

ARITHMETIC GROUPS WITH ISOMORPHIC FINITE QUOTIENTS

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ABSTRACT. Two finitely generated groups have the same set of finite quotients if and only if their profinite completions are isomorphic. Consider the map which sends (the isomorphism class of) an S -arithmetic group to (the isomorphism class of) its profinite completion. We show that for a wide class of S -arithmetic groups, this map is finite to one, while the fibers are of unbounded size.

1. INTRODUCTION

A classical topic in number theory is the study of integral quadratic forms of the same genus, i.e., those who become isomorphic integrally in every local completion. In the 70's Grunewald, Pickel and Segal (cf. [GPS80] and the references therein) studied non-commutative analogues, establishing:

Theorem. [GPS80] *Let \mathcal{P} be the family of polycyclic-by-finite groups and for $\Gamma \in \mathcal{P}$ let $\mathcal{P}_\Gamma := \{\Lambda \in \mathcal{P} : \widehat{\Lambda} \cong \widehat{\Gamma}\}$ where $\widehat{\Gamma}$ denotes the profinite completion of Γ . Then, for any $\Gamma \in \mathcal{P}$, \mathcal{P}_Γ is a finite union of isomorphism classes.*

Note that for finitely generated groups, having isomorphic profinite completion is equivalent to having the same set of finite quotients (see subsection 2.3). Thus, in other words, the Theorem says that in the family \mathcal{P} an element is determined up to finitely many options by the set of its finite quotients.

Recently, Grunewald and Zalesskii [GZ11] revisited the topic. We will follow their terminology: Let \mathcal{C} be a family of groups and $\Gamma \in \mathcal{C}$. The \mathcal{C} -genus of Γ is

$$g(\mathcal{C}, \Gamma) := \text{IsoClasses}(\{\Lambda \in \mathcal{C} : \widehat{\Lambda} \cong \widehat{\Gamma}\})$$

which is the set of isomorphism classes of groups from \mathcal{C} whose profinite completion is isomorphic to $\widehat{\Gamma}$. In this paper we study the genus of S -arithmetic subgroups of simple algebraic groups. Let us recall (following [PR94, 4.1]) their definition:

Let \mathcal{A} be the family of groups Γ such that:

- (1) There exists $n \in \mathbb{N}$, a number field k and a k -algebraic subgroup $G \subset GL_n$ such that G is simply-connected, almost and absolutely simple.
- (2) The group Γ is isomorphic to a subgroup of $G(\bar{k})$ and commensurable to $G(\mathcal{O}_{k,S}) := G \cap GL_n(\mathcal{O}_{k,S})$ where S is a finite set of places of k

containing the Archimedean ones and $\mathcal{O}_{k,s}$ is the ring of S -integers (defined below).

- (3) The S -rank of G is ≥ 2 , i.e., $\sum_{v \in S} \text{rank}_{k_v}(G) \geq 2$ where $\text{rank}_{k_v}(G)$ is the dimension of a maximal k_v -split tori in $G(k_v)$.

Recall that two groups Γ and Λ are called *commensurable* if they have finite index subgroups $\Gamma_1 < \Gamma, \Lambda_1 < \Lambda$ with $\Gamma_1 \cong \Lambda_1$ (see also Definitions 5 and 6 below).

The group structure of elements of \mathcal{A} encodes, in some sense, the arithmetic information that is used to define them. Many rigidity results about members of \mathcal{A} have been proved, in particular Margulis' super-rigidity [Mar91] implies that whenever two elements of \mathcal{A} are isomorphic, there is a unique isomorphism between their ambient groups over a unique isomorphism of their fields of definition. In contrast to Margulis super rigidity, we find in Section 4 and in [Aka11] pairs of groups $\Gamma, \Lambda \in \mathcal{A}$ that have isomorphic profinite completion but are not isomorphic. Moreover, the associated algebraic and arithmetic information that defines Γ and Λ can be quite different. Nevertheless, for $\Gamma \in \mathcal{A}$ let

$$\mathcal{A}_\Gamma := \{\Lambda \in \mathcal{A} : \widehat{\Lambda} \cong \widehat{\Gamma}\},$$

and we conjecture the following:

Conjecture 1. *For all $\Gamma \in \mathcal{A}$, the set \mathcal{A}_Γ is a disjoint union of finitely many isomorphism classes.*

In this paper we prove:

Theorem 2. *Assume that $\Gamma \in \mathcal{A}$ has the congruence subgroup property (see 2.5) then the set \mathcal{A}_Γ is a disjoint union of finitely many isomorphism classes.*

Note that Serre's conjecture [PR94, 9.5] states that all elements of \mathcal{A} posses the congruence subgroup property and indeed this conjecture has been proved for most Γ 's in \mathcal{A} (see [PR08] for the current state of this conjecture). Without assuming the congruence subgroup property we show:

Theorem 3. *For all $\Gamma \in \mathcal{A}$, the set \mathcal{A}_Γ is contained in the union of finitely many commensurability classes.*

The profinite completions of $\Gamma \in \mathcal{A}$ is strongly related to the p -adic closure of Γ . Thus, in some sense, the proofs of the above results relies on group and representation theoretic methods that are used for extracting this "local" information about Γ , and then using some local-to-global finiteness results.

Theorem 2 answers Problem I of [GZ11] for the family of S -arithmetic groups, and the examples we give in §4 answer problems II and III for the family of S -arithmetic groups. Moreover, the proofs of Theorems 2,3 can give bounds on the size of $g(\mathcal{A}, \Gamma)$ for various $\Gamma \in \mathcal{A}$. In particular we show,

Theorem 4. *For any $n \in \mathbb{N}$ there exists $\Gamma \in \mathcal{A}$ with $|g(\mathcal{A}, \Gamma)| > n$.*

This paper is organized as follows: After giving the necessary definitions and preliminaries in §2, we prove Theorem 3 in §3. In §4 we give examples of non-isomorphic elements of \mathcal{A} that share the same profinite completion. These examples show that our proof of Theorem 3 is “tight” in the sense that all the steps of the proof are reflected in non-trivial examples. Moreover, these examples provide three different types of families which establish Theorem 4. In §5 we restrict ourselves to a fixed commensurability class \mathcal{C} whose elements admit the Congruence Subgroup Property and show that $\mathcal{C}_\Gamma = \{\Lambda \in \mathcal{C} : \widehat{\Gamma} \cong \widehat{\Lambda}\}$ is a disjoint union of finitely many isomorphism classes. Finally, we deduce Theorem 2 in section 6.

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2. NOTATION, CONVENTIONS AND PRELIMINARIES

We work with the definitions and notation from [PR94]; relevant results can be found there (especially in chapters 1 and 5).

2.1. Field and group-theoretic notation. For a number field k we let \mathcal{O}_k be the ring of integers of k , and V^k, V_f^k, V_∞^k - the set of all places of k , non-Archimedean places of k , Archimedean places of k , respectively. For $v \in V^k$, we let k_v be the corresponding completion, and \mathcal{O}_v is the ring of integers of k_v . The letter S always denotes a finite subset of V^k which contains V_∞^k . Let $\mathcal{O}_{k,S} := \{x \in \mathcal{O}_k : v(x) \geq 0, \forall v \notin S\}$ be the ring of S -integers. Finally, \mathbb{A}_k denoted the Adeles ring of k and $\mathbb{A}_{k,S}$ the ring of S -Adeles (See section 2.4 and [PR94, 1.25]). When Γ_1 is of finite-index in Γ_2 , we write $\Gamma_1 <_{fi} \Gamma_2$.

2.2. A “working” definition for the family \mathcal{A} . In order to make the reduction steps of the proof of Theorem 2 precise, we will need a more technical definition of the family \mathcal{A} . We begin by defining the notion of commensurability:

Definition 5. Two subgroups Γ_1, Γ_2 of a group H are said to be *commensurable* if $\Gamma_1 \cap \Gamma_2$ is of finite-index in both Γ_1 and Γ_2 . Two groups Γ_1, Γ_2 are said to be *abstractly commensurable* if there is a group H containing isomorphic copies Γ'_1, Γ'_2 of Γ_1, Γ_2 respectively, such that Γ'_1 and Γ'_2 are commensurable.

Since isomorphic $\Gamma, \Gamma' \in \mathcal{A}$ may have different ambient information, it will be convenient to us to record the ambient information as part of the definition of \mathcal{A} :

Definition 6. An element of \mathcal{A} is a quadruple $(\Gamma, (G, \rho), k, S)$ such that

- (1) k is a number field, i.e., a finite extension of \mathbb{Q} , and $G \xrightarrow{\rho} GL_n$ is a faithful k -representation of G as a matrix group, in which G is a simply-connected, almost and absolutely simple and defined over k . We identify G with its image under ρ .
- (2) The group $\Gamma \subset G(\bar{k})$ is commensurable to $G(\mathcal{O}_{k,S}) := G \cap GL_n(\mathcal{O}_{k,S})$ where S is a finite subset of V^k containing V_∞^k .
- (3) The S -rank of G is ≥ 2 , i.e., $\sum_{v \in S} \text{Rank}_{k_v}(G) \geq 2$ where $\text{Rank}_{k_v}(G)$ is the dimension of a maximal k_v -split tori in $G(k_v)$.

We call (G, k, S) the ambient information of Γ , G the ambient group of Γ , k the field of definition of Γ and S the valuations of Γ , and denote them by $G_\Gamma, k_\Gamma, S_\Gamma$ respectively. Often, we will leave them implicit. The *commensurability class* of an element $\Gamma \in \mathcal{A}$ is the set of all elements of \mathcal{A} which are abstractly commensurable to it.

We wish to make some further assumption on S_Γ without restricting the generality of the family \mathcal{A} .

Lemma 7. *Let $\Gamma = (\Gamma, G, k, S) \in \mathcal{A}$ and let $C := \{v \in V_f^k : G(k_v) \text{ is compact}\}$. Then $G(\mathcal{O}_{k,S \setminus C})$ is a finite-index subgroup of $G(\mathcal{O}_{k,S})$. In particular, $\Gamma = (\Gamma, G, k, S \setminus C) \in \mathcal{A}$ can also serve as ambient information for Γ .*

Proof. Recall that $\mathcal{O}_{k,T} := \{x \in k : x \in \mathcal{O}_v, \forall v \notin T\}$. Using this, one can see that the diagonal embedding $G(\mathcal{O}_{k,S}) \rightarrow \prod_{v \in S} G(k_v)$ induces the inequality

$$\begin{aligned} [G(\mathcal{O}_{k,S}) : G(\mathcal{O}_{k,S \setminus C})] &\leq \left[\prod_{v \in S} G(k_v) : \prod_{v \in S \setminus C} G(k_v) \times \prod_{v \in C} G(\mathcal{O}_v) \right] \\ &= \prod_{v \in C} [G(k_v) : G(\mathcal{O}_v)]. \end{aligned}$$

Each factor of the latter is the number of cosets of an open subgroup of a compact group, hence finite. \square

From now on, whenever an element $\Gamma = (\Gamma, G, k, S) \in \mathcal{A}$ is specified, we will always assume that S does not contain a valuation v for which $G(k_v)$ is compact.

Lemma 8. *Let $(\Gamma, G, k, S) \in \mathcal{A}$ and let $\varphi : G \rightarrow G'$ a homomorphism of algebraic groups where G' is also defined over k and φ is a k -isomorphism. Then, $\varphi(\Gamma)$ is commensurable to $G'(\mathcal{O}_{k,S})$.*

Proof. See [PR94, Theorem 4.1]. \square

We will have a particular interest in the rational primes that has no valuation in S above them; given a finite subset $S \subset V^k$ we denote by S^{full} the set of rational primes p for which $v|p$ imply that $v \notin S$. We call a prime number p *full for Γ* if $p \in S_\Gamma^{full} := (S_\Gamma)^{full}$.

Note that we do not impose the condition that Γ lie in $G_\Gamma(k_\Gamma)$. If it does lie in it, i.e. if $\Gamma \subset G_\Gamma(k_\Gamma)$, we say that Γ is *rational*. Note that in many cases, maximal arithmetic subgroups lie outside $G_\Gamma(k_\Gamma)$.

2.3. Profinite completion of a group. For background on profinite groups the reader can consult the book of Ribes and Zalesskii [RZ10]. For completeness, we record here the definition of the profinite completion of a group. Let Γ be a finitely generated group. The profinite topology on Γ is defined by taking as a fundamental system of neighborhoods of the identity the collection of all normal subgroups N of Γ such that Γ/N is finite. One can easily show that a subgroup H is open in Γ if and only if $H <_{fi} \Gamma$. We can complete Γ with respect to this topology to get

$$\widehat{\Gamma} := \varprojlim \{\Gamma/N : N <_{fi} \Gamma\};$$

this is the profinite completion of Γ . There is a natural homomorphism $i : \Gamma \rightarrow \widehat{\Gamma}$ given by $i(\gamma) = \varprojlim(\gamma N)$. A group is called *residually finite* if i is injective. One can show ([RZ10, Corollary 3.2.8]) that for finitely generated groups $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ if and only if their set of finite quotients are equal.

Lemma 9. *Let Γ be a residually finite group. There is a one-to-one correspondence between the set \mathcal{X} of all finite-index subgroups of Γ and the set \mathcal{Y} of all open subgroups of $\widehat{\Gamma}$, given by*

$$\begin{aligned} X &\mapsto cl(X) \quad (X \in \mathcal{X}) \\ Y &\mapsto Y \cap \Gamma \quad (Y \in \mathcal{Y}) \end{aligned}$$

where $cl(X)$ denotes the closure of X in $\widehat{\Gamma}$. Moreover,

$$[\Gamma : X] = [\widehat{\Gamma} : cl(X)].$$

Proof. See [LS03, Window: profinite groups, Proposition 16.4.3] □

As members of \mathcal{A} are residually finite we can use Lemma 9, to make the following definition:

Definition 10. Given an isomorphism $\Phi : \widehat{\Gamma} \rightarrow \widehat{\Lambda}$ we say that $\Lambda^0 <_{fi} \Lambda$ correspond to $\Gamma^0 <_{fi} \Gamma$ via Φ if $\Lambda^0 = \Phi(\widehat{\Gamma^0}) \cap \Lambda$ and denote $\Phi_*(\Gamma^0) = \Lambda^0$.

This sets up a correspondence between finite-index subgroups of Γ and finite-index subgroups of Λ which respects the relation of containment. Moreover, note that by Lemma 9 we have $[\Lambda : \Phi_*(\Gamma^0)] = [\Gamma : \Gamma^0]$.

2.4. Adeles groups. For an algebraic group $G \subset GL_n$ that satisfies the assumptions in Definition 6, let

$$G(\mathbb{A}_k) := \prod_{v \in V^k}^* G(k_v), \quad G(\mathbb{A}_{k,S}) := \prod_{v \in V^k, v \notin S}^* G(k_v)$$

where $*$ denotes the restricted product of $G(k_v)$ for all $v \in V^k$ with respect to $G(\mathcal{O}_v) := G \cap GL_n(\mathcal{O}_v)$ (see [PR94, §5.1] for details). For $v \notin S$, we

denote by $\pi_v : G(\mathbb{A}_{k,S}) \rightarrow G(k_v)$ and $\pi_p : G(\mathbb{A}_{k,S}) \rightarrow \prod_{v \notin S, v|p} G(k_v)$ the natural projections and we denote by $i_v : G(k_v) \rightarrow G(\mathbb{A}_{k,S})$ and by $i_p : \prod_{v \notin S, v|p} G(k_v) \rightarrow G(\mathbb{A}_{k,S})$ the natural injections.

2.5. The congruence subgroup property. We give here a brief survey; for proofs of the assertions below the reader can consult [PR08].

Given an element of \mathcal{A} of the form $G(\mathcal{O}_{k,S})$ (as always, there is an implicit embedding of G to GL_n), for any non-zero ideal I of $\mathcal{O}_{k,S}$ let K_I be the kernel of the map

$$(2.1) \quad G(\mathcal{O}_{k,S}) \rightarrow G(\mathcal{O}_{k,S}/I).$$

The completion of $G(\mathcal{O}_{k,S})$, with respect to the topology in which these kernels form a system of neighborhoods of the identity, is called the congruence completion. We denote this completion by $\overline{G(\mathcal{O}_{k,S})}$. By strong approximation for G w.r.t. S , the maps in 2.1 are surjective for all but finitely many ideals I (See [PR94, 7.4]). Using this, one can show that $\overline{G(\mathcal{O}_{k,S})}$ is naturally isomorphic to the open compact subgroup $\prod_{v \notin S} G(\mathcal{O}_v)$ of $G(\mathbb{A}_{k,S})$. We have following the short exact sequence

$$1 \rightarrow C(S, G) \rightarrow \widehat{G(\mathcal{O}_{k,S})} \rightarrow \overline{G(\mathcal{O}_{k,S})} \rightarrow 1$$

and $C(S, G)$ is called the congruence kernel of G w.r.t. S . If $C(S, G)$ is finite, we say that G admits the congruence subgroup property. Finally, if Γ is commensurable to $G(\mathcal{O}_{k,S})$ then the completion of Γ w.r.t. the family $\{K_I \cap \Gamma\}_{I \triangleleft \mathcal{O}_{k,S}}$ is called the congruence completion of Γ and is denoted by $\overline{\Gamma}$. One has a similar short exact sequence

$$1 \rightarrow C(S, \Gamma) \rightarrow \widehat{\Gamma} \rightarrow \overline{\Gamma} \rightarrow 1.$$

Definition 11. We say that $\Gamma \in \mathcal{A}$ has the congruence subgroup property if $|C(S, \Gamma)| < \infty$, and we denote by \mathcal{A}_{CSP} the set of elements of \mathcal{A} admitting the congruence subgroup property.

Remark 12. We used Lemma 7 to show that we can assume that for any $\Gamma = (\Gamma, G, k, S) \in \mathcal{A}$, S does not contain a valuation v for which $G(k_v)$ is compact. This assumption affect what we mean by “the congruence completion”. As first noted by Raghunathan, when S contain a valuation v for which $G(k_v)$ is compact, $G(\mathcal{O}_{k,S})$ has finite-index subgroups corresponding to its embedding in $G(k_v)$ that are not consider traditionally as congruence subgroups. For example, given an element of the form $G(\mathbb{Z}) \in \mathcal{A}$ such that $G(\mathbb{Q}_{p_0})$ is compact, we’ve seen in Lemma 7 that $G(\mathbb{Z}) <_{fi} G(\mathbb{Z}[\frac{1}{p_0}])$. Considering $G(\mathbb{Z}[\frac{1}{p_0}])$ with the ambient information $(G, \mathbb{Q}, \{\infty, p_0\})$ the congruence completion of $G(\mathbb{Z}[\frac{1}{p_0}])$ is equal to $\prod_{q \neq p_0} G(\mathbb{Z}_q)$. On the other hand, as $G(\mathbb{Z}) <_{fi} G(\mathbb{Z}[\frac{1}{p_0}])$ we can also consider $G(\mathbb{Z}[\frac{1}{p_0}])$ with the ambient information $(G, \mathbb{Q}, \{\infty\})$; the congruence completion of $G(\mathbb{Z}[\frac{1}{p_0}])$ in this case is equal to $G(\mathbb{Q}_p) \times \prod_{q \neq p_0} G(\mathbb{Z}_q)$. See [SW09, Page 42] for more details and for related notion of the “adelic completion”.

2.6. A number theoretic Lemma. We record a consequence of Chebotarev density Theorem which characterize the normal closure of a number field in terms of its splitting primes:

Lemma 13. *Let $Spl(k)$ be the set rational primes which are of totally split in k . If the symmetric difference of $Spl(k)$ with $Spl(k')$ is a finite set then k and k' have the same normal closure.*

Proof. See [Mar77, Theorem 43 and Chapter 8 exercise 1]. □

3. FINITENESS UP-TO COMMENSURABILITY

In this section we prove Theorem 3. The main step in our proof is to gather enough information about the following Lie algebras: Given $(\Gamma, G, k, S) \in \mathcal{A}$, for any prime number p let L_p^Γ be the Lie algebra of

$$\prod_{v \notin S, v|p} G(k_v)$$

considered over \mathbb{Q}_p . The following proposition shows that these Lie algebras are encoded in the group structure of the profinite completion:

Proposition 14. *Let $\Lambda \in \mathcal{A}_\Gamma$. Then $L_p^\Gamma \cong L_p^\Lambda$ for all primes p .*

We prove Proposition 14 at the end of this section using Margulis Super-Rigidity Theorem. We will first deduce Theorem 3 from Proposition 14 by extracting arithmetic and algebraic information from the Lie algebras appearing in Proposition 14.

3.1. The Lie algebra L_p^Γ . In this subsection we collect the arithmetic and geometric information that can be deduced from knowing L_p^Γ for all p .

Lemma 15. *Let $(\Gamma, G, k, S) \in \mathcal{A}$. Then,*

- (1) *The Lie algebra L_p^Γ is semi-simple. Its simple components are $Lie_{\mathbb{Q}_p} G(k_v)$ for all $v \notin S, v|p$.*
- (2) *Let n_p^Γ be the number of simple components of L_p^Γ , and let $N_\Gamma := \max_p \{n_p^\Gamma\}$. Then n_p^Γ equals the number of valuations $v \notin S$ that lies over p . In particular, $N_\Gamma = [k : \mathbb{Q}]$. Furthermore, $n_p^\Gamma = N_\Gamma$ if and only if p is a full prime for Γ that splits completely in k .*
- (3) *For a full prime p , the Lie algebra L_p^Γ is naturally isomorphic to the the \mathbb{Q}_p -Lie algebra of $\text{Res}_{k/\mathbb{Q}} G(\mathbb{Q}_p)$.*
- (4) *Let $D_p^\Gamma := \dim_{\mathbb{Q}_p}(L_p^\Gamma)$, $D_\Gamma = \max_p(D_p^\Gamma)$ and $\dim(G) := \dim_{\mathbb{C}}(G(\mathbb{C}))$. Then*

$$D_p^\Gamma = \dim(G) \cdot \sum_{v \notin S, v|p} [k_v : \mathbb{Q}_p] \quad \text{and} \quad D_\Gamma = \dim(G) \cdot [k : \mathbb{Q}].$$

Furthermore, p is full if and only if $D_p^\Gamma = D_\Gamma$.

Proof. We write $Lie_{k_v}(G(k_v))$ for the Lie algebra $Lie(G(k_v))$ considered over k_v and $Lie_{\mathbb{Q}_p}(G(k_v))$ for the same Lie algebra but considered over \mathbb{Q}_p . As G is absolutely simple, $Lie_{k_v}(G(k_v))$ is simple. As $[k_v : \mathbb{Q}_p]$ is finite, it is shown in [Bou98, §6.10] that $Lie_{\mathbb{Q}_p}(G(k_v))$ is also simple, so the claims of (1) follow. This also shows that each valuation $v \notin S$ that lies over p corresponds to a simple components of L_p^Γ . Thus, using the fundamental identity $[k : \mathbb{Q}] = \sum_{v \in V^k, v|p} [k_v : \mathbb{Q}_p]$, the further claims of (2) also follow. The assertion in (3) follow directly from the definition of restriction of scalars and proved in [PR94, 2.1.2].

For (4), first note that as G is absolutely simple, we know that for any valuation v , $dim_{k_v}(Lie_{k_v}(G(k_v))) = dim(G)$. So,

$$dim_{\mathbb{Q}_p}(Lie_{\mathbb{Q}_p}(G(k_v))) = [k_v : \mathbb{Q}_p] \cdot dim_{k_v}(Lie_{k_v}(G(k_v))) = [k_v : \mathbb{Q}_p] \cdot dim(G).$$

Using again the identity $[k : \mathbb{Q}] = \sum_{v \in V^k, v|p} [k_v : \mathbb{Q}_p]$, we see that

$$D_p^\Gamma = dim(G) \cdot \sum_{v \notin S, v|p} [k_v : \mathbb{Q}_p] \leq dim(G) \cdot [k : \mathbb{Q}],$$

with equality if and only if p is full. The latter bound does not depend on p so $D_\Gamma = dim(G) \cdot [k : \mathbb{Q}]$. \square

Corollary 16. *Fix $\Gamma \in \mathcal{A}$. Then, for all $\Lambda \in \mathcal{A}_\Gamma$ we have*

- (1) $[k_\Gamma : \mathbb{Q}] = [k_\Lambda : \mathbb{Q}]$, and the normal closure of k_Γ is equal to the normal closure of k_Λ ,
- (2) $dim(G_\Gamma) = dim(G_\Lambda)$,
- (3) $S_\Gamma^{full} = S_\Lambda^{full}$,
- (4) For all full primes p , we have $Res_{k_\Lambda/\mathbb{Q}}(G_\Lambda) \cong Res_{k_\Gamma/\mathbb{Q}}(G_\Gamma)$ as algebraic groups over \mathbb{Q}_p .

Proof. We use the results and notation of Lemma 15 freely in this proof. From Proposition 14 we know that for all p , $n_p^\Gamma = n_p^\Lambda$ and $D_p^\Gamma = D_p^\Lambda$. We immediately get

$$[k_\Gamma : \mathbb{Q}] = N_\Gamma = N_\Lambda = [k_\Lambda : \mathbb{Q}]$$

and likewise

$$[k_\Gamma : \mathbb{Q}] \cdot dim(G_\Gamma) = D_\Gamma = D_\Lambda = [k_\Lambda : \mathbb{Q}] \cdot dim(G_\Lambda)$$

so $dim(G_\Gamma) = dim(G_\Lambda)$. Furthermore,

$$S_\Gamma^{full} = \{p : D_p^\Gamma = D_\Gamma\} = \{p : D_p^\Lambda = D_\Lambda\} = S_\Lambda^{full}.$$

We now show that k_Γ and k_Λ has the same normal closure. Let A be the set of split primes in k_Γ and B be the set of split primes in k_Λ . Recall that a full prime p splits completely in k_Γ if and only if $n_p^\Gamma = N_\Gamma$. Clearly, $n_p^\Gamma = N_\Gamma$ if and only if $n_p^\Lambda = N_\Lambda$. Thus, a full prime p splits completely in k_Γ if and only if it splits completely in k_Λ . Therefore, the symmetric difference of A and B is contained in the complement of S_Γ^{full} , which is a finite set. By Lemma

13, k_Γ and k_Λ has the same normal closure. Finally, for part (4), let p be a full prime and note that from Lemma 15.3 we have

$$\mathrm{Lie}(\mathrm{Res}_{k_\Lambda/\mathbb{Q}}(G_\Lambda)(\mathbb{Q}_p)) \cong L_p^\Lambda \cong L_p^\Gamma \cong \mathrm{Lie}(\mathrm{Res}_{k_\Gamma/\mathbb{Q}}(G_\Gamma)(\mathbb{Q}_p)).$$

As both groups are simply connected, this Lie algebra isomorphism implies that the groups are isomorphic over \mathbb{Q}_p , as claimed. \square

3.2. Proof of Theorem 3 assuming Proposition 14.

Proof. Let us fix $\Gamma = (\Gamma, G, k, S) \in \mathcal{A}$ and denote the elements of \mathcal{A}_Γ by

$$\mathcal{A}_\Gamma = \{(\Lambda_i, G_i, k_i, S_i)\}_{i \in I}.$$

We advise the reader to recall the conventions and definitions from Section 2.2. For brevity, we say that a subset of \mathcal{A}_Γ is CFIN if it is contained in finitely many commensurability classes. Thus our aim is to show that \mathcal{A}_Γ itself is CFIN. By Corollary 16.1, $\{k_i\}_{i \in I}$ is a finite set of fields as all of them have the same normal closure. Thus we can divide \mathcal{A}_Γ into finitely many subsets, according to their fields of definition. It is enough to prove that each such subset is CFIN.

Let $C = \{(\Lambda_i, G_i, k, S_i)\}_{i \in I_1}$ be such a subset. Part 3 of Corollary 16, implies that for any $i \in I_1$, $S_\Gamma^{\mathrm{full}} = S_i^{\mathrm{full}}$. This imply that

$$S_i \subset \{v \in V^k : v|p \notin S_\Gamma^{\mathrm{full}}\}, \quad i \in I_1.$$

Note that as Γ is fixed throughout, S_Γ^{full} is a fixed set of rational primes, so $\{S_i\}_{i \in I_1}$ is a finite set (of subsets of V^k). Thus, we can divide C into finitely many subsets, according to the sets of the corresponding valuations. It is enough to prove that each such subset is CFIN .

Let $D = \{(\Lambda_i, G_i, k, S)\}_{i \in I_2}$ be such a subset. Corollary 16.2 shows that for any $i \in I_2$, $\dim(G_i) = \dim(G)$. As there are only finitely many semisimple groups of given dimension over \mathbb{C} , we deduce that the groups G_i become isomorphic over the algebraic closure to one of finitely many algebraic groups¹. We can divide D into finitely many subsets, according to the ambient groups. It is enough to prove that each such subset is CFIN.

Let $E = \{(\Lambda_i, G_i, k, S)\}_{i \in I_3}$ be such a subset. Corollary 16.4 implies that for any $i, j \in I_3$ we have $\mathrm{Res}_{k/\mathbb{Q}}(G_i)(\mathbb{Q}_p) \cong \mathrm{Res}_{k/\mathbb{Q}}(G_j)(\mathbb{Q}_p)$ for all $p \in S_\Gamma^{\mathrm{full}}$. All but finitely many primes are in S_Γ^{full} . Thus, all the groups in $\{\mathrm{Res}_{k/\mathbb{Q}}(G_i)\}_{i \in I_3}$ agree locally in all but finitely many places. Using [BS64], this shows that there are finitely many \mathbb{Q} -isomorphism classes of \mathbb{Q} -algebraic groups in $\{\mathrm{Res}_{k/\mathbb{Q}}(G_i)\}_{i \in I_3}$. Let $F := \{(\Lambda_i, G_i, k, S)\}_{i \in I_4} \subset E$ such that the elements of $\{\mathrm{Res}_{k/\mathbb{Q}}(G_i)\}_{i \in I_4}$ belong to the same \mathbb{Q} -isomorphism class. We claim that all the elements of $\{\Lambda_i\}_{i \in I_4}$ belong to the same commensurability class. Indeed, by general properties of the restriction of scalars functor, for any $i, j \in I_4$, the isomorphism $\mathrm{Res}_{k/\mathbb{Q}}(G_i) \cong \mathrm{Res}_{k/\mathbb{Q}}(G_j)$ comes from a

¹In fact, one can easily see that *all* of them become isomorphic to $G(\bar{k})$ but we do not use this fact.

unique k -isomorphism of G_i with G_j over a unique automorphism of the field k . The reader can check (see [PR94, Proposition 4.1]) that the image of Λ_i under such k -isomorphism and under an automorphism of k is a subgroup of $G_j(\bar{k})$ which is commensurable to Λ_j . This shows that all the elements of F are contained in one commensurability class and thus E is CFIN. By the reductions made above, this concludes the proof. \square

3.3. Proof of Proposition 14. As seen above, Proposition 14 is the main ingredient in the proof of Theorem 3. The main idea of its proof is that given $\Gamma \in \mathcal{A}$, we will show that L_p^Γ is the Lie algebra of a maximal p -adic analytic quotient of $\widehat{\Gamma}$. For example, for $\Gamma = SL_n(\mathcal{O}_k)$ where k is not a totally imaginary number field, one has $\widehat{\Gamma} = \prod_{v \in V^k} SL_n(\mathcal{O}_v)$ and $\prod_{v|p} SL_n(\mathcal{O}_v)$ is a p -adic analytic quotient of $\widehat{\Gamma}$ having L_p^Γ as its Lie algebra. The reader can check that it is also *maximal* p -adic analytic quotient of $\widehat{\Gamma}$ in the sense we will define momentarily.

The above example uses the congruence subgroup property, but the following proof does not. The substitute is the fact that an element $\Gamma \in \mathcal{A}$ satisfy the so-called Margulis' super-rigidity theorem [Mar91, Theorem 6, page 5]:

Theorem 17. [Margulis' super rigidity] *Let $\Gamma \in \mathcal{A}$ and H be an algebraic group which is defined over a field l with $\text{char}(l) = 0$. Then for any homomorphism $\delta : \Gamma \rightarrow H(l)$ there exist a finite index subgroup $\Gamma' < \Gamma$ such that $\delta|_{\Gamma'}$ factor as*

$$\begin{array}{ccc} \Gamma' & \xrightarrow{\delta|_{\Gamma'}} & H(l) \\ & \searrow & \nearrow \phi \\ & G(k) = (Res_{k/\mathbb{Q}}G)(\mathbb{Q}) & \end{array}$$

where ϕ is a (uniquely determined) l -morphism of algebraic groups between $Res_{k/\mathbb{Q}}G$ to the Zariski closure of $\delta(\Gamma')$.

Proof. Elements of \mathcal{A} satisfy the conditions appearing in [Mar91, Theorem 6, page 5] which states the above claim. \square

Fix a prime number p and let $\mathcal{N}(\Gamma, p)$ be the set of normal subgroups N such that $\widehat{\Gamma}/N$ is a p -adic analytic group. Note that any such p -adic analytic group is a finitely generated group which can be realized as a compact open subgroup of some algebraic group defined of \mathbb{Q}_p . For any $N \in \mathcal{N}(\Gamma, p)$ we fix such a realization and identify $\widehat{\Gamma}/N$ with this realization. We call an element $N \in \mathcal{N}(\Gamma, p)$ *maximal* if for any $M \in \mathcal{N}(\Gamma, p)$,

$$\dim(\widehat{\Gamma}/N) \geq \dim(\widehat{\Gamma}/M)$$

where \dim is the dimension as a p -adic analytic group (i.e., the dimension of the underlying analytic manifold). We denote the set of maximal elements

of $\mathcal{N}(\Gamma, p)$ by $\mathcal{N}_0(\Gamma, p)$. Let $(\Gamma, G, k, S) \in \mathcal{A}$ and $\pi : \widehat{\Gamma} \rightarrow \bar{\Gamma}$ be the natural map between the profinite completion and the congruence completion of Γ . The group $\bar{\Gamma}$ is an open compact subgroup of the Adele group $G(\mathbb{A}_{k,S})$. Let π_p be the projection from $G(\mathbb{A}_{k,S})$ to $\prod_{v \notin S, v|p} G(k_v)$.

Proposition 18. *Let $(\Gamma, G, k, S) \in \mathcal{A}$, $N_p := \ker(\pi_p \circ \pi)$. Then $N_p \in \mathcal{N}_0(\Gamma, p)$. Moreover, for any $N \in \mathcal{N}_0(\Gamma, p)$ there exist an isomorphism of Lie algebras between $\text{Lie}(\widehat{\Gamma}/N)$ and $\text{Lie}(\widehat{\Gamma}/N_p)$.*

Proof. It is clear that $N_p \in \mathcal{N}(\Gamma, p)$ and also the quotient $\widehat{\Gamma}/N_p$ is naturally contained in

$$\prod_{v|p, v \notin S} G(k_v) \cong \prod_{v|p, v \notin S} (\text{Res}_{k_v/\mathbb{Q}_p} G)(\mathbb{Q}_p)$$

and commensurable to

$$D := \prod_{v|p, v \notin S} G(\mathcal{O}_v) \cong \prod_{v|p, v \notin S} (\text{Res}_{k_v/\mathbb{Q}_p} G)(\mathbb{Z}_p).$$

These properties of the restriction of scalars functor are proved in [PR94, 2.1.2].

Now, in order to show that $N_p \in \mathcal{N}_0(\Gamma, p)$, we need to verify the maximality condition. As $\widehat{\Gamma}/N_p$ is commensurable to D , they have the same dimension as p -adic analytic groups. The group D is open in $\prod_{v|p, v \notin S} G(k_v)$, and its dimension as a p -adic analytic group is equal to the dimension over \mathbb{Q}_p of the group $\prod_{v|p, v \notin S} G(k_v)$. The latter is equal to $\sum_{v|p, v \notin S} [k_v : \mathbb{Q}_p] \cdot \dim(G)$. Note that when $p \in S_{\Gamma}^{full}$, this dimension is equal to $[k : \mathbb{Q}] \cdot \dim(G)$.

We now use Margulis' super-rigidity (Theorem 17) to show that the dimension of $\widehat{\Gamma}/N_p$ is the maximal dimension of a p -adic analytic quotient of $\widehat{\Gamma}$ that may occur. Let $M \in \mathcal{N}(\Gamma, p)$, i.e., $M \triangleleft \widehat{\Gamma}$ such that $\widehat{\Gamma}/M$ is a p -adic analytic group which can be realized as an open compact subgroup of $H(\mathbb{Q}_p)$ where H is an algebraic group which is defined over \mathbb{Q}_p ; Moreover, $\widehat{\Gamma}/M$ is and Zariski dense in $H(\mathbb{Q}_p)$. The openness condition implies that the dimension of H as an algebraic group is equal to the dimension of $\widehat{\Gamma}/M$ as a p -adic analytic subgroup, so our goal is to show that

$$(3.1) \quad \dim(H) \leq \sum_{v \notin S} [k_v : \mathbb{Q}_p] \cdot \dim(G).$$

By Theorem 17 the map

$$\Gamma \rightarrow \widehat{\Gamma} \rightarrow \widehat{\Gamma}/M \subset H(\mathbb{Q}_p)$$

factor, on a finite-index subgroup Γ' of Γ , as

$$\begin{array}{ccccc} \Gamma' \subset & \longrightarrow & \widehat{\Gamma} & \longrightarrow & \widehat{\Gamma}/M \subset H(\mathbb{Q}_p) \\ \downarrow & & & & \nearrow \phi \\ G(k) = & \text{Res}_{k/\mathbb{Q}} G(\mathbb{Q}) \subset & \longrightarrow & \text{Res}_{k/\mathbb{Q}} G(\mathbb{Q}_p) & \end{array}$$

where $\phi : Res_{k/\mathbb{Q}}(G) \rightarrow H$ is a surjective \mathbb{Q}_p -morphism of algebraic groups. Note that for $p \in S_\Gamma^{full}$, the desired inequality in (3.1) follows as the surjectivity of ϕ yields that

$$\dim(H) \leq \dim(Res_{k/\mathbb{Q}}(G)) = [k : \mathbb{Q}] \cdot \dim(G) = \dim(\widehat{\Gamma}/N_p).$$

For the general case, note that

$$(3.2) \quad (Res_{k/\mathbb{Q}}G)(\mathbb{Q}_p) \cong \prod_{v|p} (Res_{k_v/\mathbb{Q}_p}G)(\mathbb{Q}_p) \cong \prod_{v|p} G(k_v)$$

and we conclude by showing that ϕ induces a surjective map from

$$\prod_{v|p, v \notin S} (Res_{k_v/\mathbb{Q}_p}G) \rightarrow H,$$

which gives the desired inequality (3.1) as above.

Let C be the closure of $\Gamma' \cap G(\mathcal{O}_{k,S})$ in (3.2). We claim that $C = C_1 \times C_2$ where

$$C_1 <_{fi} \prod_{v|p, v \notin S} G(\mathcal{O}_v) = \prod_{v|p, v \notin S} (Res_{k_v/\mathbb{Q}_p}G)(\mathbb{Z}_p)$$

$$C_2 = \prod_{v|p, v \in S} G(k_v) = \prod_{v|p, v \in S} (Res_{k_v/\mathbb{Q}_p}G)(\mathbb{Q}_p)$$

Indeed, by our assumption on the S -rank of G , G satisfy the strong approximation property w.r.t. S (see [PR94, Theorem 7.12 and Proposition 7.2(2)]) which assert the above. Note that C_1 is compact and C_2 is an affine semisimple algebraic groups without any compact factor, by the assumption made on S after Lemma 7. The map ϕ induces a map from C to the compact group $\widehat{\Gamma}/M$, and the following Lemma shows that $\{e\} \times C_2$ is mapped to the identity of the compact group $\widehat{\Gamma}/M$.

Lemma 19. *Let L be a linear algebraic group such that there exist a finite collection of one-dimensional unipotent subgroups $\{U_i\}_{i=1}^l$ with*

$$L(\mathbb{Q}_p) = U_1(\mathbb{Q}_p) \cdots U_l(\mathbb{Q}_p).$$

Then, any polynomial map φ from $G(\mathbb{Q}_p)$ to a compact variety is a constant map.

Proof. [Proof of Lemma 19] Using the ultrametric on \mathbb{Q}_p one can easily show that any polynomial $f(x)$ with coefficients in \mathbb{Q}_p is either constant or unbounded. This imply that $\varphi := \varphi|_{U_i(\mathbb{Q}_p)}$ is a constant map for $i = 1, \dots, l$. As all the maps φ_i agree on the identity, they agree everywhere. Finally, since $L(\mathbb{Q}_p) = U_1(\mathbb{Q}_p) \cdots U_l(\mathbb{Q}_p)$, φ is the constant map. \square

By [Hum75, Thm 27.5(e) & Proposition 7.5(b)] $\{e\} \times C_2$ satisfy the conditions of Lemma 19 with $L = C_2$ and the map ϕ (which is a \mathbb{Q}_p -polynomial

map). Thus $\{e\} \times C_2$ is in the kernel of ϕ . Dividing by $\{e\} \times C_2$, ϕ induces a surjective morphism

$$\tilde{\phi}: \prod_{v|p, v \notin S} \text{Res}_{k_v/\mathbb{Q}_p} G \rightarrow H$$

so $\dim(H)$ cannot exceed

$$\dim_{\mathbb{Q}_p} \left(\prod_{v|p, v \notin S} \text{Res}_{k_v/\mathbb{Q}_p} G \right) = \sum_{v \notin S} [k_v : \mathbb{Q}_p] \cdot \dim(G).$$

as desired.

The above shows that taking $M = N$ for some $N \in \mathcal{N}_0(\Gamma, p)$ we get a surjective morphism between $\tilde{\phi}: \prod_{v|p, v \notin S} \text{Res}_{k_v/\mathbb{Q}_p} G \rightarrow H$ and from the maximality condition, H has the same dimension as $\text{Res}_{k_v/\mathbb{Q}_p} G$. Thus $\tilde{\phi}$ induces an isomorphism between $\text{Lie}(\text{Res}_{k_v/\mathbb{Q}_p} G) = \text{Lie}(\widehat{\Gamma}/N_p)$ and $\text{Lie}(H) = \text{Lie}(\widehat{\Gamma}/N)$, as claimed. \square

We now finally prove Proposition 14:

Proof. Let $\Phi: \widehat{\Gamma} \rightarrow \widehat{\Lambda}$ be a given isomorphism, p a prime number and $N \in \mathcal{N}_0(\Gamma, p)$. Note that $L_p^\Gamma = \text{Lie}(\widehat{\Gamma}/N_p)$ and therefore by Proposition 18 $L_p^\Gamma = \text{Lie}(\widehat{\Gamma}/N)$ for any $N \in \mathcal{N}_0(\Gamma, p)$. The reader can easily verify that assigning to each subgroup of $\widehat{\Gamma}$ its image by Φ , we get a bijection between $\mathcal{N}(\Gamma, p)$ and $\mathcal{N}(\Lambda, p)$ which restricts to a bijection between $\mathcal{N}_0(\Gamma, p)$ and $\mathcal{N}_0(\Lambda, p)$. Moreover, it is clear that for any $N \triangleleft \widehat{\Gamma}$, we have $\widehat{\Gamma}/N \cong \widehat{\Lambda}/\Phi(N)$. Therefore

$$L_p^\Gamma = \text{Lie}_{\mathbb{Q}_p}(\widehat{\Gamma}/N) \cong \text{Lie}_{\mathbb{Q}_p}(\widehat{\Lambda}/\Phi(N)) = L_p^\Lambda.$$

\square

4. EXAMPLES

In the course of the proof of Theorem 3 we deduced from $\widehat{\Gamma} \cong \widehat{\Lambda}$ the following facts: the groups G_Γ and G_Λ are forms of each other, the fields F_Γ and F_Λ have the same normal closure and the sets S_Γ and S_Λ have the same set of full primes.

The purpose of this section is to demonstrate that all the steps of the proof of Theorem 3 are reflected in non-trivial examples. This is done by giving explicit examples of non-isomorphic arithmetic groups which are profinitely isomorphic. The examples in 4.1 and 4.3 exhibit an explicit family of groups for which, in terms of [GZ11], have unbounded genus, hence give an answer to problem III of [GZ11].

4.1. Real forms, Profinite properties and Kazhdan's property (T).

A property \mathcal{P} of finitely generated residually finite groups is called a *profinite property* if the following is satisfied: if Γ_1 and Γ_2 are such groups with $\widehat{\Gamma}_1 \cong \widehat{\Gamma}_2$ then Γ_1 has \mathcal{P} if and only if Γ_2 has \mathcal{P} . See [Aka11] and the references within for various interesting profinite properties and various non-profinite properties. Kassabov showed that that property (τ) is not a profinite property, and he asked whether Kazhdan property (T) is a profinite property. In [Aka11], it is shown that Kazhdan property (T) is not a profinite property. Explicitly, the following is proved:

Theorem 20. *Let D be a positive square-free integer, $k := \mathbb{Q}(\sqrt{D})$ and \mathcal{O}_k its ring of integers. Fix an integer $n > 7$ and let $\Gamma = Spin(1, n)(\mathcal{O}_k)$ and $\Lambda = Spin(5, n-4)(\mathcal{O}_k)$. Then, there exist finite-index subgroups $\Gamma_0 < \Gamma$ and $\Lambda_0 < \Lambda$ such that the profinite completion of Γ_0 is isomorphic to the profinite completion of Λ_0 while Λ_0 admits property (T) and Γ_0 does not. Therefore, Kazhdan property (T) is not profinite.*

In particular, there exist non-isomorphic arithmetic groups with isomorphic profinite completions. Note that $Spin(1, n)(\mathcal{O}_k)$ (resp. $Spin(5, n-4)(\mathcal{O}_k)$) are central extensions of irreducible lattices in $H_1 := SO(1, n) \times SO(1, n)$ (resp. $H_2 := SO(5, n-4) \times SO(5, n-4)$). We can therefore deduce that H_1 and H_2 have lattices Γ_1 and Λ_1 resp. with isomorphic profinite completions. As $\mathbb{R}\text{-rank}(H_1) = 2$ while $\mathbb{R}\text{-rank}(H_2) = 10$ we see that the rank of the ambient Lie group is also not a profinite property (in contrast to well know rigidity results of Raghunathan and Margulis which showed the ambient algebraic group of isomorphic lattices have the rank).

Note that $G_\Gamma = Spin(1, n)$ and $G_\Lambda = Spin(5, n-4)$ are both forms of the complex lie group $Spin(n+1)$. All the other forms are the groups $Spin(1+4k, n-4k)$ for suitable k . Choosing suitable n and k 's the same proof show that one can construct any given number of non-isomorphic arithmetic groups with isomorphic profinite completion. Thus this family of groups has unbounded genus so this give a proof of Theorem 4. In fact, one may show that for any split G of high rank, any non-trivial element of $H^1(\mathbb{R}, G)$ can give rise to a pair of non-isomorphic arithmetic group with isomorphic profinite completion.

4.2. Arithmetic equivalence and Adelic equivalence of fields.

Let k be a number field, ζ_k its Dedekind zeta function and \mathbb{A}_k its Adele ring. Recall also that V^k is the set of all valuations on k . We define two closely related equivalence relations on number fields : k and l are said to be *arithmetic equivalent* if $\zeta_k = \zeta_l$, i.e., if their Dedekind zeta functions are equal. Similarly, k and l are said to be *Adelic equivalent* if $\mathbb{A}_k \cong \mathbb{A}_l$, i.e., if their Adele rings are isomorphic. The following proposition shows a strong relation between these two notions:

Proposition 21. (1) *Two number fields k and l are arithmetic equivalent if and only if there exist $S \subset V^k, T \subset V^l$ and a bijection*

$\phi : V^k \setminus S \longrightarrow V^l \setminus T$ such that for all $v \in V^k \setminus S$ there exist isomorphism $k_v \cong l_{\phi(v)}$.

- (2) Two number fields k and l are Adelic equivalent if and only if there exist a bijection $\phi : V^k \rightarrow V^l$ such that for all $v \in V^k$ there exist isomorphism $k_v \cong l_{\phi(v)}$.

Proof. See [Kli98, page 235 iv) and Theorem 2.3]. \square

One clearly see that being Adelic equivalent is a stronger condition.

Theorem 22. *There exist two non-isomorphic (and not totally imaginary) number fields k and l which are Adelic equivalent. Furthermore, they have the same degree, the same set of splitting primes and the same normal closure*

Proof. See [Kli98, page 239-240]. \square

We now show that non-isomorphic Adelic equivalent fields give a simple example of non-isomorphic arithmetic groups with isomorphic profinite completions. Similarly, non-isomorphic arithmetic equivalent fields give a simple example of non-isomorphic S -arithmetic groups with isomorphic profinite completions.

Proposition 23. *Fix the standard representation of SL_n ($n \geq 3$) and let k and l be Adelic equivalent non-isomorphic fields which are not totally imaginary. Set $\Gamma = SL_n(\mathcal{O}_k)$ and $\Lambda = SL_n(\mathcal{O}_l)$. Then, $\widehat{\Gamma} = \widehat{\Lambda}$ and Γ and Λ are non-isomorphic.*

Proof. As k and l are not totally imaginary, the congruence kernel of SL_n w.r.t. k and l we have that

$$\widehat{\Gamma} = \widehat{SL_n(\mathcal{O}_k)} = SL_n(\widehat{\mathcal{O}_k}) = \prod_{v \in V^k} SL_n(\mathcal{O}_v)$$

and similarly

$$\widehat{\Lambda} = \widehat{SL_n(\mathcal{O}_l)} = SL_n(\widehat{\mathcal{O}_l}) = \prod_{w \in V^l} SL_n(\mathcal{O}_w).$$

From the second part Proposition 21 we know that since k and l are Adelic equivalent, there exist a bijection $\phi : V^k \rightarrow V^l$ such that for all $v \in V^k$ there exist isomorphism between k_v and $l_{\phi(v)}$. This isomorphism induces an isomorphism from $(\mathcal{O}_k)_v$ to $(\mathcal{O}_l)_{\phi(v)}$ and as SL_n is defined over \mathbb{Q} it also induces isomorphism between $SL_n((\mathcal{O}_k)_v)$ to $SL_n((\mathcal{O}_l)_{\phi(v)})$. This shows that $\widehat{\Gamma} \cong \widehat{\Lambda}$.

If there were an isomorphism $\Psi : \Gamma \rightarrow \Lambda \subset SL_n(l)$, another version of Margulis' super-rigidity [Mar91, Theorem 5, page 5] implies that there exist an embedding of k into l , which contradicts our assumptions. \square

We note that using the first part of Proposition 21, the same proof will yield an example of non-isomorphic S -arithmetic groups with isomorphism

profinite completion from non-isomorphic arithmetic equivalent fields. Moreover, using the same group theoretic techniques that yield arithmetic equivalent pairs of fields, one can find unboundedly many fields which are arithmetic equivalent. This in turn will give rise to a family of arithmetic groups with unbounded genus as claimed in Theorem 4.

4.3. Examples involving different S 's. Let k/\mathbb{Q} be a Galois extension of degree n , p_0 be a prime that splits completely in k and $X_{p_0} := \{v_1, \dots, v_n\}$ denote the set of all valuation over p_0 . Recall that any element $\sigma \in \text{Gal}(k/\mathbb{Q})$ induces a permutation of X_{p_0} . For each $I \subset \{v_1, \dots, v_n\}$ let $\Gamma_I := \text{SL}_d(\mathcal{O}_{k, I \cup V_\infty^k})$ where $d \geq 3$.

Proposition 24. *We have $\widehat{\Gamma}_I \cong \widehat{\Gamma}_J$ if and only if $|I| = |J|$.*

Proof. As $d \geq 3$, Γ_I possesses the congruence subgroup property for any I . This implies that

$$\widehat{\Gamma}_I = \prod_{v \notin I} \text{SL}_d(\mathcal{O}_v) = \prod_{v \in X_{p_0} \setminus I} \text{SL}_d(\mathbb{Z}_{p_0}) \times \prod_{p \neq p_0} \prod_{v|p} \text{SL}_d(\mathcal{O}_v)$$

since the assumption that p_0 splits completely implies that $\mathcal{O}_v = \mathbb{Z}_{p_0}$ for all $v \in X_{p_0}$. This clearly shows that $\widehat{\Gamma}_I \cong \widehat{\Gamma}_J$ whenever $|I| = |J|$. The ‘‘only if’’ part follows from Proposition 14; indeed, $\widehat{\Gamma}_I \cong \widehat{\Gamma}_J$ implies that $L_{p_0}^{\Gamma_I} \cong L_{p_0}^{\Gamma_J}$ and these Lie algebras are isomorphic if and only if $|X_{p_0} \setminus I| = |X_{p_0} \setminus J| \iff |I| = |J|$. \square

Margulis’ super rigidity tells us when the arithmetic groups themselves, rather than merely their profinite completion, can be isomorphic:

Theorem 25. *We have $\Gamma_I \cong \Gamma_J$ if and only if there exist $\sigma \in \text{Gal}(k/\mathbb{Q})$ with $\sigma(I) = J$.*

Proof. Acting on the matrix entries (in the standard representation) of Γ_I with σ gives an isomorphism between Γ_I and $\Gamma_{\sigma(I)}$, so the if part is obvious. Now assume $\Gamma_I \cong \Gamma_J$ and map Γ_J diagonally to

$$\text{Res}_{k/\mathbb{Q}_p}(\text{SL}_n)(\mathbb{Q}_{p_0}) = \prod_{v \in X_{p_0}} \text{SL}_n(\mathbb{Q}_{p_0}).$$

Up to passing to finite-index subgroup of Γ_I , Margulis’ super rigidity gives us the following commutative diagram:

$$\begin{array}{ccccc} \Gamma_I & \hookrightarrow & \Gamma_J & \xrightarrow{\hspace{10em}} & \overline{\Gamma}_J \subset \text{Res}_{k/\mathbb{Q}}(\text{SL}_d)(\mathbb{Q}_{p_0}) \\ \downarrow & & & \nearrow \phi & \\ \text{SL}_d(k) & \cong & \text{Res}_{k/\mathbb{Q}}(\text{SL}_d)(\mathbb{Q}) & \hookrightarrow & \text{Res}_{k/\mathbb{Q}}(\text{SL}_d)(\mathbb{Q}_{p_0}) \end{array}$$

where $\phi : \text{Res}_{k/\mathbb{Q}}(\text{SL}_d) \rightarrow \text{Res}_{k/\mathbb{Q}}(\text{SL}_d)$ is a map of algebraic groups. By the universal property of the restriction of scalars functor, ϕ comes from a unique automorphism of SL_d composed with an automorphism σ of k/\mathbb{Q} .

Moreover, it must map the closure of Γ_I to the closure of Γ_J . Therefore, σ must map I to J , as claimed. \square

The following corollary gives another family of groups which prove Theorem 4.

Corollary 26. *Let I be a subset of X_p of size l with $0 < l \leq n$. Then, the set \mathcal{A}_{Γ_I} has at least $\frac{\binom{n}{l}}{n}$ isomorphism classes and is contained in at least $\frac{\binom{n}{l}}{n}$ commensurability classes.*

5. FINITENESS WITHIN A COMMENSURABILITY CLASS

We now turn to establish the desired finiteness result within a commensurability class that admits the congruence subgroup property. Recall the definition of a commensurability class given in subsection 2.2. Given a commensurability class \mathcal{C} , there exist G, k and S such that each element of \mathcal{C} has an isomorphic copy in

$$\mathcal{B}_{G,k,S} := \{\Lambda \in \mathcal{A} : (G_\Lambda, F_\Lambda, S_\Lambda) = (G, k, S)\}.$$

Note that all the elements of $\mathcal{B}_{G,k,S}$ possess the congruence subgroup property if and only if the congruence kernel of $G(\mathcal{O}_{k,S})$ is finite. In this section we prove:

Theorem 27. *Let $G(\mathcal{O}_{k,S})$ be an element of \mathcal{A}_{CSP} , and $\mathcal{B} := \mathcal{B}_{G,k,S}$. For all $\Gamma \in \mathcal{B}$, $\mathcal{B}_\Gamma := \{\Lambda \in \mathcal{B} : \widehat{\Lambda} \cong \widehat{\Gamma}\}$ is finite union of isomorphism classes.*

We will in fact show that the covolume in $G_S := \prod_{v \in S} G(k_v)$ of elements in \mathcal{B}_Γ is equal to the covolume as Γ . Using Theorem 27 in conjunction with Theorem 3 we prove Theorem 2 in the next section.

5.1. Overview of the proof. Given an isomorphism $\Phi : \widehat{\Gamma} \rightarrow \widehat{\Lambda}$, we will find in subsection 5.3 $\Gamma^0 <_{f_i} \Gamma$, $\Lambda^0 <_{f_i} \Lambda$ such that $\Phi|_{\widehat{\Gamma^0}}$ has special properties. We examine such maps in subsection 5.2 and show in particular that these properties will allow us to deduce that Γ^0 and Λ^0 has the same covolume in G_S . From this we deduce the desired result in subsection 5.4.

5.2. Product-type elements and Factor-type isomorphisms. The reader is advised to recall the notions from subsections 2.4 and 2.5 and that we called $\Gamma \in \mathcal{B}$ *rational* if $\Gamma \subset G(k)$. We say that a rational element $\Gamma \in \mathcal{B}$ is of *product-type* if Γ admits the congruence subgroup property and if $\widehat{\Gamma} = \overline{\Gamma} = \prod_{v \notin S_\Gamma} \Gamma_v \subset G(\mathbb{A}_{k,S})$ with $\Gamma_v \subset G(k_v)$. For product-type element Γ , we denote the v -factor of $\widehat{\Gamma}$ by Γ_v . We let \mathcal{B}_0 be the subset of elements of \mathcal{B} which are rational and have the trivial group as their congruence kernel. As explained in subsection 2.5 the profinite completion of an element of \mathcal{B}_0 is an open compact subgroups of $G(\mathbb{A}_{k,S})$.

Definition 28. Let $\Gamma, \Lambda \in \mathcal{B}$ be rational product-type elements, i.e. $\Gamma, \Lambda \subset G(k)$ and $\widehat{\Gamma} = \prod_{v \notin S} \Gamma_v$, $\widehat{\Lambda} = \prod_{v \notin S} \Lambda_v$. An isomorphism $\Phi : \widehat{\Gamma} \rightarrow \widehat{\Lambda}$ is called of *factor-type* if for every $v \notin S$, $\Phi|_{i_v(\Gamma_v)}$ is an isomorphism between $i_v(\Gamma_v)$ and $i_w(\Lambda_w)$ for some $w \notin S$.

We will shortly show that the existence of a factor-type isomorphism implies the equality of the covolumes of Γ and Λ in G_S , and to this end we briefly review the volume formula of Prasad. We need to use a very simple case of it which is supplied by [Pra89, Theorem 3.7]. In the terms given there, a product-type element $\Delta \in \mathcal{B}_0$ is called the principal S -arithmetic subgroup determined by the parahoric subgroups Δ_v where $\widehat{\Delta} = \prod_{v \notin S} \Delta_v$ (See [Pra89, Theorem 2.1&3.4]). Theorem 3.7 of [Pra89] shows that the volume of G_S/Δ is

$$\text{vol}(G_S/\Delta) = C(G, k) \cdot \mathcal{E}(\Delta)$$

where $C(G, k)$ is a constant that depends only on G and k and does not depend on Δ , and $\mathcal{E}(\Delta)$ is the infinite product $\prod_{v \notin S} \epsilon_v(\Delta_v)^{-1}$ where $\epsilon_v(\Delta_v)$ is the volume of Δ_v with respect to the Haar measures ω_v^* which are normalized to give measure one to any Iwahori subgroup (See [Pra89, 3.4]).

Lemma 29. *Let $\Gamma, \Lambda \in \mathcal{B}_0$ be product-type elements (as in definition 28) such that there exist a factor-type isomorphism between their profinite completions. Then, Γ and Λ have the same covolume in $G_S := \prod_{v \in S} G(k_v)$.*

Proof. We use the notation of Definition 28. As Γ, Λ are both product-type elements of \mathcal{B}_0 , it is enough to show that $\mathcal{E}(\Gamma) = \prod_{v \notin S} \epsilon_v(\Gamma_v)^{-1}$ is equal to $\mathcal{E}(\Lambda) = \prod_{v \notin S} \epsilon_v(\Lambda_v)^{-1}$. The existence of a factor-type isomorphism implies that there exists $\sigma \in \text{Sym}\{v : v \notin S\}$ such that Γ_v is isomorphic to $\Lambda_{\sigma(v)}$, for all $v \notin S$. As G is absolutely simple and simply-connected, and $k_v, k_{\sigma(v)}$ are local fields, a theorem of Pink ([Pin98, Theorem 0.3]) claims that any such isomorphism extends uniquely to an algebraic automorphism ϕ of G over a unique isomorphism of local fields $\psi : k_v \rightarrow k_{\sigma(v)}$. We claim that such isomorphism is measure-preserving. Indeed, the map ψ is measure-preserving since the measures ω_v^* are normalized to give measure one to any Iwahori subgroup and ψ maps an Iwahori subgroup to an Iwahori subgroup. Moreover, as G is simple, it is unimodular, so the automorphism ϕ is also measure-preserving. Therefore, $\epsilon_v(\Gamma_v) = \epsilon_{\sigma(v)}(\Lambda_{\sigma(v)})$. As $\sigma \in \text{Sym}\{v : v \notin S\}$ it follows that $\mathcal{E}(\Gamma) = \mathcal{E}(\Lambda)$, so the covolumes of Γ and Λ are equal. \square

In the next subsection we will see that any isomorphism induces a factor-type isomorphism on certain finite-index subgroups.

5.3. Factor-type isomorphism between finite-index subgroups. Fix a representation of G to GL_n . By Jordan's Theorem ([Dix71, page 98]) there exist a constant $j = j(n)$ such that $\langle A^j \rangle$ (the group generated by the j -th powers of A) is Abelian for any finite subgroup A of $GL_n(k_v)$ (for any valuation $v \in V^k$). The constant j is called Jordan's constant. Recall that by Lemma 9, we can define a finite-index subgroup of an element Γ

by specifying a finite-index subgroup of $\widehat{\Gamma}$. In this subsection we prove the following:

Proposition 30. *Let Γ, Λ be in \mathcal{B}_0 and assume that Γ is of product-type with $\widehat{\Gamma} = \prod_{v \notin S} \Gamma_v$. Let $D = \prod_{v \notin S} \Gamma_v^0$ where $\Gamma_v^0 = [\langle \Gamma_v^j \rangle, \langle \Gamma_v^j \rangle]$. Then $D <_{f_i} \widehat{\Gamma}$ and there exist $\Gamma^0 <_{f_i} \Gamma$ with $\widehat{\Gamma^0} = H$. Moreover, let $\Phi : \widehat{\Gamma} \rightarrow \widehat{\Lambda}$ be an isomorphism and $\Lambda^0 := \Phi_*(\Gamma^0)$ see (definition 10). Then, Λ^0 is of product-type and $\Phi|_{\widehat{\Gamma^0}}$ is a factor-type isomorphism between $\widehat{\Gamma^0}$ and $\widehat{\Phi(\Gamma^0)} = \widehat{\Lambda^0}$.*

Proof. (of Proposition 30) We first show that $D <_{f_i} \widehat{\Gamma}$. It is obvious that the closure of $[\langle \Gamma^j \rangle, \langle \Gamma^j \rangle]$ in $\widehat{\Gamma}$ is contained in D . By Margulis' normal subgroup Theorem [Mar91, Section 4.4], $[\langle \Gamma^j \rangle, \langle \Gamma^j \rangle] <_{f_i} \Gamma$ since it is an infinite normal group. Therefore $D <_{f_i} \widehat{\Gamma}$. As Γ and Λ are elements of \mathcal{B}_0 , their profinite completions are naturally compact open subgroups of $G(\mathbb{A}_{k,S})$. We introduce some useful notation: For a prime p let

$$X_p := \{v : v \notin S, v|p\}, \quad H := \prod_{v \in X_p} \Gamma_v$$

$$\mathfrak{g} := Lie(H) = Lie_{\mathbb{Q}_p} \left(\prod_{v \in X_p} G(k_v) \right), \quad H^0 := \prod_{v \in X_p} \Gamma_v^0.$$

Given a map $\Phi : A \rightarrow B$ where A and B are two subsets of $G(\mathbb{A}_{k,S})$, let $\Phi_{\beta\alpha} := \pi_\beta \circ \Phi \circ i_\alpha$ where α and β are can be valuations of k or rational primes (the maps i_α and π_β were defined in subsection 2.4).

We claim that for any rational prime p and $w \in V^k \setminus S$ such that $w \nmid p$ we have that the image of $\Phi_{wp} : H \rightarrow G(k_w)$ is a finite group. Indeed, as H is a virtually pro- p compact group and so is its image. On the other hand, its image is also a compact subgroup of $G(k_w)$ and therefore it is virtually pro- q where $w|q \neq p$, hence is must be finite.

Now we can use Jordan's Theorem to show that $\Phi_{wp}|_{H^0} : H^0 \rightarrow G(k_w)$ has trivial image: This is because $\Phi_{wp}(H^0)$ is generated by

$$\{\Phi_{wv}(\overline{[\langle \Gamma_v^j \rangle, \langle \Gamma_v^j \rangle]})\}_{v \in X_p}$$

and

$$(5.1) \quad \Phi_{wv}(\overline{[\langle \Gamma_v^j \rangle, \langle \Gamma_v^j \rangle]}) = \overline{[\langle (\Phi_{wv}(\Gamma_v))^j \rangle, \langle (\Phi_{wv}(\Gamma_v))^j \rangle]} = \{e\}$$

where last equality follows from Jordan's Theorem and since $\Phi_{wv}(\Gamma_v)$ is a finite group by the last claim.

The above shows that $\Phi_{pp}|_{H^0}$ is injective and therefore an isomorphism of the p -adic analytic group H^0 with its image $\Phi_{pp}(H^0)$. Thus the latter is a p -adic analytic compact group of dimension $\dim(H^0) = \dim(\mathfrak{g})$. Thus $\Phi_{pp}(H)$ is open in $\prod_{v \in X_p} G(k_v)$. Therefore, the derivative of $\Phi_{pp}|_{H^0}$, which is also the derivative of $\Phi_{pp}|_H$, is an automorphism of \mathfrak{g} .

Let $\mathfrak{g} = \oplus_{v \in X_p} \mathfrak{g}_v$ where $\mathfrak{g}_v = Lie(G(k_v))$. As \mathfrak{g} is semisimple, there exist a permutation $\sigma \in Sym(X_p)$ such that d is induced by an isomorphism of \mathfrak{g}_v

with $\mathfrak{g}_{\sigma(v)}$, $v \in X_p$. This imply that for any $w \neq \sigma(v)$, $\Phi_{wv} : \Gamma_v \rightarrow G(k_w)$ has finite image, and by the same argument as in (5.1) this also imply that $\Phi_{wv}|_{\Gamma_v^0} : \Gamma_v^0 \rightarrow G(k_w)$ has trivial image. Therefore, $\Phi_{\sigma(v)v} : \Gamma_v^0 \rightarrow G(k_{\sigma(v)})$ is injective on Γ_v^0 .

Let $\Delta_w := \Phi_{\sigma(v)v}(\Gamma_v^0) \subset G(k_w)$ where v and σ are the unique valuation and permutation that satisfy $\sigma(v) = w$ and supplied by the above argument. As d induce an isomorphism of \mathfrak{g}_v with \mathfrak{g}_w , Δ_w is open in $G(k_w)$. We now show that $\widehat{\Lambda}^0$ defined above is of product-type by showing that $\widehat{\Lambda}^0 = \prod_{w \notin S} \Delta_w$. Since $\widehat{\Gamma}^0 = \prod_{v \notin S} \Gamma_v^0$, the group $\Phi(\widehat{\Gamma}^0) = \widehat{\Lambda}^0$ is generated by $\{\Phi \circ i_v(\Gamma_v^0)\}_{v \notin S}$. The above shows that $\Phi \circ i_v(\Gamma_v^0)$ has trivial w -coordinate whenever $w \neq \sigma(v)$. Therefore, only the elements of Δ_w may appear as the w -coordinates of elements of $\widehat{\Lambda}^0$, so $\widehat{\Lambda}^0 \subseteq \prod_{w \notin S} \Delta_w$. Similarly, any $(\alpha_w) \in \prod_{w \notin S} \Delta_w$ is in $\widehat{\Lambda}^0$ since it is the image of $(i_{\sigma^{-1}(w)}^{-1} \circ \Phi^{-1}(\alpha_w))_{w \notin S} \in \prod_{v \notin S} \Gamma_v^0 = \widehat{\Gamma}^0$, showing $\widehat{\Lambda}^0 = \prod_{w \notin S} \Delta_w$. This also shows that $\Phi|_{\widehat{\Gamma}^0}$ is a factor-type isomorphism as it is induced from the isomorphisms $\Phi_{\sigma(v)v}|_{\Gamma_v^0} : \Gamma_v^0 \rightarrow \Delta_{\sigma(v)}$. \square

5.4. Proof of Theorem 27. In order to use Proposition 30 we need to find finite-index subgroups of general elements of \mathcal{B} that will satisfy the assumptions of Proposition 30, so we begin by the following lemma:

Lemma 31. *Let $\Gamma \in \mathcal{B}$.*

- (1) *There exist $\Gamma^0 <_{fi} \Gamma$ such that $\Gamma^0 \subset G(k)$.*
- (2) *There exist $\Gamma^0 <_{fi} \Gamma$ having the trivial group as its congruence kernel which implies that $\overline{\Gamma^0} = \widehat{\Gamma^0}$, i.e., the congruence completion of Γ^0 is equal to the profinite completion of Γ^0 .*
- (3) *Assume $\Gamma \in \mathcal{B}_0$. There exist $\Gamma^0 <_{fi} \Gamma$ which is of product-type.*

Proof.

As Γ is commensurable to $G(\mathcal{O}_{k,S})$, one can take $\Gamma^0 = \Gamma \cap G(\mathcal{O}_{k,S})$ for (1). For (2), Note that we are assuming that the congruence kernel of $G(\mathcal{O}_{k,S})$ which is denoted by $C(S, G)$ is finite. The congruence kernel of Γ , $C(S, \Gamma)$ is contained in $C(S, G)$ hence it is also finite and therefore discrete in the profinite topology on $\widehat{\Gamma}$. Therefore, there exist a finite-index subgroup $H < \widehat{\Gamma}$ such that $C(S, \Gamma) \cap H = \{e\}$. Lemma 9 implies that H is of the form $\widehat{\Gamma^0}$ with $\Gamma^0 <_{fi} \Gamma$. One can show that for any $\Delta <_{fi} \Gamma$, $C(S, \Delta) = C(S, \Gamma) \cap \widehat{\Delta}$, thus the congruence kernel of Γ^0 is

$$C(S, \Gamma) \cap \widehat{\Gamma^0} = C(S, \Gamma) \cap H = \{e\}$$

so we have $\overline{\Gamma^0} = \widehat{\Gamma^0}$.

Finally, for (3), recall that the assumption that $\Gamma \in \mathcal{B}_0$ implies that $\widehat{\Gamma} = \overline{\Gamma}$ is an open compact subgroup of $G(\mathbb{A}_{k,S})$. From the definition of the topology on $G(\mathbb{A}_{k,S})$, there exist a finite-index subgroup H of $\overline{\Gamma} \subset G(\mathbb{A}_{k,S})$ of the form

$H = \prod_{v \notin S} P_v$. Again, by Lemma 9, H is of the form $\widehat{\Gamma}^0$ with $\Gamma^0 <_{f_i} \Gamma$, so Γ^0 is of product-type. \square

Proof. (of theorem 27) Let Λ be an arbitrary element of \mathcal{B}_Γ and $\Phi : \widehat{\Gamma} \rightarrow \widehat{\Lambda}$ be an isomorphism. Using Lemma 31 we first find $\Lambda_1 <_{f_i} \Lambda$ such that $\Lambda_1 \in \mathcal{B}_0$. Let $\Gamma_1 <_{f_i} \Gamma$ such that $\Phi_*(\Gamma_1) = \Lambda_1$. Then, again using Lemma 31, we can find $\Gamma_2 <_{f_i} \Gamma_1$ with the following properties: $\Gamma_2 \in \mathcal{B}_0$ and it is of product-type. Let $\Lambda_2 := \Phi_*(\Gamma_2)$ and note that $[\Gamma : \Gamma_2] = [\Lambda : \Lambda_2]$ and that Λ_2 also belongs to \mathcal{B}_0 .

Now, the groups Γ_2 and Λ_2 together with the isomorphism $\Phi|_{\widehat{\Gamma}_2} : \widehat{\Gamma}_2 \rightarrow \widehat{\Lambda}_2$ satisfy the the conditions of Proposition 30. Therefore, we find there exist $\Gamma_3 <_{f_i} \Gamma_2$ such that Γ_3 and $\Lambda_3 := \Phi_*(\Gamma_3)$ have a factor-type isomorphism between them and thus the same covolume in G_S (Lemma 29). Moreover,

$$[\Gamma : \Gamma_3] = [\Lambda : \Phi_*(\Gamma_3)] = [\Lambda : \Lambda_3],$$

so Γ and Λ also have the same covolume in G_S . As Λ was arbitrary, it follows that *all* the elements in \mathcal{B}_Γ has the same covolume. A theorem by Borel which is a S -arithmetic extension a well-known theorem of Wang [Bor87] asserts that there are finitely many isomorphism classes of lattices of bounded covolume in G_S , which is of S -rank ≥ 2 . The elements of \mathcal{B} are such lattices so this conclude the proof of theorem 27. \square

6. PROOF OF THE MAIN THEOREM 2

By Theorem 3 there exist finitely many commensurability classes, $\mathcal{C}^1, \dots, \mathcal{C}^r$ and arithmetic groups $\Gamma_1, \dots, \Gamma_r$ with $\Gamma_i \in \mathcal{C}^i$ such that

$$\mathcal{A}_\Gamma = \cup_{i=1}^r \mathcal{C}_{\Gamma_i}^i,$$

where $\mathcal{C}_\Delta^i := \{\Lambda \in \mathcal{C}^i : \widehat{\Lambda} \cong \widehat{\Delta}\}$.

Given an element $\Delta \in \mathcal{A}$ a Theorem of Lubotzky [Lub95] characterize the property “ Δ has the Congruence Subgroup Property” in terms of certain group-theoretic properties of $\widehat{\Delta}$. Since Γ has CSP and $\widehat{\Gamma} \cong \widehat{\Gamma}_i$ it follows that Γ_i has CSP for all $i = 1, \dots, r$. Finally, for each i there exist (G_i, k_i, S_i) such that any element of \mathcal{C}^i has an isomorphic copy in $\mathcal{B}_{G_i, k_i, S_i}$. Therefore Theorem 27 applied for $\mathcal{B}_{G_i, k_i, S_i}$ implies that $\mathcal{C}_{\Gamma_i}^i$ is a finite union of isomorphism classes. As i was arbitrary, it follows that \mathcal{A}_Γ is also a finite union of isomorphism classes.

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