On the normal bundle of area minimizing surfaces in Riemannian 4-manifolds

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Abstract

Let Σ be a Riemann surface, let M be a Riemannian 4-manifold and let f : $\Sigma \to M$ be a pseudo-conformal harmonic immersion. There is a well-defined notion of image tangent plane bundle Tf and normal plane bundle Nf for f. The sum of the Euler numbers of these two bundles, $e(Tf) + e(Nf)$ is called the **twistor degree** or *adjunction number* of f. We examine the contribution of branch points to this twistor degree in the case that f is *area minimizing* (i.e. area minimizing near the origin among piecewise C^1 maps of Σ into M, cf. [14]). The formula we derive is analogous to the formula for the virtual genus of singular projective curves; the Puiseux expansion being replaced by the expansion of Micallef-White ([14]).

I. Preliminaries

In all that follows, Σ is a Riemann surface, M is a Riemannian 4-manifold and $f : \Sigma \to M$ is a branched immersion, that is, it is an immersion, except at a finite number of points, $x_1, \ldots, x_p$, called **branch points**, where the differential df vanishes.

§1. Definition of the twistor degree.

1.1 Definition

Let $G_2(M)$ be the Grassman bundle of oriented 2-planes tangent to M. Outside the branch points, the differential df lifts to a map

$$\widetilde{df} : \Sigma - \{x_1, \ldots, x_p\} \to G_2(M).$$

The map $\widetilde{df}$ extends to the whole of Σ (cf. for instance [6]); this enables us to define the image tangent bundle Tf; let us recall the formula for its Euler number:

Proposition 1.

$$e(Tf) = \chi(\Sigma) + \sum_{p_i} m_i$$

where p_i runs through the branch points of f and m_i is the branching order of p_i.

Also, by taking orthogonal complements, we define a normal bundle Nf.

Definition 1 ([5]). *The* **twistor degree** $d_+(f)$ *is*

$$d_+(f) = e(Tf) + e(Nf).$$

where e denotes the Euler number.

NB 1. In [5]'s definition, there is a factor $\frac{1}{2}$ which comes naturally from their twistorial definition; we have decided to drop it for simplicity's sake.

NB 2. Chen and Tian ([4]) call the twistor degree the *adjunction number.*

1.2. Twistor interpretation

a) Preliminaries

We denote by $\Lambda^2(M)$ the bundle of second exterior products of tangent spaces to M. We let

$$* : \Lambda^2(M) \to \Lambda^2(M)$$

be the Hodge operator and we denote by $\Lambda^+(M)$ the $(+1)$-eigenspace for $*$.

Let $Z^+(M)$ be the unit sphere bundle of $\Lambda^+(M)$; we recall (see [1]) that it is the bundle of the almost complex structures on M which preserve the orientation and metric of M; it is called the *twistor space* of M ([1], [2]). Following [9] we denote by T_F the tangent bundle to $Z^+(M)$ along the fibers and we define the 2-cohomology class in $Z^+(M)$,

$$\omega = c_1(T_F).$$

b) Lifting f to the twistor space.

The map $f : \Sigma \to M$ lifts to a map (called the **Gauss map**)

$$Z^+ f : \Sigma \to Z^+(M)$$

in the following way. Let $p \in \Sigma$ and let (e_1, e_2) be a positive orthonormal basis of $(Tf)_p$; then we have

$$Z^+ f(p) = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + *(e_1 \wedge e_2))$$

and the twistor degree of f is given by

$$d_+(f) = -\int_\Sigma (Z^+ f)^* \omega.$$

NB. Likewise we define $d_-(f)$ by lifting f to $\Lambda^-(M)$, the (-1)-eigenspace of the Hodge operator.

REMARK. The reader may get confused by finding elsewhere ([5], [17]) this formula without a *minus* sign:

$$d_+(f) = \int_\Sigma (Z^+ f)^* \omega.$$

The reason is that there are two possible orientations on T_F; when one looks at minimal surfaces in M *à la* Eells-Salamon, that is via pseudo-holomorphic curves into $Z^+(M)$, one uses the orientation opposed to the one in [9].

1.3. Historical note