with partial participation and this equilibrium also applies to the entire game; hence u' does not induce a unique full participation equilibrium.

Now consider the case where the alternative contracting scheme u' reduces the payoff to more than a single agent and from more than a single class. Assume first that for some k classes $\{l_1, ..., l_k\}$, with $k \leq m$, agents $1j$, $j \in \{l_1, ..., l_k\}$ are paid less in u' than in u and that all other agents are paid the same. It must be the case that under u' full participation is a Nash equilibrium; otherwise it is definitely not the optimal unique full participation Nash equilibrium. We assume therefore that under u' full participation is a Nash equilibrium and we shall show that it is nevertheless not a unique equilibrium. Specifically, we will show that under u' we can construct a Nash equilibrium with the set of participants being $N \backslash S_{j*}$, when S_{j*} is an arbitrary class such that $j^* \in \{l_1, ..., l_k\}$.

We first note that there exists an equilibrium in which none of the members in S_{j*} participate. Indeed, given that all agents in $N \backslash S_{j*}$ participate, participation is no longer a dominant strategy for agent $1j^*$ and hence there exists an equilibrium in which all agents in S_{j*} are staying out of the game restricted to S_{j*}, and therefore this equilibrium holds also for the entire game. Since for all agents in $N \backslash S_{j*}$ it was a best response to participate even when all members in S_{j*} (with whom they have negative externalities) participate, it is certainly still a best response for these agents when S_{j*} do not participate. Hence partial participation is a Nash equilibrium and we obtained the desired contradiction. Next, we assume a contracting scheme u' in which an arbitrary group of agents is paid less than the payoff in u. Let j^* be a class such that $j^* \in \{l_1, ..., l_k\}$ in which some agent ij is paid less. Using the argument in the first part of the proof and in Propositions 2 and 4, there is a Nash equilibrium in the game restricted to S_{j*} where there is only partial participation. Furthermore, the profile in which the set of participants is exactly

$N\backslash S_{j*}$ in addition to the subset of participants in S_{j*} is a Nash equilibrium in the entire game. This again follows from the fact that under u' the best response of each agent in $N\backslash S_{j*}$ to full participation by the rest must be to participate, and $N\backslash S_{j*}$ will definitely participate if a subset of S_{j*} does not participate.

We have shown so far that for a given contracting scheme u' in which some agents get less than u and the rest get the same as u, there is an equilibrium in which some agents do not participate. It is therefore left to show that for a given contracting scheme u' in which we reduce the payoff for some agents and increase the payoff for others either we still have an equilibrium in which some agents do not participate or alternatively the total payment of the principal increases. Consider first that u is unique; hence the optimal solution u_+^j for each positive participation problem in each class is unique. Assume that u' is achieved by lowering the payoff for the agent ranked first in class j^*, i.e., agent $1j^*$. To rule out the equilibrium in which class S_{j*} is not participating it is necessary to increase the payoff of another agent, say kj^*, in such a way that he will have a dominant strategy to participate, and induce the participation of the agent whose payoff was lowered. This means moving the kj^* agent up to become first in the class. But this must cost more to the principal as v^{j*} is unique, a contradiction. Alternatively, if u is not unique, then there are multiple solutions for each positive participation problem. The argument is very similar. Say that u was chosen from a group of decomposition contracting schemes. Again, to avoid partial participation equilibrium it is necessary to provide higher incentives to agent kj^*, who becomes first. If the total payment is identical to the payment in v^{j*} then we have reached a different optimal solution to the participation problem of class j^*. However, if we do not reach a different decomposition contracting scheme, this implies that the payment in this case is higher. The same logic can be applied to situations in which we lower the payoffs for other agents within the different classes. Therefore we conclude that the decomposition contracting scheme is the optimal contracting scheme that sustains full participation as a unique equilibrium. ∎