Uniqueness of Structure in Banach Spaces

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0 Introduction

In this chapter, we consider Banach spaces which can be represented as spaces of sequences or functions with some specific properties, and study the natural question whether such a representation is unique, up to a notion of equivalence which can vary from case to case. The typical space of sequences is derived from a Banach space X with a normalized Schauder basis $\{e_n\}_{n=1}^{\infty}$, where to each element $x = \sum_{n=1}^{\infty} a_n e_n \in X$, there corresponds the sequence of coefficients $(a_1, a_2, \cdots, a_n, \cdots)$. Of particular interest in this context will be the spaces with a normalized unconditional or symmetric basis.

The notion of uniqueness mentioned above means a different thing in each of the cases under consideration. A Banach space X will be said to have a **unique** general or unconditional or symmetric basis, up to equivalence, if, for any two normalized bases of the **same** type, there exists an automorphism T of X which takes one basis into the other.

The existence of a normalized Schauder basis $\{e_n\}_{n=1}^{\infty}$ in a Banach space X allows the representation of X as a space of **ordered** sequences. In the special case where $\{e_n\}_{n=1}^{\infty}$ is an unconditional basis any permutation $\{e_{\pi(n)}\}_{n=1}^{\infty}$ of $\{e_n\}_{n=1}^{\infty}$ is again a basis which need not be equivalent to $\{e_n\}_{n=1}^{\infty}$, for every permutation π, unless $\{e_n\}_{n=1}^{\infty}$ is a symmetric basis. Therefore, in the case of spaces with a normalized unconditional basis (which is not symmetric) we have the option of considering a different type of uniqueness of the normalized unconditional basis, namely that of uniqueness up to a permutation and equivalence. More precisely, a space X has a **unique** normalized unconditional basis, up to a **permutation** (and equivalence) whenever, for each pair of normalized unconditional bases of X, there exists an automorphism T of X which takes the first basis into a **permutation** of the second. Of course, both notions of uniqueness coincide in the case of symmetric bases.

In the continuous case, we typically consider Banach lattices which can be represented as spaces of measurable functions on a suitable measure space. Most interesting is the special case of the so-called rearrangement invariant (r.i.) function spaces over a measure space (Ω, Σ, μ). The main property of such a space X of functions is that automorphisms of the underlying measure space transform elements of X into elements of X with the **same** norm. In the Basic Concepts article such spaces are called **symmetric lattices** .

The notion of uniqueness can be also studied in the context of finite dimensional spaces, but, in this case, we have to be more careful since for spaces of the same dimension all the structures are obviously unique. However, for families of Banach spaces $\{X_n\}_{n=1}^{\infty}$, with $\dim X_n = n$, for all n it makes perfect sense to inquire whether, for any two normalized bases of X_n, of the **same** type, there exists an automorphism T_n of X_n which takes the first basis into the second and, most importantly, the quantity $\|T_n\| \cdot \|T_n^{-1}\|$ is bounded by a constant independent of n but possibly dependent on the structure constant (by structure constant we mean either the basis constant or unconditional or symmetric constant, according to the case under consideration).

1 Uniqueness of general and unconditional bases

It is quite easy to prove that a separable Hilbert space has a unique normalized unconditional basis, up to equivalence. Indeed, if $\{u_n\}_{n=1}^{\infty}$ is a normalized K-unconditional basis in ℓ_2 then, by the parallelogram identity,

$$\mathop{Ave}_{\sigma_n=\pm 1} \|\sum_{n=1}^{N} a_n \sigma_n u_n\|^2 = \sum_{n=1}^{N} |a_n|^2$$

for every choice of N and scalars $\{a_n\}_{n=1}^{2^N}$. This of course implies that

$$K^{-1}\Big(\sum_{n=1}^{\infty}|a_n|^2\Big)^{1/2} \le \|\sum_{n=1}^{\infty} a_n u_n\| \le K\left(\sum_{n=1}^{\infty}|a_n|^2\right)^{1/2},$$

for any choice of $\{a_n\}_{n=1}^{\infty}$, i.e. $\{u_n\}_{n=1}^{\infty}$ is K-equivalent to the unit vector basis of ℓ_2.

It turns out that a separable Hilbert space does not have unique basis, up to equivalence. This fact, which is less trivial, was first proved by Babenko [B]. His construction is based on the simple observation that the characters $\{e^{int}\}_{n\in\mathbb{Z}}$, in the order $\{1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \cdots\}$, form a basis in the space $L_p(\mathbb{T}, W(t))$, where $W(t)$ is an integrable weight function on the circle $\mathbb{T}$, if and only if the Riesz projection, defined by

$$P_+\left(\sum_{n=-\infty}^{+\infty} a_n e^{int}\right) = \sum_{n=0}^{\infty} a_n e^{int}$$

is bounded on this space. This latter question has been extensively studied in harmonic analysis and a well known necessary and sufficient condition for the boundedness of the Riesz projection (or, as a matter of fact, of the Hilbert transform) in the space $L_p(\mathbb{T}, W(t))$ is the so-called A_p-condition (see e.g. [G p.254])

$$\sup_I\Big(\frac{1}{\mu(I)}\int_I W(t)dt\Big)\ \Big(\frac{1}{\mu(I)}\int_I \Big(\frac{1}{W(t)}\Big)^{\frac{1}{p-1}}dt\Big)^{p-1} < \infty,$$

where μ is the Lebesgue measure and the supremum is taken over all intervals $I \subset \mathbb{T}$ of positive measure.

The weight function considered by Babenko is

$$W_\alpha(t) = |t|^{2\alpha},$$

where α is a suitable number. In order to ensure the integrability of $W_\alpha(t)$, one has to require that $2\alpha + 1 > 0$ i.e. that $\alpha > -1/2$. Furthermore, it is easily verified that the A_2-condition is satisfied by W_α iff $1 - 2\alpha > 0$, i.e. when $\alpha < 1/2$. It follows that, for $-1/2 < \alpha < 1/2$, the characters $\{e^{int}\}_{n\in\mathbb{Z}}$, in the order described above, form a basis of the Hilbert space $L_2(\mathbb{T}, W_\alpha(t))$. Equivalently, the sequence

$$\Big\{ \Big(\frac{2\alpha+1}{2\pi^{2\alpha+1}}\Big)^{1/2} |t|^\alpha e^{int} \Big\}_{n\in\mathbb{Z}}$$

forms a normalized Schauder basis in $L_2(\mathbb{T})$, for any value of $-1/2 < \alpha < 1/2$. For $\alpha = 0$, we recover the (unique) unconditional basis of the separable Hilbert space while, for the remaining values of α, we obtain mutually non-equivalent conditional bases of the Hilbert space. Indeed, for $\alpha \neq \beta$ in the interval $(-1/2, 1/2)$ and $0 < \lambda < \pi$, we can find scalars $\{a_n\}_{n\in\mathbb{Z}}$ so that

$$f(t) = \sum_{n=-\infty}^{+\infty} a_n e^{int} = \chi_{[0,\lambda)}(t).$$

in $L_2(\mathbb{T})$ and a.e. on $\mathbb{T}$. Then the fact that

$$\frac{\|f|t|^\alpha\|}{\|f|t|^\beta\|} = \lambda^{\alpha-\beta}; \;\; 0 < \lambda < \pi,$$

shows that the bases $\{|t|^\alpha e^{int}\}_{n\in\mathbb{Z}}$ and $\{|t|^\beta e^{int}\}_{n\in\mathbb{Z}}$ are not equivalent.

Another construction of conditional bases in a separable Hilbert space, due to C.A. McCarthy and J. Schwartz [Mc - S], is presented in detail in [L - T I p. 74].

The discussion above can be summarized in the following proposition.

Proposition 1.1 *A separable Hilbert space has, up to equivalence, a unique normalized unconditional basis and uncountably many mutually non-equivalent conditional basis.*

In fact, the following more general result was proved by Pelczynski and Singer [P - S].

Theorem 1.2 *Any Banach space with an infinite Schauder basis has uncountably many mutually non-equivalent bases.*

It turns out that also the spaces ℓ_1 and c_0 have a unique unconditional basis, up to equivalence, but this fact, which was proved by J. Lindenstrauss and A. Pelczynski [L - P], is more difficult and requires some consequences of the famous Grothendieck inequality. We refer to well-known corollaries of this inequality that every bounded linear operator $T\colon c_o \to \ell_1$ is 2-absolutely summing and its 2-summing norm $\pi_2(T)$ satisfies the inequality $\pi_2(T) \leq K_G\|T\|$, and that every bounded linear operator $U\colon \ell_1 \to \ell_2$ is absolutely summing and its 1-summing norm $\pi_1(U)$ satisfies $\pi_1(U) \leq K_G\|U\|$. In both inequalities above, K_G denotes as usual the Grothendieck constant. The precise statement of Grothendieck's inequality together with a simple proof can be found in the Basic Concepts article.

In order to prove e.g. that ℓ_1 has a unique unconditional basis, up to equivalence, let $\{x_n\}_{n=1}^\infty$ be a normalized K-unconditional basis of ℓ_1, for some $K \geq 1$, and fix a sequence of scalars $\{a_n\}_{n=1}^\infty$ such that the series $\sum\limits_{n=1}^\infty a_n x_n$ converges. Then notice that the operator $T\colon c_o \to \ell_1$, defined by $Tt = \sum\limits_{n=1}^\infty a_n t_n x_n$, for $t = \{t_n\}_{n=1}^\infty \in c_0$, is of norm $\|T\| \leq 2K\|\sum\limits_{n=1}^\infty a_n x_n\|$ (in the real case, $2K$ can be replaced by K). Hence, by the aforementioned estimate of the 2-summing norm of T, we conclude that $\pi_2(T) \leq 2K_G K\|\sum\limits_{n=1}^\infty a_n x_n\|$. Then the definition of $\pi_2(T)$, applied for the unit vectors in c_0, together with the fact that $\{x_n\}_{n=1}^\infty$ is assumed to be normalized imply that

$$\left(\sum_{n=1}^\infty |a_n|^2\right)^{1/2} = \left(\sum_{n=1}^\infty \|Te_n\|^2\right)^{1/2} \leq 2K_G K\|\sum_{n=1}^\infty a_n x_n\|.$$

This inequality shows that the operator $U\colon \ell_1 \to \ell_2$, defined by $U(\sum\limits_{n=1}^\infty a_n x_n) = \{a_n\}_{n=1}^\infty$, for any convergent series $\sum\limits_{n=1}^\infty a_n x_n \in \ell_1$, is bounded by $2K_G K$. Then, by using the estimate, discussed above, of the 1-summing norm $\pi_1(U)$ of U, it follows that $\pi_1(U) \leq 2K_G^2 K$. Hence, for any $x = \sum\limits_{n=1}^\infty a_n x_n \in \ell_1$,

$$\sum_{n=1}^\infty |a_n| = \sum_{n=1}^\infty \|U(a_n x_n)\| \leq \pi_1(U) \sup_{\varepsilon_n = \pm 1} \|\sum_{n=1}^\infty \varepsilon_n a_n x_n\| \leq 4K_G^2 K^2\|\sum_{n=1}^\infty a_n x_n\|,$$

which completes the proof of the fact that $\{x_n\}_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ_1.

The case of c_0 is proved by using a simple duality argument.

The summary of this discussion is

Proposition 1.3 *Both spaces ℓ_1 and c_0 have, up to equivalence, a unique normalized unconditional basis.*

An alternative proof of Proposition 1.3, which does not use Grothendieck's inequality, is presented in Section 5 of the Basic Concepts article.

It turns out that ℓ_2, ℓ_1 and c_o are the only Banach spaces with a unique unconditional basis, up to equivalence. This result was proved by Lindenstrauss and Zippin [L - Z].

Theorem 1.4 *A Banach space has, up to equivalence, a unique unconditional basis if and only if it is isomorphic to one of the spaces ℓ_2, ℓ_1 or c_0.*

Instead of the original proof from [L - Z], we present a shorter proof based on an argument due to W.B. Johnson. Suppose that a space X has, up to equivalence, a unique normalized unconditional basis $\{x_n\}_{n=1}^{\infty}$. Since, for any choice of $\varepsilon_n = \pm 1, n = 1, 2, \cdots, \{\varepsilon_n x_n\}_{n=1}^{\infty}$ is an unconditional basis of X it must be equivalent to $\{x_n\}_{n=1}^{\infty}$ and, thus, $\{x_n\}_{n=1}^{\infty}$ is symmetric. Let now $\{u_m\}_{m=1}^{\infty}$ be an arbitrary normalized block basis with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ and denote by U its span.

If we prove that X is isomorphic to $X \oplus U$, then

$$\{x_1, u_1, x_2, u_2, \cdots, x_n, u_n, \cdots\}$$

forms a normalized unconditional basis of a space isomorphic to X. Hence, by the uniqueness of $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$ i.e. $\{x_n\}_{n=1}^{\infty}$ is a perfectly homogeneous basis in the terminology of [Z]. Thus, by the well known result of Zippin [Z], $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_p; $1 \leq p < \infty$ or of c_0. In order to complete the proof, recall e.g. that the Rademacher functions $\{r_n\}_{n=1}^{\infty}$ span a Hilbert space in $L_p(0,1)$, for any value of $p \geq 1$, and their span $[r_n]_{n=1}^{\infty}$ in complemented in $L_p(0,1)$, whenever $p > 1$. In particular, $\ell_p^{2^n}$ contains a uniformly complemented copy of ℓ_2^n, for any n and $p > 1$. Hence, ℓ_p contains a complemented copy of the direct sum $\left(\sum\limits_{n=1}^{\infty} \oplus \ell_2^n\right)_p$ and, by the decomposition method of Pelczynski, $\ell_p \approx \left(\sum\limits_{n=1}^{\infty} \oplus \ell_2^n\right)_p$, for $p > 1$, which shows that the unit vector basis of ℓ_p is not the unique unconditional basis of this space, up to equivalence, for $p > 1, p \neq 2$.

In order to prove that $X \approx X \oplus U$, Johnson suggested to use the decomposition method, as follows: Let V be the span of normalized blocks with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ which has the universality property that every possible normalized block with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ appears infinitely many times in the sequences generating V. Then, since any block basis with constant coefficients spans a complemented subspace in a space with a symmetric basis (see e.g. [L] or [L - T I p. 123] $X \oplus V$ is isomorphic to a complemented in X, i.e. that

$$X \approx X \oplus V \oplus W,$$

for some space W. Since $V \oplus U \approx V$ because of the above universality property, it follows that

$$X \approx X \oplus V \oplus U \oplus W \approx X \oplus U,$$

which completes the argument.
Remark: It can be shown that any space with an unconditional basis which is not unique, up to equivalence, actually has uncountably many mutually non equivalent unconditional bases.

Suppose now that X is a space with a unique, up to equivalence, normalized Schauder basis $\{x_n\}_{n=1}^{\infty}$. Then $\{x_n\}_{n=1}^{\infty}$ is equivalent to $\{\varepsilon_n x_n\}_{n=1}^{\infty}$, for every choice of $\varepsilon_n = \pm 1$, $n = 1, 2, \cdots$, i.e. $\{x_n\}_{n=1}^{\infty}$, is unconditional and thus equivalent to the unit vector basis of ℓ_2, ℓ_1 or c_o. However, in each of these three cases, X also has a conditional basis. Simple examples of conditional bases are $\{e_n - e_{n-1}\}_{n=1}^{\infty}$ in ℓ_1 (where $e_0 = 0$ and $\{e_n\}_{n=1}^{\infty}$ denotes the unit vector basis in ℓ_1) and the summing basis $\{\sum\limits_{i=n}^{\infty} e_n\}_{n=1}^{\infty}$ in c, which of course is isomorphic to c_0. These facts together with Proposition 1.1 provide a partial proof of Theorem 1.2.

2 Uniqueness of symmetric bases

An immediate consequence of Theorem 1.3 is the fact that ℓ_2, ℓ_1 and c_0 have also a unique symmetric basis, up to equivalence. It turns out that there are considerably more Banach spaces with a unique, up to equivalence, symmetric basis.

Theorem 2.1 *The spaces c_0 and $\ell_p; 1 \leq p < \infty$, have, up to equivalence, a unique symmetric basis.*

Proof: Fix $p > 1$, let $\{e_n\}_{n=1}^{\infty}$, denote the unit vector basis of $\ell_p, \{e_n^*\}_{n=1}^{\infty}$ the biorthogonal sequence associated to $\{e_n\}_{n=1}^{\infty}$ and let $\{x_m\}_{m=1}^{\infty}$ be another normalized symmetric basis of ℓ_p. By a simple diagonalization argument, one can find a subsequence $\{x_{m_i}\}_{i=1}^{\infty}$ of $\{x_m\}_{m=1}^{\infty}$ such that $\lim\limits_{i\to\infty} e_n^* x_{m_i}$ exists for any choice of n. If all these limits are equal to zero then one can find a further subsequence of $\{x_m\}_{m=1}^{\infty}$, which is equivalent to a block basis of $\{e_n\}_{n=1}^{\infty}$. Since any normalized block basis of $\{e_n\}_{n=1}^{\infty}$ is equivalent to $\{e_n\}_{n=1}^{\infty}$ itself in ℓ_p it follows that $\{x_m\}_{m=1}^{\infty}$ is equivalent to $\{e_n\}_{n=1}^{\infty}$.

In case there exists a value of n for which $\lim\limits_{i\to\infty} e_n^* x_{m_i} = \alpha \neq 0$ then one can easily find an infinite subsequence of $\{x_m\}_{m=1}^{\infty}$ which is equivalent to the unit vector basis of ℓ_1, thus completing the proof. $\square$

The class of spaces with a unique symmetric basis is considerably larger than that of ℓ_p-spaces and it contains e.g. all the Orlicz sequence spaces ℓ_M for which the limit $\lim\limits_{t\to 0} tM'(t)/M(t)$ exists. A concrete such function is $M(t) = t^p/(1 + |\log t|); p \geq 1$. Another class of spaces with a unique symmetric basis is that of Lorentz sequence spaces $d(w, p)$ where $p \geq 1$ and $w = \{w_n\}_{n=1}^{\infty}$ is a non-increasing sequence of positive weights satisfying $w_1 = 1$, $\lim\limits_{n\to\infty} w_n = 0$ and $\sum\limits_{n=1}^{\infty} w_n = \infty$. Recall that $d(w, p)$ in the space of all sequences $\alpha = \{\alpha_n\}_{n=1}^{\infty}$ of scalars so that

$$\|\alpha\| = \sup_{\pi} \left(\sum_{n=1}^{\infty} |\alpha_{\pi(n)}^{p} w_n| \right)^{1/p} < \infty,$$

where the supremum is taken over all permutation π of the integers.

The proofs that these two classes of spaces do have a unique symmetric basis are not very hard but we omit them. They can be found e.g. in [L - T I Section 4].

An example of a space with "many" mutually non-equivalent symmetric bases is the so-called space U_1 of Pelczynski [P], which is universal for all unconditional bases in the sense that it has an unconditional basis $\{u_n\}_{i=1}^{\infty}$, and, quite remarkably, every unconditional basic sequence $\{v_j\}_{j=1}^{\infty}$ in any Banach space V is equivalent to a subsequence of $\{u_n\}_{n=1}^{\infty}$. A simple application of the decomposition method shows that the property defining U_1 characterizes it up to isomorphism.

A simple way of constructing this space was suggested by Schechtman [S]. To this end, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions which is dense in the space $C(0,1)$ of all continuous functions on $[0,1]$ and, for any sequence $a = \{a_n\}_{n=1}^{\infty}$ which is eventually zero, i. e. $a \in c_{oo}$, define

$$|||a||| = \sup\{\| \sum_{n=1}^{\infty} \varepsilon_n a_n f_n \|_{C(0,1)};\ \ \varepsilon_n = \pm 1, n = 1, 2 \cdots \}.$$

The unit vectors $\{u_n\}_{n=1}^{\infty}$ clearly form an unconditional basis in the completion U_1 of c_{oo} relative to the norm $|||\cdot|||$. If $\{v_k\}_{k=1}^{\infty}$ is an unconditional basic sequence in an arbitrary Banach space V one can assume without loss of generality that V is a subspace of $C(0,1)$. Hence, one can find a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ so that $\|v_k - f_{n_k}\|_{C(0,1)} \to 0$, as $k \to \infty$, "fast enough" as to imply that $\{v_k\}_{k=1}^{\infty}$ is equivalent to $\{f_{n_k}\}_{k=1}^{\infty}$. Therefore, $\{f_{n_k}\}_{k=1}^{\infty}$ is unconditional and thus equivalent to $\{u_{n_k}\}_{k=1}^{\infty}$, which completes the proof.

In order to show that U_1 has a symmetric basis, a fact which is not a priori obvious, one needs an interpolation argument of Davis [D]: let $\{x_n\}_{n=1}^{\infty}$ be a normalized 1-unconditional basis in a Banach space X and, for $m > 1$ and $p \geq 1$, define the Orlicz function $M_p(t) = t^p/(1 + |\log t|)$; $t > 0$, and the norm

$$\|\alpha\|_{m,p} = \inf\{(\|\beta\|^2_{\ell_{M_p}} + \|\gamma\|^2_{\ell_p})^{1/2}; \alpha = \beta m^{-1} + \gamma m,\ \text{ with } \beta \in \ell_{M_p} \text{ and } \gamma \in \ell_p\},$$

for all $\alpha \in \ell_{M_p}$. Then, whenever $\{m_n\}_{n=1}^{\infty}$ is a sequence of numbers > 1 satisfying the condition $\sum\limits_{n=1}^{\infty} m_n^{-1} < \infty$, the expression

$$\|\alpha\|^{(p)} = \| \sum_{n=1}^{\infty} \|\alpha\|_{m_n,p} x_n \|_X,$$

defines a norm on a space $\widetilde{Y}_p$ whose unit vectors form a 1-symmetric basis in the completion Y_p of $\widetilde{Y}_p$ so that

$$K_p^{-1} \|\alpha\|_{\ell_{M_p}} \leq \|\alpha\|^{(p)} \leq K_p \|\alpha\|_{\ell_p},$$

for all $\alpha \in \ell_p$. It turns out that $\{m_n\}_{n=1}^\infty$ can be selected as to ensure that $\{x_n\}_{n=1}^\infty$ is equivalent to a block basis with constant coefficients of the unit vector basis of Y_p. This proves that X is isomorphic to a complemented subspace of the space Y_p, which of course has a symmetric basis. This fact can be also proved by using the fact proved in [L] that every space with an unconditional basis is isomorphic to a complemented subspsce of a space with a symmetric basis. The advantage of the present proof is that it shows that X can be complementably embedded in uncountably many spaces with mutually non-equivalent symmetric bases.

By applying this argument to U_1, one constructs, for each value of $p \geq 1$, a space Z_p with a symmetric basis $\{z_n^{(p)}\}_{n=1}^\infty$ so that U_1 is isomorphic to a complemented subspace of Z_p. Since both U_1 and Z_p are isomorphic to their own square one can apply the decomposition method and conclude that $U_1 \approx Z_p$, for any $p \geq 1$. This further implies that U_1 has, quite surprisingly, a symmetric basis which is equivalent to $\{z_n^{(p)}\}_{n=1}^\infty$. As we have noticed before, for $p \neq q$, $\{z_n^{(p)}\}_{n=1}^\infty$ is not equivalent to $\{z_n^{(q)}\}_{n=1}^\infty$, i.e. U_1 has even a continuum of mutually non-equivalent symmetric bases.

With some additional effort, one can construct more "natural" spaces, namely Orlicz sequence spaces, which also have a continuum of mutually non-equivalent symmetric bases (cf. [L - T O III] or [L - T I p. 153]). The fact that all the examples of spaces with a symmetric basis considered so far have either a unique symmetric basis, up to equivalence, or uncountably many mutually non-equivalent symmetric bases lead inevitably to the question whether this is always the case. It turns out that the answer is negative: C. Read [R] has constructed, for every $n = 1, 2, \cdots$ or $n = \aleph_0$, a space X_n with a symmetric basis which has precisely n different symmetric bases.

It should be added that we are very far form being able to characterize the class of spaces with a unique symmetric basis, up to equivalence. In fact, we do not have even a reasonable conjecture.

3 Uniqueness of unconditional bases, up to a permutation

The notion of uniqueness of the unconditional basis of a Banach space X can be interpreted as asserting that X can be represented in a **unique** manner as a space of sequences $a = \{a_n\}_{n=1}^\infty \in X$ with the property that

$$\{a_n\}_{n=1}^\infty \in X \Rightarrow \{\varepsilon_n a_n\}_{n=1}^\infty \in X,$$

for any choice of $\varepsilon_n = \pm 1$; $n = 1, 2, \cdots$. Obviously, in this representation X is considered as a set of **ordered** sequences.

Now, if instead we consider X as a space of **unordered** sequences with the property described above then uniqueness of the representation has a different meaning: uniqueness up to permutation and equivalence. Recall that, as it was mentioned in the introduction, a Banach space X with a normalized unconditional basis $\{x_n\}_{n=1}^{\infty}$ is said to have **unique** unconditional basis, **up to equivalence and permutation**, if, whenever $\{y_n\}_{n=1}^{\infty}$ is another normalized unconditional basis of X then $\{y_n\}_{n=1}^{\infty}$ is equivalent to $\{x_{\pi(n)}\}_{n=1}^{\infty}$, for some permutation π of the integers.

It turns out that there are considerably more spaces with a unique unconditional basis, up to permutation, than the three space ℓ_2, ℓ_1 and c_o, which are known to have, up to equivalence, a unique unconditional basis.

One such class of spaces was found by I.S. Edelstein and P. Wojtaszczyk [E - W], who showed that finite direct sums of ℓ_2, ℓ_1 and c_0 have the uniqueness property mentioned before.

Theorem 3.1 *Each of the Banach spaces $\ell_1 \oplus \ell_2$, $\ell_1 \oplus c_0$, $\ell_2 \oplus c_0$ and $\ell_1 \oplus \ell_2 \oplus c_0$ has, up to permutation and equivalence, a unique unconditional basis.*

In order to describe the ideas used in the proof, consider e.g. the case of the space $\ell_1 \oplus \ell_2$ and assume that $\{z_n = x_n + y_n\}_{n=1}^{\infty}$, where $x_n \in \ell_1$ and $y_n \in \ell_2$ for all n, is a normalized K-unconditional basis of this space. Put

$$N_1 = \{n; \|x_n\| \leq 1/2K\} \text{ and } \quad N_2 = \{n; \|x_n\| > 1/2k\}$$

and notice that both $[z_n]_{n \in N_1}$ and $[z_n]_{n \in N_2}$ are complemented subspaces in $\ell_1 \oplus \ell_2$ whose direct sum is the whole space. In order to study these two complemented subspaces of $\ell_1 \oplus \ell_2$, we need a result which "rotates" any complemented subspace of a "nice" direct sum into a "correct" position. To this end, recall that an operator T from a Banach space X into a space Y is **strictly singular** if the restriction of T to any infinite dimensional subspace of X is not an isomorphism. Compact operators are of course strictly singular but these two notions are very different. Clearly, every bounded linear operator from ℓ_1 into ℓ_2 or, vice-versa, from ℓ_2 into ℓ_1 is strictly singular.

Now, we can state the theorem of Edelstein and Wojtaszczyk [E - W] which provide the "rotation" into a "correct" position.

Theorem 3.2 *Suppose that X and Y are two Banach spaces so that every operator from X into Y is strictly singular, and let P be a bounded projection form $X \oplus Y$ onto a subspace Z. Then one can find an automorphism S of $X \oplus Y$ and complemented subspaces X_0 of X and Y_0 of Y such that*

$$SZ = X_0 \oplus Y_0.$$

We omit the proof of the theorem which is based on a good understanding of several facts on Fredholm operators.

We return now to the proof of Theorem 3.1 in the case of the direct sum $\ell_1 \oplus \ell_2$. The proof will be completed once we show e.g. that $[z_n]_{n \in N_1} \approx \ell_2$ and $[z_n]_{n \in N_2} \approx \ell_1$ since both ℓ_1 and ℓ_2 have a unique unconditional basis, up to equivalence. Notice that if

$$x_n = \sum_{j=1}^{\infty} a_{n,j} z_j \text{ and } y_n = \sum_{j=1}^{\infty} b_{n,j} z_j$$

then $a_{n,n} + b_{n,n} = 1$, for all n. However, for $n \in N_1$,

$$|a_{n,n}| \leq K\|x_n\| \leq 1/2,$$

and thus $|b_{n,n}| > 1/2$.

Now consider the linear operator U from the subspace $[\varepsilon_n y_n]_{n\in N_1}$ of $L_\infty(\ell_2)$, where $\varepsilon_n(t)$; $n = 1, 2, \cdots$, denote the Rademacher functions, into $[z_n]_{n\in N_1}$ which maps $\varepsilon_n y_n$ to z_n, for all $n \in N_1$. Since both $\{\varepsilon_n y_n\}_{n\in N_1}$ and $\{z_n\}_{n\in N_1}$ are unconditional bases, a well known diagonal argument shows that the corresponding diagonal operator is bounded by $K\|U\|$. This means that

$$\|\sum_{n\in N_1} c_n b_{n,n} z_n\| \leq K\|U\| \sup_{\substack{\varepsilon_n=\pm 1\\ n=1,2,\cdots}} \|\sum_{n\in N_1} c_n \varepsilon_n y_n\|,$$

for any choice of $\{c_n\}_{n\in N_1}$.

By Theorem 3.2, if $[z_n]_{n\in N_1}$ is not isomorphic to ℓ_2 then it contains a complemented subspace isomorphic to ℓ_1. Hence, one can find a normalized block basis $w_j = \sum_{\ell\in\sigma_j} d_\ell z_\ell$; $j = 1, 2, \cdots$, of $\{z_n\}_{n\in N_1}$, which is equivalent to the unit vector basis in ℓ_1. By passing to a subsequence if necessary, one can assume that the subspaces $[y_\ell]_{\ell\in\sigma_j}; j = 1, 2, \cdots$ are "almost" supported on mutually orthogonal subspaces of ℓ_2 in the sense that the unit vector basis $\{e_n\}_{n=1}^\infty$ of ℓ_2 can be split into disjoint subsets $\{e_n\}_{n\in n_j} : j = 1, 2, \cdots$ so that, essentially speaking, $[y_\ell]_{\ell\in\sigma_j} \subset [e_n]_{n\in\eta_j}$, for all j. Therefore, for any sequence $\{c_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty |c_j|^2 < \infty$ and any choice of $\varepsilon_n = \pm 1$, for all n,

$$\|\sum_{j=1}^\infty c_j \sum_{\ell\in\sigma_j} d_\ell \varepsilon_\ell y_\ell\| \sim \left(\sum_{j=1}^\infty |c_j|^2 \|\sum_{\ell\in\sigma_j} d_\ell \varepsilon_\ell y_\ell\|^2\right)^{1/2} \leq K \left(\sum_{j=1}^\infty |c_j|^2\right)^{1/2}.$$

Hence, the series $\sum_{j=1}^\infty c_j \sum_{\ell\in\sigma_j} d_\ell \varepsilon_\ell y_\ell$ converges unconditionally and so does the series $\sum_{j=1}^\infty c_j \sum_{\ell\in\sigma_j} b_{\ell,\ell} d_\ell z_\ell$. This implies the convergence of the series $\sum_{j=1}^\infty c_j w_j$, whenever $\sum_{j=1}^\infty |c_j|^2 < \infty$, which of course contradicts the fact that $\{w_j\}_{j=1}^\infty$ is equivalent to the unit vector basis of ℓ_1. Consequently $\{z_n\}_{n\in N_1}$ is equivalent to the unit vector basis of ℓ_2.

In a similar manner but using also a duality argument, one proves that $[z_n]_{n\in N_2} \approx \ell_1$, and thus that $\{z_n\}_{n\in N_2}$ is equivalent to the unit vector basis of ℓ_1.

While the finite direct sums constructed out of the three spaces ℓ_1, ℓ_2 and c_0 have, up to equivalence and permutation, a unique unconditional basis, this is not always the case for infinite direct sums constructed out of the same set of spaces. On the positive side we have e.g. the following result from [B - C - L - T].

Theorem 3.3 *Every normalized unconditional basis of an infinite dimensional complemented subspace of the direct sum $(\ell_2 \oplus \ell_2 \oplus \cdots \oplus \ell_2 \oplus \cdots)_0$, is equivalent to a permutation of the unit vector basis of one of the following six spaces:*

$$\ell_2, c_0, \ell_2 \oplus c_0, \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_0, \ell_2 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_0, (\ell_2 \oplus \ell_2 \oplus \cdots \oplus \ell_2 \oplus \cdots)_0.$$

A similar statement holds for complemented subspaces of the dual space $(\ell_2 \oplus \ell_2 \oplus \cdots \oplus \ell_2 \oplus \cdots)_1$.

Consequently, the six spaces appearing above and their duals have, up to permutation and equivalence, a unique unconditional basis.

The proof of Theorem 3.3 is not extremely difficult but still beyond the scope of this article and therefore we omit it here. Another result from [B - C - L - T] is:

Theorem 3.4 *Every normalized unconditional basis of an infinite dimensional complemented subspace of the direct sum $(\ell_1 \oplus \ell_1 \oplus \cdots \oplus \ell_1 \oplus \cdots)_0$ is equivalent to a permutation of the unit vector basis of one of the following six spaces:*

$$c_0, \ell_1, c_0 \oplus \ell_1, \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_0, \ell_1 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_0, (\ell_1 \oplus \ell_1 \oplus \cdots \oplus \ell_1 \oplus)_0$$

Consequently, each of these six spaces has, up to permutation and equivalence, a unique unconditional basis.

Though Theorems 3.3 and 3.4 have a similar formulation, the proof of 3.4 in [B - C - L - T] is considerably harder than that of 3.3. This is reflected also by the fact that the function $M = M(K)$ (so that if $\{x_n\}_{n=1}^{\infty}$ is a normalized K-unconditional basis of one of the spaces X appearing above, then $\{x_n\}_{n-1}^{\infty}$ is $M(K)$-equivalent to a permutation of the unit vector basis of X) behaves differently in the two cases discussed above. While the proof of 3.3 gives $M(K)$ as a power of K, the proof of 3.4 yields $M = M(K)$ as an exponential function of K and examples show that exponential function is really needed.

Very recently, Casazza and Kalton [C - K 2] provided a simpler proof of the uniqueness, up to a permutation, of the unit vector basis in the space $(\ell_1 \oplus \ell_1 \oplus \cdots \oplus \ell_1 \oplus \cdots)_0$ and its dual.

Quite surprisingly, the other infinite direct sums which can be constructed out the the three spaces ℓ_1, ℓ_2 and c_o do not have the the uniqueness property exhibited in Theorems 3.3 and 3.4.

Theorem 3.5 (cf. [B - C - L - T]) *The direct sums*

$$\left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_2, \quad c_0 \oplus \left(\sum_{n=1}^{\infty} \oplus \ell_\infty^n\right)_2, \quad (c_0 \oplus c_0 \oplus \cdots \oplus c_0 \oplus \cdots)_2$$

and their duals fail to have a unique unconditional basis, up to equivalence and permutation.

Proof: We shall treat here only the case of the dual direct sum $X = (\sum_{n=1}^{\infty} \oplus \ell_1^n)_2$; the other cases can be easily derived from it without too much difficulty.

In order to exhibit a normalized unconditional basis of X which is not permutatively equivalent to the unit vector basis of X, we first fix n and let $\mathcal{F}_i = \{A_{i,j}\}_{j=1}^n$; $1 \le i \le n$, the independent partitions of the interval $[0,1]$ into sets of measure equal to $1/n$ (i.e., for $A \in \mathcal{F}_i$, $B \in \mathcal{F}_j$ and $i \ne j, \mu(A \cap B) = \mu(A)\mu(B) = 1/n^2$). With $\{e_i\}_{i=1}^n$ denoting the unit vector basis of the space ℓ_1^n, consider the vector valued functions

$$f_{i,j}(t) = n^{1/2}\chi_{A_{i,j}}(t)e_i;\ \ 1 \le i,j \le n,$$

which form a normalized 1-unconditional basis in a subspace Y_n of the space $L_2([0,1],\ell_1^n)$ since

$$\|\sum_{i,j=1}^n a_{i,j}f_{i,j}\| = n^{1/2}\left(\int_0^1 \left(\sum_{i=1}^n |\sum_{j=1}^n a_{i,j}\chi_{A_{i,j}}(t)|\right)^2 dt\right)^{1/2},$$

for any choice of $\{a_{i,j}\}_{i,j=1}^n$. Indeed, for each $1 \le i \le n$, the functions

$$|\sum_{j=1}^n a_{i,j}\chi_{A_{i,j}}(t)|$$

and

$$|\sum_{j=1}^n |a_{i,j}|\chi_{A_{i,j}}(t)|$$

are both $\mathcal{F}_i$ measurable and have the same distribution. Hence, the independence of $\mathcal{F}_i, 1 \le i \le n$ ensures that that the functions $|\sum_{j=1}^n a_{i,j}\chi_{A_{i,j}}(t)|$ and $\sum_{j=1}^n |a_{i,j}|\chi_{A_{i,j}}(t)|$ have the same norm in the space $L_2([0,1],\ell_1^n)$. Now notice that if $\mathbb{E}_i$ denotes the conditional expectation operator associated to the partition $\mathcal{F}_i$, i.e. the operator defined by

$$\mathbb{E}_i\varphi = \sum_{j=1}^n \left(\int_{A_{i,j}} \varphi(t)dt\right)\chi_{A_{i,j}},$$

for $\varphi \in L_1(0,1)$, then one can define a projection P_n from $L_2([0,1],\ell_1^n)$ onto Y_n by setting

$$P_n = \sum_{i=1}^n (\mathbb{E}_i\varphi_i)e_i,$$

whenever $f(t) = \sum_{i=1}^n \varphi_i(t)e_i$ is an element of $L_2([0,1],\ell_1^n)$. Then, by using the independence of the partitions $\{\mathcal{F}_i\}_{i=1}^n$, one gets that

$$\|P_n f\|^2 = \int_0^1 (\sum_{i=1}^n |\mathbb{E}_i\varphi_i|)^2 dt = \sum_{i=1}^n \int_0^1 |\mathbb{E}_i\varphi_i|^2 dt + \sum_{\substack{i,j=1\\ i\neq j}}^n \int_0^1 |E_i\varphi_i||E_j\varphi_j| dt = \sum_{i=1}^n \int_0^1 |E_i\varphi_i|^2 dt + \sum_{\substack{i,j=1\\ i\neq j}}^n (\int_0^1 |E_i\varphi_i| dt)(\int_0^1 |E_j\varphi_j| dt),$$

for any f as above. Hence,

$$\|P_n f\|^2 \leq 2\sum_{i=1}^n \int_0^1 |\varphi_i|^2 dt = 2\|f\|^2,$$

i.e. $\|P_n\| \leq \sqrt{2}$, which shows that Y_n is $\sqrt{2}$-complemented in $L_2([0,1], \ell_1^n)$, or even in the subspace $(\ell_1^n \oplus \ell_1^n \oplus \cdots \oplus \ell_1^n)_2$ (n^n factors) of $L_2([0,1], \ell_1^n)$. It follows that the space $Y = (\sum_{n=1}^{\infty} \oplus Y_n)_2$ is isomorphic to a complemented subspace of X.

On the other hand, the block basis $y_i = n^{-1/2} \sum_{j=1}^n f_{i,j};\quad 1 \leq i \leq n$, of $\{f_{i,j}\}_{i,j=1}^n$ is 1-equivalent to the unit vector basis of ℓ_1^n, and thus its span is complemented in Y_n, since ℓ_1 is block injective, i.e. 1-complemented, whenever it is embedded as a block basis of a 1-unconditional basis. Consequently, Y contains a 1-complemented copy of X and thus, by the decomposition method, $X \approx Y$.

The proof will be completed once we show that the natural basis of Y is not equivalent to a permutation of the unit vector basis in X. To this end, notice that the unit vector basis of X has the property that any of its subsets of finite codimension contains, for each n, a subsequence 1-equivalent to the unit vector basis of ℓ_1^n. The natural basis of Y has, however, a diametrically opposite behavior. Indeed, for each $\varepsilon > 0$ and each integer k, there exists an integer $n = n(\varepsilon, k)$ such that any subset of $\{f_{i,j}\}_{i,j=1}^n$ of cardinality k is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_2^k and, moreover, this property remains valid in Y provided that, for any such k, we eliminate a suitable finite set of vector from the basis of Y.

Another quite well-known space for which the question of uniqueness, up to a permutation, could be settled is the so-called Tsirelson space, introduced in [T] (see also [F - J]). This space is the completion of the space of sequences $x \in c_{0,0}$ under the minimal norm $\|\cdot\|_T$ satisfying the conditions:

$$(i) \quad \|x\|_T \geq \|x\|_0,$$

for all $x \in c_{00}$, and

$$(ii) \quad \|\sum_{i=1}^{\ell} x_i\|_T = \frac{1}{2}\sum_{i=1}^{\ell} \|x_i\|_T,$$

whenever $\ell \leq$ support $x_1 <$ support $x_2 < \cdots <$ support x_ℓ; $\ell = 1, 2, \cdots,$. The fact that such a norm exists is proved in [F - J].

Theorem 3.6 *Every complemented subspace of T with an unconditional basis has, up to permutation and equivalence, a unique unconditional basis.*

Actually, this result was preceded by a theorem proved in [B - C - L - T] which asserts that the 2-convexification $T^{(2)}$ of T, as well as its complemented subspaces with an unconditional basis, have a unique unconditional basis, up to equivalence and permutation (recall that $T^{(2)}$ is obtained from T by defining the norm in it as follows: $\|x\|_{T^{(2)}} = \| \ |x|^2 \ \|_T^{1/2}$, for vectors x having the property that the square of their absolute value belongs to T). The proof is quite difficult. The proof for T (i.e. of Theorem 3.6), due to Casazza and Kalton [C - K 1], is the byproduct of a more general study of the uniqueness property in spaces which do not contain uniformly complemented copies of ℓ_2^n, for all n.

For $1 < p \neq 2$, the p-convexification $T^{(p)}$ of T does not have a unique unconditional basis, up to permutation and equivalence, since, as was pointed out by Kalton, $T^{(p)}$ can be represented a Tsirelson sum of ℓ_p^n's and, in this sum, the factor ℓ_p^n has an unconditional basis containing among its vectors an ℓ_2^k with $k \sim \log n$.

The Banach spaces with a unique unconditional basis, up to permutation and equivalence, which were considered so far in this section have the additional property that also their complemented subspaces with an unconditional basis share the uniqueness feature. For such spaces one can introduce the notion of genus (cf. [B - C - L - T]).

A Banach space with an unconditional basis is said to be of **genus** n if in all its complemented subspaces with an unconditional basis, the normalized unconditional basis is unique, up to equivalence and permutation, and there are exactly n different isomorphic types of complemented subspaces with an unconditional basis.

The spaces ℓ_1, ℓ_2 and c_0 are clearly of genus 1. It turns out (cf. [B - C - L - T]) that these three spaces are the only ones of genus 1.

Theorem 3.7 *ℓ_1, ℓ_2 and c_0 are the only spaces with a (unique) unconditional basis of genus* 1.

The idea of the proof from [B - C - L - T] is the following: if X is a space with a normalized unconditional basis $\{x_n\}_{n=1}^\infty$ of genus 1 then every infinite subsequence of $\{x_n\}_{n=1}^\infty$ is equivalent to a permutation of $\{x_n\}_{n=1}^\infty$. Then, by using a sort of compactification argument involving spreading models and ultraproducts, one can show that $\{x_n\}_{n=1}^\infty$ is, up to permutation, a subsymmetric basis. Therefore, one can assume w.l.o.g.. that $\{x_n\}_{n=1}^\infty$ itself is subsymmetric.

Since $\{x_n\}_{n=1}^\infty$ is subsymmetric any block basis with constant coefficients of $\{x_n\}_{n=1}^\infty$ spans a complemented subspace U of X. Moreover, by a slightly more complicated argument than that used in the case when $\{x_n\}_{n=1}^\infty$ is symmetric, it can be easily shown that $X \approx X \oplus U$. Hence, any normalized block basis with constant coefficients of $\{x_n\}_{n=1}^{\infty}$ is equivalent to $\{x_{\pi(n)}\}_{n=1}^{\infty}$, for some permutation π of the integers. Then, by a simple modification of Zippin's characterization of perfectly homogeneous bases from [Z], one concludes that $\{x_n\}_{n=1}^{\infty}$ is equivalent to the unit vector basis in c_0 or ℓ_p, for some $p \geq 1$. The cases when $1 < p \neq 2$ can be easily dismissed, as shown before.

Theorems 3.3 and 3.4 above show that the sums $\left(\sum_{n=1}^{\infty} \oplus \ell_2^n\right)_0$, $\left(\sum_{n=1}^{\infty} \oplus \ell_1^n\right)_0$ and their duals are spaces of genus 2. However, it is not known if there exist other spaces of genus 2.

The spaces $\ell_1 \oplus \ell_2$, $\ell_1 \oplus c_0$ and $\ell_2 \oplus c_0$ are of genus 3 but, again, we do not have a complete characterization of this class.

It is quite possible that the class of spaces of finite genus coincides with that of spaces which are obtained from Hilbert space by taking repeated finite or infinite direct sums in the sense of c_0 or ℓ_1. Casazza and Lammers [C - L] obtained many results on spaces of finite genus, e.g. that the unconditional basis contains a subsequence equivalent to the unit vector basis in either ℓ_1, or ℓ_2 or c_0. There is a feeling that this class will be well understood once spaces of genus 2 are characterized.

The Tsirelson space T and its 2-convexification are of infinite genus.

The question whether the uniqueness of the unconditional basis, up to permutation and equivalence, is a hereditary property has a negative answer. More precisely, Casazza and Kalton [C - K 1] constructed the first example of a Banach space with a unique unconditional basis, up to equivalence and permutation, which has complemented subspaces with an unconditional basis lacking the uniqueness property. Their starting point in the Orlicz sequence space ℓ_M, where $M(t) = t/(1 + |\log t|)$, for $t \geq 0$. In this, as in any other Orlicz space, a normalized block basis with constant coefficients of the from $u_j = \left(\sum_{n=q_{j-1}+1}^{q_{j+1}} e_n\right) / \|\sum_{n=q_{j-1}+1}^{q_{j+1}} e_n\|$; $j = 1, 2, \ldots$ generates a so called modular space $\ell_M[s_j]$, where $s_j = 1/\|\sum_{n=q_{j-1}+1}^{q_{j+1}} e_n\|$, for all j. This subspace, which is clearly complemented in ℓ_M, can be described as the space of all sequences $a = (a_1, a_2, \ldots, a_j, \ldots)$ so that $\sum_{j=1}^{\infty} M_{s_j}(|a_j|) < \infty$, where $M_s(t) = M(st)/M(s)$.

It can be easily verified that if $s_j \to 0$, very fast as $j \to \infty$, then $\ell_M[s_j]$ is isomorphic to ℓ_1, and, on the other hand, if $\inf_j s_j > 0$ then $\ell_M[s_j] \approx \ell_M$. By manipulating between these diametrically opposite situations, one can select a normalized block basis $\{u_j\}_{j=1}^{\infty}$ of the unit vector basis in ℓ_M which is permutatively equivalent to its square but $\ell_M[s_j]$ is not isomorphic to either ℓ_1 or ℓ_M itself. Casazza and Kalton have shown in [C - K 1] that in this case $\{u_j\}_{j=1}^{\infty}$, is, up to permutation and equivalence, the unique unconditional basis of $\ell_M[s_j]$ and, moreover, $\ell_M[s_j]$ contains complemented subspaces with a non-unique unconditional basis.

The fact that direct sums in the sense of c_o of spaces such as ℓ_2 or ℓ_1, which do have a unique unconditional basis, have also the uniqueness property, led to the question (stated explicitly in [B - C - L - T]) whether, whenever a space X has unique unconditional basis, up to permutation, then $(X \oplus X \oplus \cdots \oplus X \oplus \cdots)_0$ also has a unique unconditional basis. It turns out that the answer to this question is negative: Casazza and Kalton [C - K 1] have proved that direct sums of T or $T^{(2)}$ in the c_0-sense do not have a unique unconditional basis, up to permutation, in spite of the fact mentioned above that both T and $T^{(2)}$ have this property.

4 Uniqueness in finite dimensional spaces

Since any two bases of a finite dimensional space are always equivalent the question of uniqueness of the basis makes no sense in the framework of a single finite dimensional Banach space but rather in that of families of such spaces. As in the case of infinite dimensional spaces, the most interesting results are obtained for families of spaces with an unconditional or symmetric basis.

Definition 4.1 *Let $\mathcal{F}$ be a family of finite dimensional spaces each of which has a normalized 1-unconditional basis. We say that the members of $\mathcal{F}$ have a unique unconditional basis, up to equivalence (and permutation), if there exists a function $\psi\colon [1,\infty) \to [1,\infty)$ such that, whenever a space $E \in \mathcal{F}$ has another normalized unconditional basis $\{e_j\}_{n=1}^n$ whose unconditional constant is $\leq K$, then $\{e_j\}_{j=1}^n$ is $\psi(K)$-equivalent to (a permutation of) the given 1-unconditional basis of E.*

By replacing in the above definition the word "unconditional" with "symmetric", one defines the notion of uniqueness of the symmetric basis for the elements of $\mathcal{F}$.

Typical families studied in this section are

$$\mathcal{F}_p = \{\ell_p^n; n = 1, 2, \dots\}; \;\; 1 \leq p \leq \infty.$$

The same argument, involving the parallelogram identity, which was used in the case of ℓ_2, shows that the members of $\mathcal{F}_2$ have a unique unconditional basis, up to equivalence. Furthermore, by using the same argument involving 1-summing and 2-summing operators, as in the case of ℓ_1 and c_0, one can show that also $\mathcal{F}_1$ and $\mathcal{F}_\infty$ have the same uniqueness property as $\mathcal{F}_2$.

Proposition 4.2 *The members of the families $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_\infty$ have a unique unconditional basis, up to equivalence.*

The analogy with the infinite dimensional case might lead one to believe that these three families above are the only ones having, up to equivalence, a unique unconditional basis.

It turns out that this fact is false. In order to produce examples of other families whose member have a unique unconditional basis, up to equivalence, one needs a result of Dor and Starbird [D - S] asserting that, for $p > 2$, a normalized unconditional basis of ℓ_p is either equivalent to the unit vector basis of ℓ_p or it admits, for each n, a block basis which is 2-equivalent to the unit vector basis of ℓ_2^n. By using this fact together with the known assertion that Hilbert space is uniformly complemented in L_p-spaces, for $p > 2$ (cf. [P - R] or [M]), it is deduced in [J - M - S - T p. 48] that:

Proposition 4.3 *There exist constants $C < \infty$ and $M < \infty$ such that every normalized K-unconditional basis of ℓ_p; $p > 1$, with*

$$K \leq C \max\{\sqrt{p},\ \sqrt{p/(p-1)}\},$$

for some constant $C < \infty$, is M-equivalent to the unit vector basis of ℓ_p.

We omit the details of the proof but just mention that the bound for K, appearing in the right hand side of the above inequality, is actually equal to $\gamma_p(\ell_2)/\sqrt{2}$, where $\gamma_p(\ell_2)$ denotes as usual the factorization constant of ℓ_2 through L_p-spaces.

An immediate consequence of Proposition 4.3 is the fact that, for each sequence $p_n \to \infty$, the members of the family

$$\mathcal{F} = \{\ell_{p_n}^n;\ \ n = 1, 2, \ldots\}$$

have a unique unconditional basis, up to equivalence.

We pass now to some questions concerning the uniqueness of the symmetric basis. We begin with subspaces of L_p. Among the finite dimensional subspaces of L_p with a symmetric basis one can find the families $\mathcal{F}_p$ and $\mathcal{F}_2$. Other interesting subspaces of L_p are the "diagonals" of $\ell_p^n \oplus \ell_2^n$ generated by vectors of the form $\{e_j^{(p)} + w_j e_j^{(2)}\}_{j=1}^n$, where $\{e_j^{(p)}\}_{j=1}^n$ and $\{e_j^{(2)}\}_{j=1}^n$ denote the unit vectors in ℓ_p^n, respectively ℓ_2^n, and $\{w_j\}_{j=1}^n$ is an arbitrary sequence of scalars. Spaces of this type were studies by H.P. Rosenthal [Ro] who coined for them the name X_p-space. If $w_j = w$; $j = 1, 2, \ldots n$ then obviously we deal with symmetric X_p-space.

It turns out that every symmetric basic sequence in L_p; $p > 2$, is equivalent to a symmetric X_p-space. This characterization has been proved in the Memoir [J - M - S - T p.34].

Theorem 4.4 *For every $p > 2$ and $K \geq 1$, one can find a constant $D = D(p, K) < \infty$ so that any normalized basic sequence $\{x_j\}_{j=1}^n$ in $L_p(0,1)$, whose symmetry constant is $\leq K$, is D-equivalent to the symmetric X_p-space generated by $\{e_j^{(p)} + we_j^{(2)}\}_{j=1}^n$, where*

$$w = \|\sum_{j=1}^n x_j\|/\sqrt{n}.$$

Proof: In the first step of the proof, the symmetric basic sequence in L_p is replaced by a sequence of so-called "symmetrically exchangeable" random variables, i.e. a sequence of functions in L_p whose joint distribution in $\mathbb{R}^n$ remains invariant under permutation and change of sign. In order to describe this construction, let $\{x_j\}_{j=1}^n$ be a normalized K-symmetric basic sequence in $L_p(0,1)$; $p>2$, and H_n the family of all distinct $n!2^n$-pairs $\{\pi,(\varepsilon_j)_{j=1}^n\}$, where π is a permutation of $\{1,2,\dots,n\}$ and $\varepsilon_j=\pm1$, for all $1\le j\le n$.

The elements of H_n can be put in a one-to-one correspondence with unit subintervals of $[0,n!2^n]$ of the form $[k,k+1]$, with k being an integer. If a pair $\{\pi,(\varepsilon_j)_{j=1}^n\}$ is in correspondence with the subinterval I of $[0,n!2^n]$ then we define

$$f_j(t)=\varepsilon_j x_{\pi(j)}(t);\ t\in I;\ 1\le j\le n.$$

In order to make the functions $\{f_j\}_{j=1}^n$, which are defined on the interval $[0,n!2^n]$, into an exchangeable sequence we compress the interval $[0,n!2^n]$ into $[0,1]$ in the obvious linear way thus obtaining a new sequence $\{g_j\}_{j=1}^n$ which is symmetrically exchangeable and K-equivalent to the original sequence $\{x_j\}_{j=1}^n$.

It is easily seen that there is no loss of generality in assuming that each point of $[0,1]$ belongs to the support of at least one of the functions $\{x_j\}_{j=1}^n$. Then, by a corresponding change of density, one can make the expression $(\Sigma_{j=1}^n|x_j|^2)^{1/2}$ into a constant C. This however implies that also the square function $(\Sigma_{j=1}^n|g_j|^2)^{1/2}$ is a constant equal to $C=\|(\Sigma_{j=1}^n|x_j|^2)^{1/2}\|$.

Since $\{g_j\}_{j=1}^n$ is a 1-unconditional basic sequence in $L_p(0,1)$ and $L_p(0,1)$ is of cotype p, for $p>2$, it follows that

$$\|\sum_{j=1}^n a_jg_j\|\ge\left(\sum_{j=1}^n|a_j|^p\right)^{1/p},$$

for any choice if $\{a_j\}_{j=1}^n$. On the other hand, since $\{g_j\}_{j=1}^n$ is an orthogonal sequence in $L_2(0,1)$ with $\|g_j\|_2=C/\sqrt{n}$, for all $1\le j\le n$, we get that

$$\|\sum_{j=1}^n a_jg_j\|\ge\|\sum_{j=1}^n a_jg_j\|_2=\left(\sum_{j=1}^n|a_j|^2\|g_j\|_2^2\right)^{1/2}=\frac{C}{\sqrt{n}}\left(\sum_{j=1}^n|a_j|^2\right)^{\frac12},$$

for all $1\le j\le n$. This proves one part of 4.3 since

$$\|\sum_{j=1}^n a_jx_j\|\ge K^{-1}\|\sum_{j=1}^n a_jg_j\|\ge K^{-1}\max\left\{\left(\sum_{j=1}^n|a_j|^p\right)^{1/p},\ \frac{C}{\sqrt{n}}\left(\sum_{j=1}^n|a_j|^2\right)^{1/2}\right\},$$

for all $\{a_j\}_{j=1}^n$, and, by Khinchine's inequality,

$$\|\sum_{j=1}^n x_j\|\le K\left(\int\|\sum_{j=1}^n\varepsilon_jx_j\|^pd\varepsilon\right)^{1/p}=K\|\left(\int|\sum\varepsilon_jx_j|^pd\varepsilon\right)^{1/p}\|\le KB_pC.$$

The opposite inequality is more difficult and requires some ideas of H.P. Rosenthal [Ro] and a certain averaging procedure. We omit this argument which is described in detail in [J - M - S - T].

The significance of Theorem 4.4 becomes clear only after we fully understand the meaning of the expression $w = \|\Sigma_{j=1}^n x_j\|/\sqrt{n}$.

Proposition 4.5 *For every $K \geq 1$, there exists a constant $M = M(K)$ such that, whenever $\{x_j\}_{j=1}^n$ is a normalized K-symmetric basis in a finite dimensional subspace X of $L_p(0,1)$, then the Banach-Mazur distance $d(X, \ell_2^n)$ is M-equivalent to the expression $\|\Sigma_{j=1}^n x_j\|/\sqrt{n}$ if $1 \leq p \leq 2$ and to $\sqrt{n}/\|\Sigma_{j=1}^n x_j\|$ if $p > 2$.*

Proof: Suppose that $p > 2$. It follows from 4.4 that

$$\|\sum_{j=1}^n a_j x_j\| \geq D(K,p)^{-1} \frac{\|\sum\limits_{j=1}^n x_j\|}{\sqrt{n}} \left(\sum_{j=1}^n |a_j|^2\right)^{1/2},$$

for any choice of $\{a_j\}_{j=1}^n$. Furthermore, by using Khinchine's inequality in $L_p(0,1)$ and the 2-convexity of $L_p(0,1)$, for $p > 2$, we also get that

$$\begin{aligned}\|\sum_{j=1}^n a_j x_j\| &\leq K(\textstyle\int \|\sum\limits_{j=1}^n a_j \varepsilon_j x_j\|^p d\varepsilon)^{1/p} = K\|(\int |\sum\limits_{j=1}^n a_j \varepsilon_j x_j|^p d\varepsilon)^{1/p}\| \leq \\ &\leq KB_p\|(\sum_{j=1}^n |a_j|^2 |x_j|^2)^{1/2}\| \leq KB_p(\sum_{j=1}^n \alpha_j|^2)^{1/2},\end{aligned}$$

again for any choice of $\{a_j\}_{j=1}^n$. Combining these two inequalities, we conclude that

$$d(X, \ell_2^n) \leq KB_p D(K) \frac{\sqrt{n}}{\|\sum\limits_{j=1}^n x_j\|}.$$

Next notice that, for every sequence $\{a_j\}_{j=1}^n$ of scalars, we also get

$$\frac{1}{2K} \max_{1\leq j\leq n} |a_j| \leq \|\sum_{j=1}^n a_j x_j\| \leq K\|\sum_{j=1}^n x_j\| \max_{1\leq j\leq n} |a_j|$$

i.e.

$$d(X, \ell_\infty^n) \leq 2K^2\|\sum_{j=1}^n x_j\|.$$

Hence,

$$\sqrt{n} \leq d(X, \ell_2^n) d(X, \ell_\infty^n) \leq 2K^2 \|\sum_{j=1}^n x_j\| d(X, \ell_2^n)$$

i.e.

$$d(X, \ell_2^n) \geq (2K^2)^{-1} \frac{\sqrt{n}}{\|\sum\limits_{j=1}^n x_j\|},$$

which completes the proof for $p > 2$. The case $1 \leq p \leq 2$ is treated in a similar manner. $\square$

Proposition 4.5 shows that, for a given finite dimensional subspace X of $L_p(0,1); 1 \leq p \leq \infty$, with a symmetric basis, the expression $\| \sum_{j=1}^{n} x_j \|$ is, up to a constant depending only on the symmetry constant of $\{x_j\}_{j=1}^n$, an invariant of the space X rather than that of the particular symmetric basis $\{x_j\}_{j=1}^n$. This fact together with Theorem 4.3 imply the following consequence from [J - M - S - T p.39].

Corollary 4.6 *For $p > 2$, let G_p denote the family of all finite dimensional subspaces of $L_p(0,1)$ which have a 1-symmetric basis. Then the members of G_p have for fixed p a unique symmetric basis, up to equivalence.*

A trivial duality argument, together with 4.2, shows that, for $1 \leq p \leq \infty$, each member of the family $\mathcal{F}_p$ has, up to equivalence, a unique symmetric basis. This means that, for every $1 \leq p \leq \infty$, there is a function $\psi_p(K)$ which corresponds to the family $\mathcal{F}_p$ by Definition 4.1. It turns out that these functions can be selected as not to depend on p and the following result from [J - M - S - T p.39] is true.

Theorem 4.7 *The members of the family $\cup_{1 \leq p \leq \infty} \mathcal{F}_p$ have, up to equivalence, a unique symmetric basis.*

We omit the proof which is given in detail in the Memoir [J - M - S - T].

We already see that the class of families whose members have, up to equivalence, a unique symmetric basis is quite large. In an attempt to discover the largest family of space with a unique symmetric basis, Schütt studied in [Schu] the family $\mathcal{D}_\alpha$ of all finite dimensional Banach spaces X with a normalized 1-symmetric basis $\{x_j\}_{j=1}^n$ which satisfy the condition

$$d(X, \ell_2^n) \geq n^\alpha,$$

and proved the following beautiful result which we quote here without giving its proof.

Theorem 4.8 *For any $\alpha > 0$, each member of the family $\mathcal{D}_\alpha$ has, up to equivalence, a unique symmetric basis.*

The abundance of families of finite dimensional spaces with a unique symmetric basis leads naturally to the question, which was raised in the Memoir [J - M - S - T], whether this is not always the case. Theorem 4.8, mentioned above, shows that the construction of a counterexample is not an easy matter. Eventually, Gowers produced in [Go] ingenious examples of finite dimensional normed spaces with two (asymptotically) non-equivalent symmetric bases. The precise statement is as follows.

Theorem 4.9 *For each integer k, there exists a Banach space of dimension $n = 2^k$ with two 2-symmetric normalized symmetric bases $\{e_i\}_{i=i}^n$ and $\{e_2'\}_{i=1}^n$ whose constant of equivalence is at least* $\exp(\log\log n / 8 \log\log\log n)$.

We shall describe here only the main ideas of Gowers' construction; the details can be found in the paper [G].

Fix n and let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map defined by an orthogonal matrix which will be specified later. Let $\{e_i\}_{i=i}^n$ denote the unit vector basis of $\mathbb{R}^n$ and put $e_i' = Ae_i$, for all $1 \le i \le n$. These will be the two bases under consideration.

The next step is to define a norm $\|\cdot\|$ on $\mathbb{R}^n$ so that both $\{e_i\}_{i=i}^n$ and $\{e_i'\}_{i=1}^n$ become 2-symmetric bases. To this end, let Ω be the group of symmetries associated to the first basis $\{e_i\}_{i=1}^n$, i.e. of the linear maps w of the form

$$w\left(\sum_{i=1}^n a_i e_i\right) = \sum_{i=1}^n \varepsilon_i a_i e_{\pi(i)},$$

where $\varepsilon_i = \pm 1$, for all $1 \le i \le n$, and π is a permutation of the integers $\{e_i\}_{i=1}^n$. To the second basis $\{e_i'\}_{i=1}^n$, we associate the group $\Omega' = \{AwA^{-1}; w \in \Omega\}$. Then put

$$X_0 = \{\varepsilon_i e_i; \varepsilon_i = \pm 1,\ 1 \le i \le n\}$$

and define sets $\{X_j\}_{j=1}^\infty$ by induction, as follows: if $j \ge 1$ is odd then

$$X_j = \{w'x; x \in X_{j-1}, w' \in \Omega'\},$$

and if $j \ge 1$ is even, then

$$X_j = \{wx; x \in X_{j-1}, w \in \Omega\}.$$

Once the sets $\{X_j\}_{j=0}^\infty$ have been introduced, we define the norm of a vector $x \in \mathbb{R}^n$ by setting

$$\|x\| = \max\{2^{-j}|\langle x, x_j\rangle|; x_j \in X_j,\ j = 0, 1, 2, \ldots\}.$$

The fact that $\|\cdot\|$ is indeed a norm on $\mathbb{R}^n$ is trivial since the above maximum is always attained on a finite set of indices depending of course on the vector x under consideration.

In order to prove that $\{e_i\}_{i=1}^n$ is a 2-symmetric basis, fix a vector $x = \sum_{i=1}^n a_i e_i \in \mathbb{R}^n$ and assume that $\|x\| = 2^{-j}|\langle x, x_j\rangle|$, for some integer j and some $x_j \in X_j$. If j is odd then, for any $w \in \Omega, wx_j \in X_{j+1}$ so that

$$\|wx\| \ge 2^{-(j+1)}|\langle wx, wx_j\rangle| = 2^{-(j+1)}|\langle x, x_j\rangle| = 2^{-1}\|x\|.$$

On the other hand, if j is even then $x_j = \tilde{w}x_{j-1}$, for some $\tilde{w} \in \Omega$ and $x_{j-1} \in X_{j-1}$. Hence, for any $w \in \Omega, wx_j = (w\tilde{w})x_{j-1} \in X_j$ so that

$$\|wx\| \ge 2^{-j}|\langle wx, wx_j\rangle| = 2^{-j}|\langle x, x_j\rangle| = \|x\|,$$

which proves that $\{e_i\}_{i=1}^n$ is indeed 2-symmetric. The proof of the fact that $\{e_i'\}_{2=1}^n$ is also 2-symmetric is done in exactly the same way.

The above construction is independent of the particular choice of the orthogonal matrix A. The idea now is to select such a matrix A so that $\|e_1'\| = \|Ae_1\|$

is as small as possible and it turns out that one can construct an orthogonal matrix A for which the corresponding vector e'_1 satisfies

$$\|e'_1\| \leq \exp(-\log\log n / 8 \log\log\log n).$$

This will suffice since it is not too difficult to show that $\{e_i/\|e_i\|\}_{i=1}^n$ and $\{e'_i/\|e'_i\|\}_{u=1}^n$ are at best $\frac{1}{\|e'_1\|}$-equivalent; this fact follows by comparing the expectations $\frac{1}{\|e_1\|}\mathbb{E}\,\|\sum\limits_{i=1}^{n} g_i e_i\|$ and $\frac{1}{\|e'_1\|}\mathbb{E}\,\|\sum\limits_{i=1}^{n} g_i e'_i\|$, where $\{g_i\|_{i=1}^n$ is a sequence of independent identically distributed random variables.

The construction of the orthogonal matrix A so that $\|e'_1\| = \|Ae_1\|$ is "small" is done by induction on its size. For $k = 0$ and therefore $n = 2^0 = 1$, we let $A'_0 = (1)$ while, for $k > 0$, A'_k is defined by

$$A'_k = \begin{pmatrix} A'_{k-1} & I_{k-1} \\ I_{k-1} & -A'_{k-1} \end{pmatrix},$$

where I_{k-1} denotes the $2^{k-1} \times 2^{k-1}$-identity matrix. Once the $2^k \times 2^k$-matrix A'_k is defined, we put $A_k = k^{-1/2}A'_k$. It is easily seen that the matrices $\{A_k\}_{k=0}^{\infty}$ are not only orthogonal but also symmetric.

The most difficult part of the proof is to show that, for $n = 2^k$, the matrix $A = A_k$ has the property that $\|e'_1\| \leq \exp(-\log\log n/8\log\log\log n)$. This argument is quite technical and it relies on a lemma of Harper [H], Bernstein [Be], Hart [Ha] and Lindsey [Li] which gives an estimate from below for the number of edges joining vertices of the k-cube of a given cardinality r. We omit these details which, as we mentioned before, can be found in [Go].

We conclude this section with some remarks on two notions of uniqueness which are in the spirit of the so-called "proportional" theory of finite dimensional spaces. The main definition introduced in [C - K - T] is the following.

Definition 4.10 *Let $\mathcal{F}$ be a family of finite dimensional spaces each of which has a normalized 1-unconditional basis. We say that the members of $\mathcal{F}$ have an almost (somewhat) unique unconditional basis provided there exists a function $\varphi(K,\lambda)$, defined for all $K \geq 1$ and $0 < \lambda < 1$, such that, whenever $X \in \mathcal{F}$ with the given normalized 1-unconditional basis $\{x_i\}_{i=1}^n$ has also another normalized K-unconditional basis $\{y_i\}_{i=1}^n$ then, for any (some) $0 < \lambda < 1$, there exist a subset $\sigma \subset \{1,2,\cdot,n\}$ and a one to one function $\pi : \sigma \to \{1,2,\cdot,n\}$ so that $\{x_i\}_{i\in\sigma}$ is $\varphi(K,\lambda)$-equivalent to $\{y_{\pi(i)}\}_{i\in\sigma}$.*

These notions are obviously a generalization of the notion of unique unconditional basis, up to equivalence and permutation, introduced above. A thorough study of almost and somewhat uniqueness of unconditional basis is made in the paper [C - K - T].

We quote here, without proof, the following result from [C - K - T], which is clearly a generalization of Theorem 4.8.

Theorem 4.11 *For any $\alpha > 0$, each member of the family D_α has an almost unique unconditional basis.*

5 Uniqueness of rearrangement invariant structures

While in the preceding sections we focused on the uniqueness question only in the setting of sequence spaces, in the present one we pass to a study of similar problems in the framework of rearrangement invariant (r.i.) function spaces on a non-atomic measure space.

The main requirement imposed on an r.i. function space X on a finite or σ-finite non-atomic measure space (Ω, Σ, μ) is that, for any automorphism τ of Ω and every measurable function $f \in X$, the function $f(\tau^{-1})$ also belongs to X and has the same norm as f. If the measure space (Ω, Σ, μ) is assumed to be non-atomic and separable (i.e. Σ endowed with the usual metric $\rho(\tau, \eta) = \mu(\sigma \Delta \eta); \sigma, \eta \in \Sigma$, is a separable metric space) then it is well known that (Ω, Σ, μ) is isomorphic to a finite or infinite interval endowed with the usual Lebesgue measure. Hence, in principle, we can restrict our attention to the canonical cases $\Omega = [0,1]$ and $\Omega = [0, \infty)$, both endowed with the Lebesgue measure. In the Basic Concepts article such spaces are called symmetric lattices.

In the case when a function space X is invariant with respect to the automorphisms of Ω then the same is true for X', the subspace of the dual X^* of X which consists of "integrals", i.e. of functionals of the form

$$x_g^*(t) = \int_\Omega f g d\mu, \quad f \in X.$$

In most of the interesting cases that appear in analysis, X' is a norming subspace of X^*. This occurs if and only if $0 \leq f_n(w) \uparrow f(w)$ a.e. on Ω implies that $\lim_{n \to \infty} \|f_n\| = \|f\|$. The proof of this simple assertion can be found in [L - T II 1.b.18]. For convenience, we shall assume here that this is always the case. The formal definition of the notion of r.i. function space, which will be used in the sequel, is the following.

Definition 5.1 *A r.i. function space X on the interval $\Omega = [0,1]$ or on the interval $\Omega = [0, \infty)$ is a Banach space of classes of equivalence of measurable functions on Ω such that:*

(i) For any automorphism τ of Ω, a function $f \in X$ if and only if $f(\tau^{-1}) \in X$, and if this is the case then $\|f(\tau^{-1})\| = \|f\|$.

(ii) X' is a norming subspace of the dual X^* of X and thus X is order isometric to a subspace of X''. As a subspace of X'', the space X is either minimal (i.e. X is the closure of the simple integrable functions on Ω) or it is maximal (i.e. $X = X''$).

(iii) If $\Omega = [0,1]$ then

$$L_\infty(0,1) \subset X \subset L_1(0,1),$$

with the inclusion maps being of norm one, i.e.

$$\|f\|_1 \leq \|f\|_X \leq \|f\|_\infty,$$

for all $f \in L_\infty(0,1)$.

(iii') *If* $\Omega = [0,\infty)$ *then*

$$L_1(0,\infty) \cap L_\infty(0,\infty) \subset X \subset L_1(0,\infty) + L_\infty(0,\infty),$$

again with the inclusion maps being of norm one.

Recall that the norm of a function f in the space $L_1(0,\infty) \cap L_\infty(0,\infty)$ is defined as

$$\|f\| = \max(\|f\|_1\ , \|f\|_\infty).$$

The space $L_1(0,\infty) + L_\infty(0,\infty)$ is often used in interpolation theory and the norm of a function f in this space is usually defined by the formula

$$\|f\| = \inf\{\|g\|_1 + \|h\|_\infty;\ \ f = g + h\},$$

the infimum being taken over all decompositions $f = g + h$, with $g \in L_1(0,\infty)$ and $h \in L_\infty(0,\infty)$. It is easily verified that if $Y = L_1(0,\infty) + L_\infty(0,\infty)$ then $Y' = L_1(0,\infty) \cap L_\infty(0,\infty)$. The norm in the space Y can be alternatively defined with the aid of the notion of **decreasing rearrangement** of a function f on either $\Omega = [0,1]$ or on $\Omega = [0,\infty)$. The decreasing rearrangement f^* of a function $f \geq 0$ is defined as the right continuous inverse of the distribution function d_f of f, which is defined by

$$d_f(t) = \mu\{w \in \Omega;\ \ f(w) > t\}.$$

In other words,

$$f^*(x) = \inf\{t > 0; d_f(t) \leq x\};\ \ 0 \leq x < \mu(\Omega).$$

If f is not ≥ 0 then f^* is defined as the decreasing rearrangement of the absolute value $|f|$ of f.

It turns out that the norm of a function $f \in Y = L_1(0,\infty) + L_\infty(0,\infty)$ is equal to

$$\int_0^1 f^*(x)dx.$$

Indeed, for any decomposition $f = g + h$, as above, and any subset $\sigma \subset [0,\infty)$, we have that

$$\int_\sigma |f(t)|dt \leq \|g\|_1 + \|h\|_\infty \mu(\sigma).$$

Hence,

$$\int_0^1 f^*(x)dx = \sup\{\int_\sigma |f(t)|dt; \mu(\sigma) = 1\} \leq \|f\|.$$

Conversely, fix $f \in Y$ and put $\lambda = \|f^* - f^*\chi_{[0,1]}\|_\infty$. Then

$$\begin{aligned}\|f\| &= \|f^*\| \le \|f^* - \min(\lambda, f^*)\|_1 + \|\min(\lambda, f)\|_\infty = \\ &= \|\,(f^* - \lambda)\chi_{[0,1]}\|_1 + \lambda = \int_0^1 f^*(x)dx.\end{aligned}$$

As in the case of spaces with a symmetric basis, the question of uniqueness (this time of the r.i. structure) is studied most extensively in L_p-spaces. A result on *r.i.* function spaces on [0, 1], which is quite useful in the sdudy of uniqueness in L_p-spaces, was proved in the Memoir [J - M - S - T p.41].

Theorem 5.2 *A r.i. function space X on $[0,1]$, which is isomorphic to a subspace of $L_p(0,1); p > 2$, coincides with either $L_p(0,1)$ or $L_2(0,1)$, up to an equivalent renorming.*

Proof: Let X be an r.i. function space on $[0,1]$ and let T be an isomorphism from X into $L_p(0,1)$; $p > 2$. For every n and $1 \le i \le 2^n$, denote by $\varphi_{n,i}$ the characteristic function of the interval $[(i-1)/2^n, i/2^n)$. Since, for every n, the sequence $\{T\varphi_{n,i}\}_{i=1}^{2^n}$ is a K-symmetric basic sequence in $L_p(0,1)$, with $K \le \|T\| \cdot \|T^{-1}\|$, one can use Theorem 4.4 from the previous section and conclude the existence of a constant $D < \infty$, depending only on p and on T, so that, for every choice of scalars $\{a_i\}_{i=1}^{2^n}$, we have

$$\|\sum_{i=1}^{2^n} a_i \frac{\varphi_{n,i}}{\|\varphi_{n,i}\|_X}\|_X \overset{D}{\sim} \max\left\{\left(\sum_{i=1}^{2^n}|a_i|^p\right)^{1/p},\ w\left(\sum_{i=1}^{2^n}|a_i|^2\right)^{1/2}\right\},$$

where

$$w = \frac{\|\sum_{i=1}^{2^n}\frac{\varphi_{n,i}}{\|\varphi_{n,i}\|_X}\|_X}{\sqrt{2^n}} = \frac{1}{\|\varphi_{n,1}\|_X\sqrt{2^n}}.$$

Hence, for any simple function f on the field generated by the intervals $[(i-1)/2^n, i/2^n)$; $1 \le i \le 2^n$, we have that

$$\|f\|_X \overset{D}{\sim} \max\{\|\varphi_{n,1}\|_X 2^{n/p}\|f\|_p,\ \|f\|_2\}.$$

Taking $f \equiv 1$ we get that the sequence $\{\|\varphi_{n,1}\|_X 2^{n/p}\}_{n=1}^\infty$ is bounded by D and thus, with

$$\alpha = \liminf_{n\to\infty}\|\varphi_{n,1}\|_X 2^{n/p},$$

one concludes that,

$$\|f\|_X \overset{D}{\sim} \max\{\alpha\|f\|_p,\ \|f\|_2\},$$

for any simple function f over the dyadic intervals in $[0,1]$. If $\alpha = 0$ then obviously $X = L_2(0,1)$ and if $\alpha > 0$ then, since $p > 2$, $X = L_p(0,1)$, both equalities up to an equivalent renorming.

Theorem 5.2 has been generalized in [H - K] to a quite large class of pairs X and Y, where Y is an r.i. function space on $[0,\infty)$ and X is a non-atomic Banach lattice isomorphic to a subspace of Y. Under different conditions on Y, stated mostly in terms of p-convexity and q-concavity-notions which are described in the Basic Concepts aticle-it is shown in [H - K] that X is isomorphic to a *sublattice* of Y. For instance, this is the case when Y is p-convex and q-concave, for some $p > 2$ and $q < \infty$, and X is r-convex, for some $r > 2$. The same type of assumptions imply that if Y is an r.i. function space on $[0,1]$ then X contains a non-trivial band lattice isomorphic to a sublattice of Y. The paper [H - K] contains also some results on complemented spaces. For instance, if Y is a separable r.i. function space on either $[0,1]$ or $[0,\infty)$, which contains no ℓ_2 as a complemented sublattice, and X is a p-convex Banach lattice, for some $p > 2$, which is isomorphic to a complemented subspace of Y then X is even lattice-isomorphic to a complemented sublattice of Y.

In exactly the same manner as in the proof of Theorem 5.2, one can prove the following version for $[0,\infty)$ (cf. [J - M - S - T p.43]).

Theorem 5.3 *An r.i. function space* X *on* $[0,\infty)$*, which is isomorphic to a subspace of* $L_p(0,\infty)$*;* $p > 2$*, coincides with one of the spaces* $L_p(0,\infty)$*,* $L_2(0,\infty)$ *or* $L_2(0,\infty) \cap L_p(0,\infty)$*, up to an equivalent renorming.*

The norm of a function $f \in L_2(0,\infty) \cap L_p(0,\infty)$ is of course defined by $\|f\| = \max(\|f\|_2, \|f\|_p)$.

Theorem 5..2 above implies the uniqueness of the r.i. structure on $[0,1]$ of $L_p(0,1)$, for $p > 2$, and thus, by duality, also of $1 < p < 2$. The uniqueness of the r.i. structure of $L_2(0,1)$ is quite trivial, in view of 1.1 above. The uniqueness of the r.i. structure on $[0,1]$ of $L_\infty(0,1)$ can be easily reduced to that of $L_1(0,1)$.

In order to prove the uniqueness of the r.i. structure on $[0,1]$ of the space $L_1(0,1)$, we need the fact that if X is an r.i. function space on $[0,1]$, which is isomorphic to $L_1(0,1)$, then, for every n, the sequence of characteristic functions $\varphi_{n,i} = \chi_{[(i-1)/2^n, i/2^n)}$; $1 \le i \le 2^n$, forms a 1-unconditional basic sequence in X whose span is 1-complemented in X, by the conditional expectation operator relative to the field generated by the intervals $[(i-1)/2^n, i/2^n)$; $1 \le i \le 2^n$ (i.e. by the operator defined by $\mathbb{E}_n f = 2^n \sum_{i=1}^{2^n} (\int_{(i-1)/2^n}^{i/2^n} f dx)\varphi_{n,i})$. Then, the same argument, as the one used in section 1 to prove that ℓ_1 has a unique unconditional basis, shows that $\{\varphi_{n,i}/\|\varphi_{n,i}\|_X\}_{i=1}^{2^n}$ is equivalent to the unit vector basis of $\ell_1^{2^n}$, which constant of equivalence independent of n.

We summarize the above observations in the following theorem which, again, has been proved in [J - M - S - T].

Theorem 5.4 *The space* $L_p(0,1)$ *has a unique structure as an r.i. function space on* $[0,1]$*, for any value of* $1 \le p \le \infty$*.*

Remark: In the case $p = 1$, Kalton [K] proved a much stronger result: an r.i. function space on $[0,1]$, which contains a copy of $L_1(0,1)$, already coincides with $L_1(0,1)$, up to an equivalent norm, provided it does not contain uniformly isomorphic copies of ℓ_∞^n.

Contrary to an initial belief, the space $L_p(0,\infty)$ does not have a unique structure as an r.i. function space on $[0,\infty)$, unless $p = 1, 2$ or ∞. In fact, the following result can be easily deduced from Theorem 5.3.

Theorem 5.5 *The space $L_p(0,\infty); 1 < p \neq 2 < \infty$, has exactly two distinct representations as an r.i. function space on $[0,\infty)$:*

$$\begin{array}{ll} L_p(0,\infty) \text{ and } L_2(0,\infty)\cap L_p(0,\infty), & \text{if } p > 2, \text{ and} \\ L_p(0,\infty) \text{ and } L_2(0,\infty)+L_p(0,\infty), & \text{if } 1 < p < 2. \end{array}$$

Proof: The proof of 5.5 is completed once we show that, for $p > 2$, the spaces $L_p(0,\infty)$ and $Z = L_2(0,\infty)\cap L_p(0,\infty)$ are isomorphic. To this end, notice that the restriction $Z_{|[0,1]}$ is isomorphic to $L_p(0,1)$ and thus Z contains a complemented subspace isomorphic to $L_p(0,\infty)$. Conversely, for any n and m, the sequence $\{\chi_{[(i-1)/2^n, i/2^n)}\}_{i=1}^{2^n}$ spans in Z a symmetric X_p-space, and thus its span embeds complementably in $L_p(0,\infty)$, in a uniform manner. Hence, by a compactness argument using, for instance, ultraproducts, one can show that Z is isomorphic to a complemented subspace of $L_p(0,\infty)$. Then the decomposition method shows that $Z \approx L_p(0,\infty)$.

Many other classes of spaces which admit a unique representation as an r.i. function space on $[0,1]$ were exhibited in the Memoir [J - M - S - T] and [K]. We shall state some of these results without proof.

Theorem 5.6 *A r.i. function space X on $[0,1]$, which is q-concave for some $q < 2$, has unique structure as an r.i. function space on $[0,1]$.*

The other result that we want to state below involves the notion of the Haar basis. Recall that the Haar system $\{\chi_n\}_{n=1}^{\infty}$ on $[0,1]$ is defined by $\chi_1(t) \equiv 1$ and, for $\ell = 1, 2, \cdots, 2^k$ and $k = 0, 1, \cdots$, by

$$\chi_{2^k+\ell}(t) = \begin{cases} 1 & \text{if } \ t \in [(2\ell-2)/2^{k+1}, (2\ell-1)/2^{k+1}) \\ -1 & \text{if } \ t \in [(2\ell-1)(2^{k+1}, 2\ell/2^{k+1}) \\ 0 & \text{otherwise.} \end{cases}$$

It is an immediate consequence of basic interpolation theorems that the Haar system forms a (monotone) basis in any separable r.i. function space on $[0,1]$.

We now present a result on uniqueness which was originally proved in [J - M - S - T] under some additional assumptions and whose definitive form, stated below, is due to Kalton [K].

Theorem 5.7 *Let X be a separable r.i. function space on $[0,1]$ such that the Haar basis of X is not equivalent to a sequence of mutually disjoint functions in X whose linear span is complemented in X. Then X has unique structure as a r.i. function space on $[0,1]$.*

For using interpolation between two L_p-spaces other than L_1 and L_∞ it is useful to consider the so-called Boyd indices. In order to define these indices for an r.i. function space X on $[0,1]$, we need the dilation operator D_s restricted to $[0,1]$, which is defined for $0 < s < \infty$ and f on $[0,1]$, by

$$(D_s f)(t) = \begin{cases} f(t/s) & 0 \le t \le \min(1,s) \\ 0 & s < t \le 1 \ \text{(in the case } \ s < 1). \end{cases}$$

In fact, the operator D_s is a dilation by the ratio $s: 1$ in the positive direction of the t-axis followed by restriction to $[0,1]$. It is easily verified that, for every choice of r and s, and every r.i. function space X on $[0,1]$,

$$\|D_{rs}\|_X \le \|D_r\|_X \|D_s\|_X,$$

which eventually ensures that the following limits exist:

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|_X} = \sup_{s>1} \frac{\log s}{\log \|D_s\|_X}$$

and

$$q_X = \lim_{s\to 0+} \frac{\log s}{\log \|D_s\|_X} = \sup_{0<s<1} \frac{\log s}{\log \|D_s\|_X}.$$

The importance of these indices stems from the fact, proved by Boyd [Bo], that, whenever $1 \le p < p_X$ and $q_X < q \le \infty$ for some r.i. function space X on $[0,1]$, then every linear operator T, which is bounded on both $L_p(0,1)$ and $L_q(0,1)$, is also bounded on X.

A result due to Paley [Pa], proved in detail e.g. in [L - T II p.155], (cf. also the article of Burkholder and the article of Figiel and Wojtaszczyk in this handbook) asserts that the Haar system forms an unconditional basis in every $L_p(0,1)$-space, if $1 < p < \infty$. It follows immediately that the Haar system is also unconditional in every separable r.i. function space X on $[0,1]$, whose Boyd indices are non-trivial i.e. $1 < p_X$ and $q_X < \infty$.

In the case that one of the Boyd indices is trivial, Kalton [K] proved the following uniqueness result.

Theorem 5.8 *Let X be a separable r.i. function space on $[0,1]$ such that either $p_X = 1$ or $q_X = \infty$. Then X has unique structure as an r.i. function space on $[0,1]$.*

Finally, we mention the following result which was proved in [J - M - S - T p.169] , in the reflexive case, and follows from 5.8 in the non reflexive one.

Theorem 5.9 *A separable Orlicz function space on $[0,1]$ has unique structure as an r.i. function space on $[0,1]$.*

It turns out that there are also many r.i. function spaces on $[0,1]$ which fail to have a unique structure as a r.i. function space on $[0,1]$. The construction of such examples has something in common with that of a space with "many" mutually non-equivalent symmetric bases, presented in section 2.

We start, as in section 2, with the space U_1 defined there, whose normalized unconditional basis $\{u_n\}_{n=1}^\infty$ is universal for all normalized unconditional basic sequences in the sense that each such sequence is equivalent to a subsequence of $\{u_n\}_{n=1}^\infty$.

We could proceed as in section 2 and interpolate U_1 between two distinct L_p-spaces but the r.i. space constructed in this way will not be isomorphic to U_1. In order to avoid this problem, we shall first p-convexify and 2-concavify the space U_1, for some $1 < p < 2$, thus obtaining a new space whose unconditional basis is universal for all p-convex and 2-concave normalized unconditional basic sequences. To this end, with $1 < p < 2$ and $q = p^* = p/(p-1)$, we introduce a new norm $|||\cdot|||$ on U_1 by setting for $u \in U_1$,

$$|||u||| = \sup\left\{ \left(\sum_{k=1}^n \|v^{(k)}\|^q\right)^{1/2} ;\ u = \left(\sum_{k=1}^n |v^{(k)}|^q\right)^{1/2} \right\},$$

where the supremum is taken over all decompositions of u, as above. It is easily checked that $\{u_n\}_{n=1}^\infty$ is in $(U_1, |||\cdot|||)$ q-concave and universal for all q-concave normalized unconditional basic sequences. Hence, the biorthogonal functionals form in the dual V of $(U_1, |||\cdot|||)$ a p-convex unconditional basis which is universal for all p-convex normalized unconditional basic sequences. Repeating this procedure, this time with 2 instead of q, we eventually get a space W_p, with a normalized unconditional basis $\{w_n^{(p)}\}_{n=1}^\infty$, which is p-convex and 2-concave and, moreover, each p-convex and 2-concave normalized unconditional sequence is equivalent to a subsequence of $\{w_n^{(p)}\}_{n=1}^\infty$.

Next, with $p \le r < 2$, we interpolate W_p between $L_r(0,1)$ and $L_2(0,1)$ thus obtaining a p-convex and 2-concave r.i. function space $Y_{p,r}$ on $[0,1]$. The Boyd indices of this space are non-trivial and therefore the Haar system in $Y_{p,r}$ is unconditional and this unconditional structure is p-convex and 2-concave. Hence, $Y_{p,r}$ is isomorphic to a complemented subspace of W_p.

The interpolation procedure discussed above is done by choosing a sequence $\{n_k\}_{k=1}^\infty$ and by defining

$$\|f\|_k = \inf\{n_k\|g\|_r + n_k^{-1}\|h\|_2;\ \ f = g + h\}$$

and

$$\|f\|_{Y_{p,r}} = \|\sum_{\ell=1}^\infty \|f\|_k w_k^{(p)}\|_{W_p},$$

for any $f \in L_r(0,1)$. One can easily show that if $\{n_k\}_{k=1}^\infty$ tends "fast" to ∞ then the corresponding space $Y_{p,r}$ contains a complemented subspace isomorphic to W_p. Then, by using the decomposition method, we conclude that $Y_{p,r} \approx W_p$, for any choice of $p < r < 2$. Finally, it is not very hard to verify that, for different values of $p < r < 2$, we obtain mutually non-equivalent r.i. function spaces $Y_{p,r}$ on $[0,1]$, which are all isomorphic to W_p.

While the class of r..i. function spaces on a non-atomic measure space can be considered as a continuous variant of the class of spaces with a symmetric basis, non-atomic Banach lattices are the continuous analogue of spaces with an unconditional basis. The first result on uniqueness of structure for non-atomic Banach lattices was proved by Abramovich and Wojtaszczyk [A - W] who showed that L_1 and L_2 have unique structure as non-atomic Banach lattices.

It turns out that, contrary to the case of spaces with unique unconditional basis, there exist other spaces with unique non-atomic structure. For example, the Orlicz space $L_M(0,1)$, with $M(t) \sim t(\log t)^\alpha$, for large t and $0 < \alpha < \frac{1}{2}$, is such an example (cf. [K]). In particular, this is an example of an r.i. function space on $[0,1]$ which is not isomorphic to an r.i. function space on $[0,\infty]$.

6 Uniqueness of bases in non-Banach spaces

As we have seen in the previous sections, it is quite rare for an unconditional basis in a Banach space to be unique, even up to a permutation. It turns out that in spaces other than Banach spaces one can find relatively many unconditional bases having the uniqueness property and , moreover, some of these uniqueness results have quite interesting applications.

One class of spaces with a rich structure is that of so-called **quasi-Banach** spaces. A quasi-Banach space is a vector space X endowed with a quasi-norm $\|\cdot\|$ which satisfies the usual axioms of the norm except that the triangle inequality is replaced by the inequality

$$\|x+y\| \leq C(\|x\| + \|y\|),$$

for all $x, y \in X$ and some fixed constant $C \geq 1$. A survey of the theory of these spaces is [K - P - R] or the article of Kalton in this handbook. In the class of quasi-Banach spaces the uniqueness of unconditional bases, up to equivalence, is quite a common occurrence. For instance, it was shown by Kalton [K 1] that the spaces ℓ_p have, up to equivalence, a unique unconditional basis, for every value of $0 < p < 1$. As we have seen in the previous sections, this result is false for $1 < p \neq 2$, even up to a permutation. In the same paper [K 1] , the author also exhibits a larger class of non-locally convex Orlicz sequence spaces with a unique unconditional basis, up to equivalence. One can consider the problem of uniqueness for the so-called non-locally convex Lorentz sequence spaces $d(w,p)$, where $p > 0$ and $w = (w_n)_{n=1}^\infty$ is a non-summable monotone decreasing sequence of positive numbers and the quasi-norm $\|x\|_{w,p}$ of an element $x \in d(w,p)$ is defined by

$$\|x\|_{w,p} = \sup_\pi \left(\sum_{n=1}^\infty |x_{\pi(n)}|^p w_n\right)^{1/p} < \infty.$$

In this definition, the supremum is taken over all permutation π of the integers.

The uniqueness of the basis in $d(w,p)$ was studied in [N - O] and further in [K - L - W]. One result in this direction, which is proved in [K - L - W] and provides the solution to an open question raised in [N - O], asserts that the space $d(w,p)$ has, up to equivalence, a unique unconditional basis, whenever

$0 < p < 1$ and the sequence of weights w satisfies the condition.

$$\lim_{n\to\infty} \frac{1}{n}(w_1 + \cdots + w_n)^{1/p} = \infty.$$

Another remarkable result proved in [K - L - W] states that also the spaces $\ell_p(\ell_q)$ have a unique unconditional basis, up to equivalence and permutation, as long as $0 < p, q < 1$. The same is true for the spaces $c_0(\ell_p)$, as long as $0 < p < 1$ (cf. [Le]).

Perhaps, the most interesting class of quasi-Banach spaces is that of the Hardy spaces $H_p(\mathbb{T}), 0 < p < 1$, where $\mathbb{T}$ denotes the circle, and their m-dimensional version $H_p(\mathbb{T}^m)$, where $\mathbb{T}^m$ denotes the m-dimensional torus. The fact that these quasi-Banach spaces have an unconditional basis is not obvious. For $m = 1$, this follows from the fact, proved in [W], that $H_p(\mathbb{T})$ is isomorphic to the **dyadic** Hardy space $H_p(0,1)$, for all $0 < p < 1$. The usual Haar basis $\{h_n\}_{n=1}^{\infty}$ forms an unconditional basis in $H_p(0,1), 0 < p < \infty$, since the dyadic $H_p(0,1)$ consists of all distribution of the form $f = \sum\limits_{n=1}^{\infty} a_n h_n$ for which the expression

$$\|f\|_p = \Big(\int_0^1 (\sum_{n=1}^{\infty} |a_n h_n|^2)^{p/2} \Big)^{1/p} < \infty.$$

For higher dimensions, one verifies that the m-times tensor product of the Haar basis is an unconditional basis in the dyadic Hardy space $H_p(0,1)^m$, for every value of $0 < p < 1$, and that this space is isomorphic to the usual space $H_p(\mathbb{T}^n)$ (cf. [W]). A remarkable result from [W1] asserts that every normalized unconditional basis in $H_p(\mathbb{T}); 0 < p < 1$, is equivalent to the Haar basis in some order i.e. the space $H_p(\mathbb{T})$ has, up to equivalence and permutation, a unique unconditional basis.

A nice application of the uniqueness result in non-locally convex Hardy spaces is the fact (proved first by Kalton, Leranoz and Wojtaszczyk [K - L - W]) that, for $0 < p < 1$, the spaces $H_p(\mathbb{T}^m), m = 1, 2, \cdots$, are mutually non-isomorphic. If, for example, $H_p(\mathbb{T}^m)$ were isomorphic to $H_p(\mathbb{T})$, for some $0 < p < 1$ and some integer m, then the corresponding normalized unconditional bases of these two spaces would be equivalent, up to a permutation. That this is false is checked directly.

One should point out that the result asserting that the spaces $H_1(\mathbb{T}^m), m = 1, 2, \cdots$, are mutually non-isomorphic was proved before by Bourgain ([Bo 1], [Bo 2]).

Uniqueness of bases can be also studied in the more general context of locally convex spaces (l.c.s.), i.e. vector spaces which admit a fundamental system of convex neighborhoods of 0. The topology of such a space can be defined with the aid of a family of semi-norms. Of particular interest are the Fréchet spaces. (recall that these are l.c.s. whose topology is both metrizable and complete).

The main difficulty faced in studying uniqueness in l.c.s. is the fact the notion of "normalized basis" does not make any sense in these spaces.

The only Fréchet spaces which have a unique unconditional basis are ω, the space of all scalar sequences, and its dual ω^*, the space of all scalar sequences which are eventually zero. The uniqueness assertion for ω and ω^* is due to Köthe and Töplitz. This is the result that in some sense initiated the research on uniqueness of bases. That ω and ω^* are the only spaces with this uniqueness property is due to Dragilev (cf. [Dr 3]).

Less restrictive is the notion of quasi-equivalence. Two unconditional bases $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ in l.c.s. X, respectively Y, are said to be **quasi-equivalent** if there exists a permutation π of the integers and a sequence $\lambda = (\lambda_n)_{n=1}^{\infty}$ of scalars such that $\{\lambda_n x_{\pi(n)}\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are equivalent, i. e. there exists an isomorphism T from X onto Y so that $y_n = T(\lambda_n x_{\pi(n)})$, for all n.

A remarkable result, also due to Dragilev [Dr 1], asserts that the Fréchet space $A(D)$ of all the functions which are analytic on the open disk D (endowed with the topology of uniform convergence on each compact subset of the disk) has the property that all its unconditional bases are quasi-equivalent. This fact is also true in the class of all nuclear power series space (cf. [Mi 1]).

A systematic study of quasi-equivalence in l.c.s. has been carried out by Dragilev [Dr 2], Dubinsky [Du], Mityagin [Mi 2] and recently by Zahariuta [Za].

References

[A-W] Abramovich, Y.A. and Wojtaszczyk, P.: *On the uniqueness of order in the spaces ℓ_p and $L_p[0,1]$*, Math. Zametki **18**, 313–325 (1975) [Russian].

[B] Babenko, K.I.: *On conjugate functions*, Dokl. Akad. Nauk. SSSR **62**, 157–160 (1948) [Russian].

[Be] Bernstein, A.J.: *Maximally connected arrays on the n-cube*, Discr. Math. **15**, 1485–1489 (1967).

[Bo] Boyd, D.B.: *Indices of function spaces and their relationship to interpolation*, Canadian J. Math., **21**, 1245–1254 (1969).

[Bo1] Bourgain, J.: *The non-isomorphism of H^1-spaces in one and several variables*, J. Funct. Anal., **46**, 45–47 (1982).

[Bo2] Bourgain, J.: *The non-isomorphism of H^1-spaces in a different number of variables*, Bull. Soc. Math. Belge. Ser. B **35**, 127–136 (1983).

[B-C-L-T] Bourgain, J., Casazza, P.G., Lindenstrauss, J. and Tzafriri, L.: *Banach spaces with a unique unconditional basis, up to permutation*, Memoir Amer. Math. Soc. **322** (1985).

[C-K 1] Casazza, P.G. and Kalton, N.J.: *Uniqueness of unconditional basis in Banach spaces*, Israel J. Math. **103**, 141–176 (1998).

[C-K 2] Casazza, P.G. and Kalton, N.J.: *Uniqueness of unconditional bases in c_o products*, preprint.

[C-K-T] Casazza, P.G., Kalton, N.J. and Tzafriri, L.: *Uniqueness of unconditional and symmetric structions in finite dimensional spaces*, Ill. J. Math. **34**, 793–836 (1990).

[C-L] Casazza, P.G. and Lammers, M.C.: *Genus N Banach spaces*, Preprint.

[D] Davis, W.J.: *Embedding spaces with unconditional bases*, Israel J. Math. **20**, 189–191 (1975).

[Dr1] Dragilev, M. M.: *Canonical forms of a basis in the space of analytic function spaces*, Uspehi Math. Nauk **15**, 181–188 (1960),

[Dr2] Dragilev, M. M.: *On regular bases in nuclear spaces*, Math. Sbornik **68**, 153–173 (1965) [Russian].

[Dr3] Dragilev, M. M.: *Topological vector spaces with equivalent bases*, Math. Zametki **28**, 947–951 (1980).

[Du] Dubinsky, E.: *The structure of nuclear Fréchet spaces* , Lect. Notes. Math. **720** (1979), Springer-Verlag.

[D-S] Dor, L.E. and Starbird, T.: *Projections of L_p onto subspaces spanned by independent random variables*, Composition Math. **39**, 141–175 (1979).

[E-W] Edelstein, I.S. and Wojtaszczyk, P.: *On projections and unconditional bases in direct sums of Banach spaces*, Studia Math. **56**, 263–276 (1976).

[F-J] Figiel, T. and Johnson, W.G.: *A uniformly convex Banach space which contains no ℓ_p*, Composition Math **29**, 179–190 (1974).

[G] Garnett, J.: *Bounded analytic functions*, Academic Press, Orlando 1981.

[Go] Gowers, W.T.: *A finite-dimensional Banach space with two non-equivalent symmetric bases*, Israel J. Math., **87**, 143–151 (1994).

[H] Harper, L.H.: *Optimal assignments to number of vertices*, SIAM J. Appl. Math., **12**, 131–135 (1964).

[Ha] Hart, S.: *A note on the edges of the n-cube*, Discr. Math. **14**, 157–163 (1976).

[H-K] Hernandez, F. and Kalton, N.J.: *Subspaces of rearrangement invariant spaces*, Canad. J. Math. **48**, 794–833 (1996).

[J-M-S-T] Johnson, W.B., Maurey, B., Schechtman, G. and Tzafriri, L.: *Symmetric structures in Banach spaces*, Memoirs of Amer. Math. Soc. **217** (1979).

[K] Kalton, N.J.: *Lattice structures on Banach spaces*, Memoirs Amer. Math. Soc., **493** (1993).

[K1] Kalton, N. J.: *Orlicz sequence spaces without local convexity*, Math. Proc. Camb. Phil. Soc. **81**, 253–278 (1977).

[K-L-W] Kalton, N. J., Leranoz, C., and Wojtaszczyk, P.: *Uniqueness of unconditional bases in quasi-Banach spaces with applications to Hardy spaces*, Israel J. Math., **72**, 299–311 (1990).

[K-P-R] Kalton, N. J., Peck, N. J. and Roberts, J. W.: *An F-space Sample*, London Math. Soc. Lect. Notes 89, Cambridge University Press (1984).

[K-T] Köthe, G. and Töplitz, O. : *Theorie der halbfiniten unendlichen Matrizen*, J. Reine und Angew. Math. **165**,116–127 (1931).

[L] Lindenstrauss, J.: *A remark on symmetric bases*, Israel J. Math. **13**, 317–320 (1972).

[Le] Leranoz, C.: *Uniqueness of unconditional bases of $c_0(\ell_p), 0 < p < 1$*, Studia Math., **102**, 193–207 (1992).

[Li] Lindsey, J.H.: *Assignments of numbers to vertices*, Amer. Math. Monthly **71**, 508–516 (1964).

[L-P] Lindenstrauss, J. and Pelczynski, A.: *Absolutely summing operator s in L_p spaces and their applications*, Studia Math. **29**, 275–326 (1968).

[L-T O III] Lindenstrauss, J. and Tzafriri, L.: *On Orlicz sequence spaces III*, Israel J. Math. **14**, 368–389 (1973).

[L-T I] Lindenstrauss, J. and Tzafriri, L.: *Classical Banach Spaces I*, Sequence Spaces, Springer Verlag, Berlin-Heidelberg-New York 1977.

[L-T II] Lindenstrauss, J. and Tzafriri, L.: *Classical Banach Spaces II*, Function Spaces, Springer Verlag, Berlin-Heidelberg-New York 1979.

[L-Z] Lindenstrauss, J. and Zippin, M.: *Banach spaces with a unique unconditional basis* J. Funct. Anal. **3**, 115–125 (1969).

[Mc-S] McCarthy, C.A. and Schwartz, J.: *On the norm of a finite Boolean algebra of projections and applications to theorems of Kreiss and Morton*, Comm. Pure Appl. Math. **18**, 191–201 (1965).

[M] Maurey, B.: *Théoremes de factorisation pour les operateures a valeurs dans un espace L^p*, Asterisque, Soc. Math. France, **11** (1974).

[Mi1] Mityagin, B. S.: *Nuclear Riesz Scales*, Dokl. Akad. Nauk SSSR, **137**, 519–522 (1961).

[Mi2] Mityagin, B. S.: *Approximative dimension and bases in nuclear spaces*, Uspehi Math. Nauk. **16**, 63–132 (1961) [Russian].

[P] Pelczynski, A.: *Universal bases*, Studia Math. **32**, 247–268 (1969).

[N-O] Nawrocki, M. and Ortynski: *The Mackey topology and complemented subspaces of Lorentz sequence spaces $d(w,p)$ for $0 < p < 1$*, Trans. Amer. Math. Soc. **287**, 713–722 (1985).

[Pa] Paley, R.E.: *A remarkable series of orthogonal functions*, Proc. London Math. Soc., **34**, 241–264 (1932).

[P-R] Pelczynski, A. and Rosenthal, H.P.: *Localization techniques in L^p-spaces*, Studia Math., **52** (1975), 263–289.

[P-S] Pelczynski, A. and Singer, I.: *On non-equivalent bases and conditional bases in Banach spaces*, Studia Math. **25**, 5–25 (1964).

[R] Read, C.J.: *A Banach space with up to equivalence, precisely two symmetric bases*, Israel J. Math. **40**, 33–53 (1981).

[Ro] Rosenthal, H.P.: *On the subspaces of L^p $(p > 2)$ spanned by sequences of independent random variables*, Israel J. Math., **8**, 273–303 (1970).

[S] Schechtman, G.: *On Pelczynski's paper, "Universal bases"*, Israel J. Math. **22**, 181–184 (1975).

[Schü] Schütt, C.: *On the uniqueness of symmetric bases in finite-dimensional Banach spaces*, Israel J. Math., **40**, 97–117 (1981).

[T] Tsirelson, B.S.: *Not every Banach space contains ℓ_p or c_0*, Funct. Anal. Appl. **8**, 138–141 (1974) (translated from Russian).

[W] Wojtaszczyk, P.: *H_p-spaces, $p \leq 1$, and spline systems*, Studia Math. **77**, 289–320 (1984).

[W1] Wojtaszczyk, P.: *Uniqueness of unconditional bases in quasi-Banach spaces with applications to Hardy spaces II* , Israel J. Math. **97**, 253–280 (1997).

[Z] Zippin, M.: *On perfectly homogeneous bases in Banach spaces*, Israel J. Math. **4**, 265–272 (1966).

[Za] Zahariuta, V. P.: *On isomorphic classification of F-spaces*, Lect. Notes Math. **1043** (1983), Springer-Verlag.