A non-abelian free pro-p group is not linear over a local field

Y. Barnea* M. Larsen†

January 28, 1998

Preprint No. 8
1997/98

Abstract

In this paper we show that a (non-abelian) free pro-p group cannot be obtained as a closed subgroup of $GL_n(F)$, where F is a non-archimedean local field and n is arbitrary. Using a theorem of Zelmanov [Ze] it is a direct corollary that the Golod-Shafarevich inequality holds for every finitely generated pro-p subgroup of $GL_n(F)$. Our main tool is a recent theorem by R. Pink characterizing compact subgroups of $GL_n(F)$.

*Partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

†Partially supported by NSF grant DMS-9400833 and a grant from the Sloan Foundation.

1 Introduction

Recently R. Pink [P] gave a qualitative characterization of compact subgroups of products of semisimple algebraic groups over arbitrary local fields. In particular the following result (Corollary 0.5) is a consequence of his (much more general) theory.

Theorem 1.1. *Consider a local field F, a positive integer n, and a compact subgroup $\Gamma \subset GL_n(F)$. There always exist closed normal subgroups $\Gamma_3 \subset \Gamma_2 \subset \Gamma_1$ of Γ such that*

1. *Γ/Γ_1 is finite.*
2. *Γ_1/Γ_2 is abelian of finite exponent.*
3. *There exists a local field E of the same characteristic and the same residue characteristic as F, a connected adjoint group H over E, with universal covering $\pi : \tilde{H} \to H$, and an open compact subgroup $\Delta \subset \tilde{H}(E)$, such that Γ_2/Γ_3 is isomorphic to $\pi(\Delta)$ as topological groups.*
4. *Γ_3 is a solvable subgroup of derived length no more then n.*

We use this theorem to study the following conjecture of Lubotzky and Shalev ([LSh], Conjecture 3.8).

Conjecture *Let Λ be a complete commutative Noetherian local ring whose residue field is finite.*

(1) A (non-abelian) free pro-p group cannot be embedded as a closed subgroup in $GL_n(\Lambda)$.

(2) Every pro-p subgroup of $GL_n(\Lambda)$ satisfies some non-trivial pro-p-identity. That is, there exists a finitely generated free pro-p group Φ and a non-trivial quotient Ψ of Φ such that every continuous homomorphism from Φ to $GL_n(\Lambda)$ factors through Ψ.

It was proved by Zubkov [Zu] that (1) holds for all Λ if and only if the same is true of (2). The conjecture extends the following theorem of Zubkov ([Zu], Theorem 4.2)

Theorem 1.2. *Let $p \neq 2$ and $\mathcal{G}$ be a pro-p group in $GL_2(\Lambda)$, where Λ is a commutative pro-finite ring. Then $\mathcal{G}$ admits a pro-p-identity, independent of Λ as well as of $\mathcal{G}$.*

We should remark that E. Zelmanov told us that he has a proof of this theorem also in the case $p = 2$.

We prove the following theorem:

Theorem 1.3. *A (non-abelian) free pro-p group cannot be embedded as a closed subgroup in $GL_n(F)$, where F is a local field.*

For the purposes of this paper, a local field is the fraction field of a complete discrete valuation ring with finite residue field, though the theorem (and likewise Theorem 1.1) remain valid also for $\mathbb{R}$ and $\mathbb{C}$. The theorem was already known in the case where F is of characteristic zero by the work of Lazard [La] and Lubotzky and Mann [LM]. We should remark here that this does not imply that every pro-p subgroup of $GL_n(F)$ satisfies some non-trivial pro-p-identity.

A pro-p group G with minimal number of generators $d = d(G)$ is said to satisfy the **Golod-Shafarevich inequality** if for any presentation (in the category of pro-p groups) of G with n generators and r relators $r \geq n + d^2/4 - d$.

Pro-p groups satisfying the Golod-Shafarevich inequality have been studied by many people. Golod and Shafarevich proved that finite p-groups satisfy the inequality. Lubotzky [Lu] extended the result to p-adic analytic pro-p groups. Wilson [W] proved it for large classes of pro-p groups such as finitely generated soluble pro-p group, while Lubotzky and Shalev [LSh] proved it for L-perfect pro-p groups.

The following is a theorem of E. Zelmanov [Ze]:

Theorem 1.4. *A pro-p group presented by a small set of relators contains a (non-abelian) free pro-p subgroup.*

For the exact definition of a small set of relators see [Ze]. The best known examples of pro-p groups G with small sets of relators are groups admitting a presentation with $d = d(G)$ generators and r relators where $r < d^2/4$.

It is known that if a pro-p group G has a presentation with n generators and r relators then G has presentation with $d(G)$ generators and $r-(n-d(G))$ relators (see [Lu] Lemma 1.1). Combining this fact and Theorems 1.3 and 1.4 we immediately deduce:

Corollary 1.5. *Let G be a finitely generated pro-p subgroup of $GL_n(F)$, where F is a local field, and $d(G) = d > 1$.. If G has presentation of n generators and r relators, then $r \geq n + d^2/4 - d$.*

Wilson's proof of corollary A' in [W] actually gives the following lemma:

Lemma 1.6. *Let G be a finitely presented pro-p group. Assume that every open subgroup of G satisfies the Golod-Shafarevich inequality. Then*

(i) there is a constant k such that $d(H) \leq k|G:H|^{1/2}$ for each open subgroup H of G, and

(ii) if N is any (closed) normal subgroup of G such that $G/N \cong \mathbb{Z}_p$, then N is finitely generated.

Thus we have the following corollary:

Corollary 1.7. *Let G be a finitely presented pro-p subgroup of $GL_n(F)$. Then*

(i) there is a constant k such that $d(H) \leq k|G:H|^{1/2}$ for each open subgroup H of G, and

(ii) if N is any (closed) normal subgroup of G such that $G/N \cong \mathbb{Z}_p$, then N is finitely generated.

If G is a finitely generated pro-p group, denote by $a_n(G)$ the number of subgroups of index n of G. Applying Lemma 4.1 from [LSh] and Corollary 1.7 (i) we get:

Corollary 1.8. *Let G be a finitely presented pro-p subgroup of $GL_n(F)$. Then for any n $a_n(G) \leq p^{cn^{1/2}}$ for some constant c depending on G.*

2 On subgroups of $GL_n(F)$

We begin with a lemma essentially asserting that morphisms between varieties over local fields are Lipschitz.

Lemma 2.1. *Let V be a discrete valuation ring with uniformizer π and fraction field K. Let X and Y be separated schemes locally of finite type and flat over V, $\phi: X \times_V K \to Y \times_V K$ a morphism of generic fibers, and $x \in X(V)$ such that $\phi(x) \in Y(V)$. Then there exists an integer $N \geq 0$ such that for all $n \geq 0$ and all $x' \in X(V)$, $x \equiv x' \pmod{\pi^{N+n}}$ implies $\phi(x') \in Y(V)$ and $\phi(x) \equiv \phi(x') \pmod{\pi^n}$.*

Proof. The question is local, so we may assume $X = \text{Spec } A$ and $Y = \text{Spec } B$, where A and B are flat finitely generated V-algebras and $\phi^* : B \times_V K \to A \times_V K$ is a K-algebra homomorphism. Let I and I' denote the ideals corresponding to x and x' respectively and ψ the restriction of ϕ^* to $B = B \otimes 1 \subset B \otimes V$. Let $b_1, \ldots, b_k$ generate B as V-algebra and $c_1, \ldots, c_m$ generate the B-ideal $\psi^{-1}(I \otimes K)$, where ψ is the restriction of ϕ to $B = B \otimes 1 \subset B \otimes K$. We choose N large enough that

$$\psi(b_i) \in A + \pi^{-N} I = V + \pi^{-N} I, \quad \psi(c_j) \in \pi^{-N} I$$

for all i, j. The condition $\phi(x') \in Y(V)$ means $\psi(B) \subset V + I' \otimes K$. If $x \equiv x'$ $(\text{mod } \pi^{N+n})$, then $I \subset \pi^{N+n} V + I'$, so $\psi(b_i) \in V + \pi^{-N} I'$ for each b_i, and thus $\psi(B) \subset V + \pi^{-N} I' \subset V + I' \otimes K$. The condition $\phi(x) \equiv \phi(x') \pmod{\pi^n}$ means

$$\psi^{-1}(I \otimes K) \subset \pi^n V + \psi^{-1}(I' \otimes K).$$

As

$$\psi(c_j) \in \pi^{-N} I \subset \pi^n V + \pi^{-N} I'.$$

this condition is satisfied. □

Proposition 2.2. *Let T be a torus of dimension $r \geq 1$ defined over a field of Laurent series $F = \mathbb{F}((t))$, where $\mathbb{F}$ is a finite field. Then for each $d \geq 0$, $T(F)$ contains a closed subgroup isomorphic to $\mathbb{Z}_p^d$.*

Proof. By the structure theorem for tori over fields ([DG] X Prop. 1.4), T is determined up to F-isomorphism by its character group $\mathsf{X}^*(T)$ together with the (locally constant) action of $\text{Gal}(F^s/F)$ on $\mathsf{X}^*(T)$, where F^s denotes a separable closure of F. The action factors through a finite quotient $\text{Gal}(K/F)$, where $K \cong k((u))$ and T is split over K. If $\mathcal{T}_1$ denotes the split torus over $k[[u]]$ with generic fiber $T \times_{\mathbb{F}[[t]]} k[[u]]$, the valuation and reduction maps give short exact sequences of $\text{Gal}(K/F)$-modules

$$0 \to \mathcal{T}_1(k[[u]]) \to T(K) \to \text{Hom}(\mathsf{X}^*(T), \mathbb{Z}) \to 0,$$
$$0 \to U_0 \to \mathcal{T}_1(k[[u]]) \to \mathcal{T}_1(k) \to 0.$$

Setting

$$U_n = \ker(\mathcal{T}_1(k[[u]])) \to \mathcal{T}_1(k[[u]]/u^{n+1}),$$

we obtain a $\mathrm{Gal}(K/F)$-stable filtration

$$U_0 \supset U_1 \supset U_2 \supset U_3 \supset \cdots$$

On the other hand, T has a Néron model (technically, an lft-Néron model) $\mathcal{T}_2$ on $\mathbb{F}[[t]]$ ([BLR] §10.2 Theorem 2). Setting

$$V_n = \ker(\mathcal{T}_2(\mathbb{F}[[t]]) \to \mathcal{T}_2(\mathbb{F}[[t]]/(t^{n+1}))),$$

we obtain the filtration

$$V_0 \supset V_1 \supset V_2 \supset V_3 \supset \cdots.$$

The tori $\mathcal{T}_1$ and $\mathcal{T}_2 \times_{\mathbb{F}[[t]]} k[[u]]$ have isomorphic generic fibers, so by Lemma 2.1, the filtrations V_0 and

$$U_{e\cdot} \cap U_0^{\mathrm{Gal}(K/F)} = U_{e\cdot}^{\mathrm{Gal}(K/F)}$$

are commensurable, where e is the ramification degree of K/F.

Now U_0/U_n is killed by p^{a_n}, where a_n has logarithmic growth. The same is therefore true of $U_0^{\mathrm{Gal}(K/F)}/U_n^{\mathrm{Gal}(K/F)}$. As U_0 is a free $\mathbb{Z}_p$-module, the same is true of $U_0^{\mathrm{Gal}(K/F)}$, and if the latter has rank $\leq d$,

$$|U_0^{\mathrm{Gal}(K/F)}/U_n^{\mathrm{Gal}(K/F)}| \leq p^{a_n d}$$

By the smoothness of Néron models, $|V_{n-1}/V_n| = |\mathbb{F}|^r$, so the logarithm of $|V_0/V_n|$ grows linearly. The proposition follows. □

Corollary 2.3. *If G is a semisimple algebraic group over $F \cong \mathbb{F}((t))$, then any open subgroup of G contains a commutative subgroup topologically isomorphic to $\mathbb{Z}_p^2$.*

Proof. By [DG] XIV Theorem 1.1, G contains a maximal torus T defined over F, and by [DG] XII Lemma 1.2, the rank of T is positive. By Prop. 2.2, any open subgroup of $T(F)$ contains an open subgroup of $\mathbb{Z}_p^2$, hence a subgroup isomorphic to $\mathbb{Z}_p^2$. □

Proof of Theorem 1.3: The case $\mathrm{char}(F) = 0$ is known because p-adic analytic pro-p groups have finite rank, and free pro-p groups have infinite rank (see [DDMS] for background on p-adic analytic pro-p groups). Hence we may assume that $F \cong \mathbb{F}((t))$. Let Γ denote a closed subgroup of $GL_n(F)$ which is non-abelian free pro-p. It is well-known that every closed subgroup of a free pro-p group is again free pro-p ([Shatz] Ch. 3 §3 Cor. 3). A non-abelian free pro-p group is never solvable, so an element of Γ that normalizes a closed abelian subgroup must centralize that subgroup. Thus the only closed normal solvable subgroup of Γ is the trivial group.

Applying Theorem 1.1, we conclude first that Γ_1 is non-abelian free pro-p. Next, that Γ_2 is non-abelian free pro-p, and finally, that $\Gamma_3 = \{1\}$. Thus, Γ_2/Γ_3 is a non-abelian free pro-p subgroup isomorphic to $\pi(\Delta)$. By Corollary 2.3, Δ contains a subgroup isomorphic to $\mathbb{Z}_p^2$, so the same is true of $\pi(\Delta)$, which is absurd since $\mathbb{Z}_p^2$ is not free pro-p. □

Acknowledgment. This work is part of the first author's Ph. D. thesis under the supervision of Professor Aner Shalev. The authors thank Richard Pink for his comments on earlier versions of this paper.

References

[BLR] S. Bosch, W. Lütkebohmert, M. Raynaud, *Néron Models*, Springer-Verlag, New York, 1990.

[DDMS] J. Dixon, M. P. F. Du Sautoy, A. Mann and D. Segal, *Analytic Pro-p Groups*, London Math. Soc. Lecture Note Series **157**, Cambridge University Press, Cambridge, 1991.

[DG] M. Demazure, A. Grothendieck *Schémas en groupes*, (SGA3) Lecture Notes in Mathematics **151-153**, Springer-Verlag, Berlin, 1970.

[La] M. Lazard, Groupes analytiques p-adiques, *Publ. Math. I. H. E. S.* **26** (1965), 389–603.

[Lu] A. Lubotzky, Group presentations, p-adic analytic groups and lattices in $SL_2(\mathbb{C})$, *Ann. Math.* **118** (1983), 115-130.

[LM] A. Lubotzky and A. Mann, Powerful p-groups. I, II. *J. of Algebra* **105** (1987), 484–515.

[LSh] A. Lubotzky and A. Shalev, On some Λ-analytic pro-p groups, *Israel J. Math.* **85** (1994), 307–337.

[P] R. Pink, Compact subgroups of linear algebraic groups, to appear in *J. of Algebra*.

[Shatz] S. S. Shatz, *Profinite Groups, Arithmetic, and Geometry*, Annals of Mathematics Studies **67**, Princeton University Press, 1972.

[W] J. S. Wilson, Finite presentations of pro-p groups and discrete groups, *Invent. Math.* **105** (1991), 177-183.

[Ze] E. I. Zelmanov, Lie ring in the theory of nilpotent groups, in *Groups '93 Galway/St. Andrews*, Vol 2 (London Math. Soc. Lecture note Series **212**, Cambridge University Press, Cambridge, 1995) 567–585.

[Zu] A. Zubkov, Non-abelian free pro-p-groups cannot be represented by 2-by-2 matrices, *Sib. Math. J.* **28** (1987) 742-747.

Authors' addresses:

Yiftach Barnea, Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem 91904, Israel

Michael Larsen, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA