Arithmetic structure of fundamental groups and actions of semisimple Lie groups.*

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Preprint No. 4
1997/98

*Research partially supported by grants from NSF and the Binational Science Foundation, Israel - USA.
**Partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany)

Introduction

The aim of this paper is to establish arithmetic properties of the fundamental group of a space on which a non-compact simple Lie group acts. In addition, we establish arithmetic properties of the actions themselves.

More precisely, let G be a connected simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$. Let M be a compact space for which covering space theory holds. We assume that we have a continuous action of G on M. Then $\widetilde{G}$ acts on any covering space $M' \to M$. We further suppose that there is a finite G-invariant ergodic measure μ on M. The action of G on M is called ($\mu-$)engaging if for every finite covering $M' \to M$, the action of $\widetilde{G}$ on M' is ergodic (with respect to the natural lift of μ to M'.) The action is called totally engaging if there is no $\widetilde{G}$ equivariant measurable section of $M' \to M$ for any non-trivial covering space of M. In general, totally engaging implies engaging. As we shall see, one or both of these conditions holds for the natural actions of G on homogeneous spaces of the form $M = H/\Gamma$ where G acts via an embedding in H, where H is a Lie group and Γ is a lattice in H. Our main results on fundamental groups are the following.

Let $\pi_1(M) \to GL(V)$ be any finite dimensional linear representation over $\mathbb{C}$. Let Γ be the image, and assume Γ is infinite.

Theorem A *Suppose the action of G on M is totally engaging. Then Γ is an arithmetic group. In fact, Γ is commensurable to $H_{\mathbb{Z}}$, where H is a $\mathbb{Q}$-group with $\mathfrak{g} \hookrightarrow \mathfrak{h}$.*

Theorem B *Suppose the action of G on M is engaging. Then Γ is $\mathfrak{s}$-arithmetic. In fact, Γ is $\mathfrak{s}$-arithmetic in a $\mathbb{Q}$-group H with $\mathfrak{g} \hookrightarrow \mathfrak{h}$.*

In fact, we show that for engaging actions of G, Γ is arithmetic (not merely $\mathfrak{s}$-arithmetic) if an only if the action is totally engaging. (See Theorem 6.1.)

Here "$\mathfrak{s}$-arithmetic" is a generalization of the standard notion of S-arithmetic group where S is a finite set of primes. In the semi-simple case, such a group is virtually a product of S-arithmetic groups. In general, they will be lattices in a product of real and totally disconnected locally compact groups. These groups are discussed in detail in section 3.

We remark that Theorem A can be viewed as a generalization of Margulis' arithmeticity theorem. The latter is essentially equivalent to Theorem A when the action of G on M is transitive. In our case, we also need to construct the group H, and an embedding of Γ in H as an arithmetic group. In general, H can be much larger than G, and need not be semi-simple. A more precise and fuller statement of Theorems A and B appear below as Theorems 5.1 and 5.2.

We present in section 1 below examples showing the necessity of the hypotheses of Theorem A and B.

In addition to these arithmeticity theorems for the fundamental group, we establish arithmetic structure of the action itself. By an arithmetic action of a group G, we mean an action on a space $N = K\backslash H/\Gamma$, where H is a real algebraic $\mathbb{Q}$-group, $\Gamma \subset H$ is an arithmetic subgroup, $K \subset H$ is a compact (perhaps trivial) subgroup, and the G action is defined by a homomorphism $\sigma : G \to H$ so that K centralizes $\sigma(G)$. In [LZ], we studied arithmetic quotients of a given action. In particular, we showed that a finite entropy action of a non-compact simple Lie group G on a space M has a canonical maximal arithmetic (virtual) quotient action, say $A(M)$. Here, we show that for engaging actions, any linear representation of $\pi_1(M)$ yields an arithmetic quotient of M (and hence of $A(M)$.)

Theorem C *Let G, M and Γ be as in Theorem B. Let Γ_∞ be the arithmetic subgroup of the $\mathfrak{s}$-arithmetic group Γ. Then M has a virtual arithmetic quotient of the form $K\backslash H/\Gamma_\infty$.*

This theorem also appears in sharper form in Theorem 5.1 below.

Some of the conclusions of Theorem B and Theorem C were obtained under stronger assumptions in [Z8]. In fact, our proof of these results incorporate ideas of [Z8]. One of the basic assumptions in [Z8] is, in the context of Theorems B and C, that Γ is either discrete or has matrix entries in $\overline{\mathbb{Q}}$. This assumption is eliminated in the present work. This is of particular importance for potential applications where one may have a geometrically constructed representation, e.g. a holonomy representation, that is a priori neither discrete nor algebraic. We also observe that our conclusions of Theorems B and C in the sharp form of Theorems 5.1 and 5.2, are stronger than those of [Z8], even for representations that satisfy the assumptions in [Z8].

The authors would like to thank Moshe Jarden, Shahar Mozes, and T. N. Venkataramana for very helpful conversations on this work. This work was carried out while A.L. was a visitor at the University of Chicago and while R.Z. was a visitor at Hebrew University. It was also supported by a grant from the Binational Science Foundation of Israel and the United States. We wish to thank both universities and the BSF for their hospitality and support.

1 Engaging and totally engaging actions.

In this section we discuss the notions of engaging and totally engaging actions. The former was introduced in [Z5]. It will be useful to consider these notions for actions on general principal bundles with discrete fiber, not only on the coverings of M.

Let $P \to M$ be a principal Γ-bundle where Γ is a discrete group. We assume G is a group that acts by principal Γ-bundle automorphisms such that the action of G on M is ergodic with respect to some G quasi-invariant measure. We remark that if V is a Γ-space with quasi-invariant measure, then the associated bundle $E_V = (P \times V)/\Gamma$ is acted upon naturally by G with a natural measure class left invariant. In particular, for $V = \Gamma/\Gamma_0$ where Γ_0 is a subgroup, one has a natural measure class on $E_{\Gamma/\Gamma_0} \cong P/\Gamma_0$.

Definition 1.1. *The G-action on P is engaging if the action on P/Γ_0 is ergodic for every finite index subgroup $\Gamma_0 \subset \Gamma$.*

We shall be concerned with G-invariant reductions to subgroups of Γ.

Definition 1.2. *If $\Lambda \subset \Gamma$, then the G-action on P is called Λ-reducible if there is a measurable G-invariant reduction of P to Λ; i.e., there is a* *measurable G- invariant section of $P/\Lambda \to M$. (Here "invariant" is taken to mean invariant modulo null sets.)*

This can be reformulated in terms of cocycles. (See [Z3], [Z4] for general background.) Namely, a measurable trivialization of $P, P \cong M \times \Gamma$, defines a cocycle $\alpha : G \times M \to \Gamma$ by the equation

$$g.(m, \gamma) = (gm, \alpha(g, m)\gamma).$$

It is then easy to verify (see [Z4], e.g.) that:

Lemma 1.3. *P is Λ-reducible if and only if α is equivalent to a cocycle β such that $\beta(G \times M) \subset \Lambda$. (Here $\alpha \sim \beta$ means $\beta(g, m) = \phi(gm)^{-1}\alpha(g, m)\phi(m)$ for some $\phi : M \to \Gamma$.)*

¿From Lemma 1.3, we now have the following consequence.

Proposition 1.4. *If the G-action on P is engaging and is also Λ-reducible, then Λ is profinitely dense in Γ. Hence, under any finite dimensional linear representation $\sigma, \sigma(\Lambda)$ is Zariski dense in $\sigma(\Gamma)$.*

Proof. We recall that profinite density is equivalent to the assertion that for any subgroup $N \subset \Gamma$ of finite index that Λ surjects onto Γ/N. If the action on P is Λ-reducible, choose the cocycle β such that $\beta(G \times M) \subset \Lambda$. Then the action of G on $M \times \Gamma/N \cong P/N$ is given by $g(m, [\gamma]) = (gm, \beta(g, m)[\gamma])$. Since $M \times \{[e]\}$ is of positive measure, ergodicity of G on P/N clearly implies $\Lambda N = \Gamma$, verifying profinite density. That profinite density implies Zariski density is a result of Margulis and Soifer [MS].

In fact, further similar argument shows:

Proposition 1.5. *The action of G on P is engaging if and only if every $\Lambda \subset \Gamma$ for which the G action is Λ-reducible is profinitely dense.*

We shall most often apply Definition 1.1 to the case of $P = \widetilde{M}$ and $\Gamma = \pi_1(M)$, or to a quotient of this bundle by a normal subgroup of Γ.

Definition 1.6. [Z5] *We say the action of G on M is engaging if the action of $\widetilde{G}$ on the principal $\pi_1(M)$-bundle $\widetilde{M} \to M$ is engaging.*

Example 1.7. Let H be a connected Lie group, $\Lambda \subset H$ a lattice, and suppose G is a semisimple Lie group without compact factors that acts ergodically on H/Λ. Then the G action on H/Λ is engaging. This follows as a consequence of the more general Proposition 1.10 below.

Definition 1.8.

i) Suppose G acts on the principal Γ-bundle $P \to M$, acting ergodically on M. We say the action is totally engaging if there is no proper subgroup $\Lambda \subset \Gamma$ for which the action is Λ-reducible.

ii) If G acts on a manifold M, we say the action is totally engaging if the action of $\widetilde{G}$ on $\widetilde{M} \to M$ is totally engaging.

Proposition 1.9. *Any totally engaging action is engaging.*

This follows from Proposition 1.5.

Proposition 1.10. *Let H be a connected Lie group, $\Lambda \subset H$ a lattice, and $G \subset H$ a semisimple Lie group with no compact factors. Then the G action on H/Λ is totally engaging.*

Proof. We can write $H/\Lambda = \widetilde{H}/\widetilde{\Lambda}$ where $\widetilde{\Lambda}$ is a lattice in $\widetilde{H}$ and is the pull back of Λ to $\widetilde{H}$. Thus, we can identify $\widetilde{H/\Lambda}$ with $\widetilde{H}$ and $\pi_1(H/\Lambda)$ with $\widetilde{\Lambda}$. Suppose the $\widetilde{G}$ action is Γ-reducible for some $\Gamma \subset \widetilde{\Lambda}$. Then the section $s : H/\Lambda \to \widetilde{H}/\Gamma$ defines a finite G-invariant ergodic measure $s_*\mu$ on $\widetilde{H}/\Gamma$ that projects to the standard measure μ on H/Λ. By Ratner's theorem, $s_*\mu$ is the measure defined by volume on an L-orbit in $\widetilde{H}/\Gamma$ for some Lie group $\widetilde{G} \subset L \subset \widetilde{H}$. Since the projection of $s_*\mu$ to H/Λ is the volume on the latter, we must have $\dim L = \dim H$; it follows that $L = \widetilde{H}$. From the fact that s is a section, it then follows easily that $\Gamma = \widetilde{\Lambda}$.

Remark. There are smooth volume preserving actions of non-compact simple Lie groups on compact manifolds that are ergodic but not engaging. These are discussed in detail by Benveniste in [Be]. These examples, among a number of illuminating properties, have fundamental groups that are not $\mathfrak{s}$-arithmetic. In particular, this demonstrates the need for some hypotheses such as engaging in Theorem B. It is a natural question as to what geometric conditions on an action would imply engaging. In particular, the results of [Be] raise the question as to whether connection-preserving actions must be engaging.

Example 1.11. We present an example which is engaging but not totally engaging. The fundamental group will be S-arithmetic but not arithmetic.

Let G be a connected simply connected semisimple $\mathbb{Q}$-group with $\mathbb{Q}$-rank $= 0$, and each simple factor of G with $\mathbb{R}$-rank ≥ 2. Suppose p is a prime with each simple factor of $G_{\mathbb{Q}_p}$ non-compact. Let K be a maximal compact open subgroup of $G_{\mathbb{Q}_p}$, X the building associated to $G_{\mathbb{Q}_p}$. Thus, we can identify $V = G_{\mathbb{Q}_p}/K \subset X$ with a set of vertices. Let $\Gamma = G_{\mathbb{Z}[1/p]}$. Then $G_{\mathbb{R}}$ acts on the compact spaces

$$(G_{\mathbb{R}} \times V)/\Gamma \;\subset\; (G_{\mathbb{R}} \times X)/\Gamma \;=\; M.$$

Endow $(G_{\mathbb{R}} \times V)/\Gamma$ with the measure defined by Haar measure on $G_{\mathbb{R}} \times G_{\mathbb{Q}_p}$. We can view this as a finite $G_{\mathbb{R}}$-invariant ergodic measure on M. We remark that the action of $G_{\mathbb{R}}$ on $(G_{\mathbb{R}} \times G_{\mathbb{Q}_p})/\Gamma$ is ergodic if and only if Γ is dense in $G_{\mathbb{Q}_p}$. Since G is simply connected, $G_{\mathbb{Q}_p}$ has no non-trivial subgroups of finite index. Thus, if $\Gamma_0 \subset \Gamma$ is of finite index, the action of $G_{\mathbb{R}}$ on $(G_{\mathbb{R}} \times X)/\Gamma_0$ is also ergodic. These are the finite covers of M, so the action of $G_{\mathbb{R}}$ on M is engaging. (With a little more work, one can easily dispense with the simple connectivity assumption.) On the other hand, let $\Gamma_\infty \subset \Gamma$ be the arithmetic group $G_{\mathbb{Z}}$. Then the embedding

$$G_{\mathbb{R}} \;\longrightarrow\; G_{\mathbb{R}} \times \{[e]\} \;\subset\; G_{\mathbb{R}} \times V$$

induces an equivariant bijection

$$G_{\mathbb{R}}/\Gamma_\infty \;\cong\; (G_{\mathbb{R}} \times V)/\Gamma \subset M.$$

Consider the covering space of M defined by the subgroup Γ_∞. This is simply

$$(G_{\mathbb{R}} \times X)/\Gamma_\infty \supset G_{\mathbb{R}}/\Gamma_\infty.$$

¿From this we see that there is a measurable $G_{\mathbb{R}}$-equivariant lift of M to $(G_{\mathbb{R}} \times X)/\Gamma_\infty$, showing that the action is not totally engaging.

We do not know an example of an engaging but not totally engaging action on a manifold.

We shall discuss these conditions further in section 6, showing the intimate connection between arithmeticity and the totally engaging condition.

We conclude this section with a general result that is very useful when dealing with engaging conditions.

There are numerous general results (some of which we discuss below) on G-invariant reductions of bundles with an algebraic structure group to a subgroup. By considering homomorphisms of Γ into various algebraic subgroups, one would like to translate this into information about reductions of Γ-bundles and hence to the engaging conditions. The basic technique for doing this is the following.

Proposition 1.12. *Suppose L is a locally compact group and $H_1, H_2 \subset L$ are closed subgroups. Let a locally compact G act ergodically on a space (M, μ). Let $\alpha : G \times M \to H_1$ be a cocycle and suppose that $i \circ \alpha$ is equivalent to a cocycle into H_2, where $i : H_1 \to L$ is the inclusion. Then:*

i. If $H_1 \backslash L / H_2$ is tame [Z4] *(i.e. the H_1 orbits on L/H_2 are locally closed), then α is equivalent to a cocycle into $H_1 \cap lH_2l^{-1}$ for some $l \in L$.*

ii. More generally, suppose $\phi : M \to L$ is such that

$$\phi(gm)^{-1}(i \circ \alpha)(g, m)\phi(m) \in H_2.$$

If $im(\phi)$ lies (a.e.) in a single $H_1 : H_2$ double coset in L, then α is equivalent to a cocycle into $H_1 \cap lH_2l^{-1}$ for some $l \in L$.

Remarks.

i. Proposition 1.12(i) follows from the cocycle reduction lemma [Z3, Lemma 5.2.11]. Proposition 1.12(ii) follows from its proof, as the first step of the proof of [Z3, Lemma 5.2.11] is to use tameness to show that $\phi(M)$ lies (a.e.) in a singe $H_1 : H_2$ double coset.

ii. Proposition 1.12 is the basis of the definition of algebraic hull of a cocycle (or action on a principal bundle) [Z3], [Z4].

iii. Suppose $H_1 = \Gamma$ is a discrete subgroup of L. If H_2 is compact, and $i \circ \alpha$ is equivalent to a cocycle into H_2, then α is equivalent to a cocycle into a finite subgroup.

iv. If H_2 is open, then Proposition 1.12(i) always applies.

v. Proposition 1.12(ii) is the basis of the cohomological application of Ratner's theorem in [Z1, Proposition 3.6].

2 $\mathfrak{s}$-arithmetic groups

In section 1, we have seen how S-arithmetic groups give rise to examples of engaging actions of semi-simple real Lie groups. Actually, there are more general examples. In order to present them, let us start with some notations and a definition.

Let k be a number field, S a finite set of primes in k including all the archimedean ones and $\mathcal{O}$ the ring of algebraic integers in k. Denote

$$\mathcal{O}_S \;=\; \{x \in k \mid v(x) \geq 0 \text{ for every } v \notin S\}.$$

(Here, as usual, we think of the primes as the valuations of k.) An arithmetic group Γ is a group commensurable to $G(\mathcal{O})$ when G is a k-algebraic group. An S-arithmetic group is one commensurable to $G(\mathcal{O}_S)$. Every arithmetic group can be defined by using $\mathbb{Q}$ alone; replace G by $H = \mathrm{Res}^k_{\mathbb{Q}}(G)$ which is a group defined over $\mathbb{Q}$ and for which $H(\mathbb{Z}) \cong \mathbb{G}(\mathcal{O})$. This is not the case for S-arithmetic groups: If the set S consists, for some rational prime p, of only a proper subset of the set of primes $\{\pi_i\}_{i \in I}$ of $\mathcal{O}$ lying above p, then $G(\mathcal{O}_S)$

is usually not isomorphic to an S'-arithmetic group for any set S' of rational primes.

To be able to work over $\mathbb{Q}$ and to have the most general notion of S-arithmetic groups we use:

Definition 2.1 *A finitely generated group Γ is called an $\mathfrak{s}$-arithmetic group if there exists a $\mathbb{Q}$-algebraic group H, with $H(\mathbb{Z})$ infinite, a finite set S of primes of $\mathbb{Q}$ and a subgroup Γ_0 of $H(\mathbb{Z}_{\mathbb{S}})$, such that*

(i) Γ_0 virtually contains $H(\mathbb{Z})$; i.e., $\Gamma_0 \cap H(\mathbb{Z})$ is of finite index in $H(\mathbb{Z})$.

(ii) Γ_0 is isomorphic to a finite index subgroup of Γ.

Remark 2.2 In Definition 2.1, $\mathfrak{s}$ is just a name which has nothing to do with the finite set of primes S.

We learned the following result from T. N. Venkataramana. It shows that for H semisimple, $\mathfrak{s}$-arithmetic groups are, up to finite index subgroups, finite products of S-arithmetic groups over number fields.

Proposition 2.3 *If H in Definition 2.1 is semi-simple, then there exists finitely many number fields $k_1, \ldots, k_l$ and for each $i = 1, \ldots, l$ an absolutely almost simple k_i-algebraic group G_i and a finite set S_i of primes of k_i such that, up to a finite index subgroup, Γ is isomorphic to $\prod_{i=1}^{l} G_i(\mathcal{O}_{S_i})$, where S_i is the ring of S_i-integers in k_i.*

We postpone the proof of (2.3) to the end of the section. We remark however, that (2.3) implies that if H is a semi-simple group then the $\mathfrak{s}$-arithmetic group Γ is a lattice in a group

$$M = \prod_{i=1}^{l} \prod_{v \in S_i} G_i((k_i)_v)$$

which is a product of a real and p-adic Lie groups.

This corollary holds in a more general context. Before showing this, let us see an example which is not semi-simple.

Example 2.4 Let $U = U_4$ be the unipotent group of 4×4 upper unipotent matrices. So a typical element of U is of the type

$$g = \begin{bmatrix} 1 & a_1 & b_1 & c \\ 0 & 1 & a_2 & b_2 \\ 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let p and l be two primes and Γ the subgroup of $U(\mathbb{Q})$ defined by the conditions:

$$g \in \Gamma \quad \text{iff} \quad \begin{cases} a_i \in \mathbb{Z}, & \text{for } i = 1,2,3, \\ b_i \in \mathbb{Z}[1/p], & \text{for } i = 1,2 \\ c \in \mathbb{Z}[1/p, 1/l] \end{cases}.$$

So Γ contains $U(\mathbb{Z})$ and it is contained in $U(\mathbb{Z}[1/p, 1/l])$. Moreover, Γ is a discrete subgroup when it is embedded diagonally in the group $U(\mathbb{R}) \times U(\mathbb{Q}_p) \times U(\mathbb{Q}_l)$. However, it is not a lattice there. The projection of Γ to $U(\mathbb{Q}_p)$ is not dense in $U(\mathbb{Q}_p)$. In fact its closure is equal to $U^+(\mathbb{Q}_p)$ where $U^+(\mathbb{Q}_p)$ is defined by the conditions

$$g \in U^+(\mathbb{Q}_p) \quad \text{iff} \quad \begin{cases} a_i \in \mathbb{Z}_p \\ b_i, c \in \mathbb{Q}_p \end{cases}.$$

Similarly the closure of Γ in $U(\mathbb{Q}_l)$ is $U^{++}(\mathbb{Q}_l)$ given by:

$$g \in U^{++}(\mathbb{Q}_l) \quad \text{iff} \quad \begin{cases} a_i, b_j \in \mathbb{Z}_l \\ c \in \mathbb{Q}_l \end{cases}.$$

Moreover, it is not difficult to verify that Γ is dense in the product $\Omega = U^+(\mathbb{Q}_p) \times U^{++}(\mathbb{Q}_l)$. One can also easily check that the discrete subgroup Γ is a lattice in

$$U(\mathbb{R}) \times \Omega = U(\mathbb{R}) \times U^+(\mathbb{Q}_p) \times U^{++}(\mathbb{Q}_l).$$

Once can easily now imagine more examples of the kind when the unipotent group U is replaced by a general algebraic group.

Theorem 5.4 below states that if a higher rank real Lie group G acts on a Γ-bundle, then, under suitable assumptions Γ is $\mathfrak{s}$-arithmetic. Moreover, there exists a $\mathbb{Q}$-algebraic group H, with a $\mathbb{R}$-embedding of G into H such that $[H, H] = H$ (and so $H = U \rtimes L$, where U is unipotent,L is semi-simple) and $L(\mathbb{Z})$ is infinite such that Γ virtually contains the group $H(\mathbb{Z})$ and is contained in $H(\mathbb{Z}_\sim)$ for some set of primes. We will now show that indeed every such $\mathfrak{s}$-arithmetic group Γ gives rise to a Γ-bundle with a G-action, and a finite measure on the base preserved by G.

Lemma 2.5 *Let $H = U \rtimes L$ be a connected $\mathbb{Q}$-algebraic group, such that U is unipotent and L is semi-simple with $L(\mathbb{Z})$ an infinite group. Let S be a finite set of primes and Γ a subgroup of $H(\mathbb{Z}_\mathbb{S})$ which virtually contains $H(\mathbb{Z})$. Let $M = \prod H(\mathbb{Q}_{|})$ where p runs over the finite primes in S, and let Ω be the closure of the projection of Γ into M. Then Γ is a lattice in $H(\mathbb{R}) \times \nleq$.*

Proof: As $H(\mathbb{Z}_\mathbb{S})$ is a discrete subgroup of $H(\mathbb{R}) \times \mathbb{M}$, Γ is clearly a discrete subgroup of $H(\mathbb{R}) \times \nleq$. We need to show that Γ is of finite covolume there. Note that

(i) By strong approximation (see [PR, p.427]), it follows that Ω contains a finite index open subgroup K of $\prod_{p\in S} H(\mathbb{Z}_{|})$.

(ii) $H(\mathbb{Z})$ is a lattice in $H(\mathbb{R})$ and so is every finite index subgroup of it. In particular, for $\Gamma_1 = \Gamma \cap H(\mathbb{Z}) \cap \mathbb{K}$, there exists a subset V of finite covolume in $H(\mathbb{R})$ such that $\Gamma_1 \cdot V = H(\mathbb{R})$.

We claim now that $\Gamma \cdot (V \times K) = H(\mathbb{R}) \times \nleq$. This shows that Γ is a lattice in $H(\mathbb{R}) \times \nleq$. Indeed, let $(g_1, g_2) \in H(\mathbb{R}) \times \nleq$. By the density of Γ in Ω one can find $\gamma \in \Gamma$ such that

$$\gamma(g_1, g_2) \;=\; (\gamma g_1, \gamma g_2) \in H(\mathbb{R}) \times \mathbb{K}.$$

Now, we can choose $\gamma_0 \in \Gamma_1 \subseteq \Gamma$ such that $\gamma_0(\gamma g_1) \in V$. Since γ_0 is by the definition of Γ_1 also in K, we have that $(\gamma_0 \cdot \gamma)(g_1, g_2) \in V \times K$ and the proof is complete.

Corollary 2.6 *With the notation of (2.5), assume further that G is a semi-simple real Lie group with a $\mathbb{Q}$-embedding into H. Then the embedding of*

G in $H(\mathbb{R})$ defines an action of G on $(H(\mathbb{R}) \times \nleq)/\lneqq$, which is the base of a Γ-bundle and of finite measure. This action will be engaging if and only if for every finite index subgroup Γ_0 of Γ, the closure of Γ_0 in Ω is Ω. This happens, for example, if H is semi-simple.

We are not sure what is the most general context in which this density property (and hence engaging) holds.

Remark 2.7 As mentioned above, our main theorem is a converse of Corollary 2.6. It says that if G is of higher rank and acts in an engaging way on a Δ-bundle with a compact base, where Δ is a linear group, then Δ is $\mathfrak{s}$-arithmetic with H as in (2.5). Note however, that it does not give the complete converse. We assume that the base is compact, but prove only that Δ is a lattice in $H(\mathbb{R}) \times \nleq$ which might be of finite covolume but not necessarily cocompact.

We will return now to the proof of Proposition 2.3 and we start with a Lemma:

Lemma 2.8 *Let G be a simply connected absolutely almost simple group defined over a number field k. Let $\Delta = G(\mathcal{O})$ where $\mathcal{O}$ is the set of integers in k, S a finite set of primes of $\mathcal{O}$ and Γ a subgroup of $G(\mathcal{O}_s)$ such that $\Gamma \cap \Delta$ is of finite index in Δ. Assume further that $G(\mathcal{O})$ is infinite. Then there exists a subset S' of S such that Γ is commensurable with $G(\mathcal{O}_{S'})$.*

Proof For $v \in S$, denote by k_v the completion of k with respect to v. Let S' be the subset of S consisting of all the archimedean ones together with those $v \in S$ for which Γ is dense in $G(k_v)$. If $v \notin S'$ then the closure of Γ in $G(k_v)$ is an open compact subgroup of $G(k_v)$. Indeed, by strong approximation [PR, p.247] and the fact that Γ virtually contains the infinite group $G(\mathcal{O})$, the closure of Γ in $G(k_v)$ is always open. In the simply connected case, every open subgroup is either compact or else is all of $G(k_v)$ [PR]. This proves that after replacing Γ by a finite index subgroup, we may assume that Γ is contained in $G(\mathcal{O}_s)$.

We will now prove that Γ is of finite index in $G(\mathcal{O}_{S'})$. Let S'_f be the set of finite primes in S' and Ω the closure of Γ in $\prod_{v\in S'_f} G(k_v)$. Clearly Ω contains the closure of $\Delta' = \Gamma \cap G(\mathcal{O})$, which is of finite index in $G(\mathcal{O})$. By strong approximation, the closure of Δ' contains a product $\prod_{v\in S'_f} M_v$ where each M_v is a compact open subgroup of $G(k_v)$. In particular, for each v the subgroup generated by M_v and its Γ-conjugates lies in Ω. The conjugation action of Γ on $M_v(\subseteq G(k_v))$ factors through the projection of Γ in $G(k_v)$. Therefore, since it is dense in $G(k_v)$ and $G(k_v)$ has no open normal proper subgroups, it follows that Ω contains $G(k_v)$ for each $v \in S'_f$. So Ω is just the product $\prod_{v\in S'_f} G(k_v)$. For the same reason, this product is also the closure of $G(\mathcal{O}_{S'})$.

Let U be the closure of $G(\mathcal{O})$ in $\prod_{v\in S'_f} G(k_v)$ and U' be the closure of $\Delta = \Gamma \cap G(\mathcal{O})$. By our assumptions $[U : U'] < \infty$ and in fact there is $r \in \mathbb{N}$ and $\delta_1, \ldots, \delta_r \in G(\mathcal{O})$ such that

$$G(\mathcal{O}) = \cup_{i=1}^{r} \Delta\delta_i \quad \text{and} \quad U = \cup_{i=1}^{r} U'\delta_i.$$

Now, U is open in $\prod_{v\in S'_f} G(k_v)$. By virtue of the density of Γ in the above product, for every $g \in G(\mathcal{O}_{S'})$ there exists $\gamma \in \Gamma$ such that $\gamma^{-1}g \in U$. Now, $\gamma^{-1}g \in G(\mathcal{O}_{S'})$ as well and $U \cap G(\mathcal{O}_S) = G(\mathcal{O})$. Thus $\gamma^{-1}g \in G(\mathcal{O})$ and so there exists $\delta \in \Delta$ and $1 \le i \le r$ such that $\gamma^{-1}g = \delta \cdot \delta_i$. Hence $g = \gamma \cdot \delta \cdot \delta_i$. Since $\gamma, \delta \in \Gamma$ we deduce that $G(\mathcal{O}_{S'}) = \cup_{i=1}^{r} \Gamma\delta_i$, which shows that $[G(\mathcal{O}_{S'}) : \Gamma] < \infty$. This proves the Lemma.

Now, once Lemma 2.8 is proven for simply connected groups one can deduce a similar result for the non-simply connected case, provided Γ is finitely generated. Indeed, if $\pi : \widetilde{G} \to G$ is the simply connected cover of G, then $\pi(\widetilde{G}(k))$ is normal in $G(k)$ and $G(k)/\pi(\widetilde{G}(k))$ is a torsion abelian group (cf. [LM]). If Γ is a finitely generated subgroup of $G(k)$, then a finite index subgroup of it is contained in $\pi(\widetilde{G}(k))$. So replacing Γ by a finite index subgroup we can assume that $\Gamma \le \pi(\widetilde{G}(k))$. Look now at $\widetilde{\Gamma} = \pi^{-1}(\Gamma)$. $\widetilde{\Gamma}$, being a subgroup of $\widetilde{G}(k)$, is a linear group and hence residually finite. K =ker(π) is a finite subgroup of $\widetilde{\Gamma}$, and so $\widetilde{\Gamma}$ has a finite index subgroup Γ' intersecting K trivially. Γ' is isomorphic, therefore, to a finite index subgroup of Γ. We can therefore now appeal to (2.8).

Let now H be a $\mathbb{Q}$-algebraic semi-simple group. Then up to a finite kernel (which can be dealt with as in the previous paragraph), we can assume H is a product of $\mathbb{Q}$-simple groups, $H = \prod_{i=1}^{l} H_i$. Each H_i is equal to $\mathrm{Res}_{\mathbb{Q}}^{k_i}(G_i)$ where k_i is a number field and G_i is an absolutely almost simple group defined over k_i. So we can assume $\Gamma \leq \prod_{i=1}^{l} G_i(k_i)$ and Γ virtually contains $H(\mathbb{Z}) = \prod_{i=1}^{l} G_i(\mathcal{O}_i)$ where $\mathcal{O}_i$ is the ring of algebraic integers of k_i. For every i, the set S_i' of finite primes of k_i for which Γ is dense in $G_i(k_i)$ (or in a finite index subgroup, when G_i is not simply connected) is finite. Let Ω be the closure of Γ in $\prod_{i=1}^{l} \prod_{v \in S_i'} G_i(k_{i,v})$. Then in a way similar to the proof of (2.8), Ω can be shown to be of finite index in the product. (It is not necessarily the whole product since if the G_i's are not simply connected they have finite index open subgroups). Then one can continue to argue as in (2.8) to deduce that Γ is commensurable with a finite index subgroup of $\prod_{i=1}^{l} G_i(\mathcal{O}_{S_i'})$. This finishes the proof of (2.3).

3 Superrigidity

In this section we summarize superrigidity for actions on principal bundles, i.e. superrigidity for cocycles, and extend this to a formulation we will need. We refer to reader to [Z3] for background on cocycles.

If $P \to M$ is a principal H-bundle on which G acts, then with respect to the trivialization of the bundle defined by a measurable section, the G action will be described by a cocycle $G \times M \to H$. For any cocycle $c : G \times M \to H$, we call c tempered if it is equivalent to a cocycle $\beta : G \times M \to H$ such that for each $g \in G$, $\beta(g, M)$ (up to null sets) is a compact subset of H. This will be the case, for example, for a cocycle coming from a continuous action of G on $P \to M$ in which M is compact. We shall call a cocycle $\beta : G \times M \to H$ superrigid if there is a homomorphism $\sigma : G \to H$, a compact subgroup $C \subset Z_H(\sigma(G))$, and a cocycle $c : G \times M \to C$ such that β is equivalent to the cocycle $(g, m) \mapsto \sigma(g)c(g, m)$. We call β totally superrigid if we can take c to be trivial. If G is a connected simple real algebraic group with $\mathbb{R}$-rank at least 2, and H is algebraic over a local field, then (perhaps by passing to a finite extension of M) any tempered β is superrigid. This is proven in [Z3], [Z6] when H is defined over a local field of characteristic 0.

However, combining these arguments with [M] or [V], one can also prove this in positive characteristic, which we shall need. More precisely:

Theorem 3.1 (Cocycle superrigidity in positive characteristic). *Let G be a connected simple Lie group with finite center and $\mathbb{R}$-rank$(G) \geq 2$. Let k be a local field with char$(k) > 0$. Let H be a group defined over k and $\alpha : G \times M \to H_k$ be a cocycle. Then α is equivalent to cocycle into a compact subgroup of H_k.*

For technical reasons, it will be useful for us to reduce to the case where superrigidity is replaced by total superrigidity.

Given a cocycle $c : G \times M \to C$ where C is compact, one can always pass to an ergodic extension, say $M' \to M$, such that the lift of c to a cocycle $c' : G \times M' \to C$ is trivial in cohomology. Namely, c is equivalent to a minimal cocycle $\lambda : G \times M \to D \subset C$ where D is a closed subgroup, which means the skew product action of G on $M \times_\lambda D$, given by $g.(m,d) = (gm, \lambda(g,m)d)$ is ergodic. (See [Z2] for a full discussion.) The lift of c to $M \times_\lambda D$ is easily seen to be trivial. As discussed in [Z2], the group D is unique up to conjugacy (and in the case of real Lie groups, coincides with the algebraic hull of c [Z3], [Z4].) However, if $P \to M$ is a Γ-bundle on which the G action is engaging, it is not immediate that the action on the pullback to M', say $P' \to M'$, is still engaging. We shall need to trivialize the cocycle c arising in superrigidity while at the same time maintaining the engaging property. The following accomplishes this when C is a compact Lie group (which is the only case we shall require.)

Lemma 3.2. *Suppose G is a locally compact group with an engaging action on a principal Γ-bundle $P \to M$ (where Γ is discrete). Suppose $c : G \times M \to C$ is a cocycle into a compact Lie group. Then there is:*

i. a finite index subgroup $\Gamma' \subset \Gamma$, with associated finite cover $M' \to M$ and principal Γ'-bundle $P \to M'$, and

ii. an ergodic skew product extension X of M' by a compact subgroup of C [Z2], such that

a. the action of G on the principal Γ'-bundle $P_X \to X$ that is the pullback to X of $P \to M'$ is engaging, and *b. the pullback $c_X : G \times X \to C$ of α is trivial in cohomology.*

Proof. For each finite index subgroup $\Lambda \subset \Gamma$, let $C_\Lambda \subset C$ be the algebraic hull of the cocycle $c_\Lambda : G \times P/\Lambda \to C$ defined by lifting c. The engaging hypothesis ensures that G is ergodic on P/Λ. If $\Delta \subset \Lambda$, then $C_\Delta \subset C_\Lambda$ up to conjugacy. By the descending chain condition on closed subgroups of C, we can choose a finite index $\Gamma' \subset \Gamma$ such that for all $\Lambda \subset \Gamma'$ we have $C_\Lambda = C_{\Gamma'}$ up to conjugacy. Set $D = C_{\Gamma'}$, $M' = P/\Gamma'$, and $\lambda : G \times M' \to D$ a cocycle equivalent to $c_{\Gamma'}$. Let $X = M' \times_\lambda D$. Then c_X is trivial in cohomology. To prove the lemma, it suffices to see that the action of G on the Γ'-bundle P_X is engaging. However, if $\Lambda \subset \Gamma'$ is of finite index then $P_X/\Lambda \cong P/\Lambda \times_\lambda D$ which is ergodic under G since $D = C_\Lambda$ for any such Λ.

Corollary 3.3. *Let G be a connected simple Lie group with $\mathbb{R}$-rank$(G) \geq 2$. Suppose G acts on a space M with a finite invariant measure, and that $P \to M$ is a principal Γ-bundle with an engaging G-action. Let $\alpha : G \times M \to H$ be a tempered cocycle into a real algebraic group. Then there is a finite index subgroup $\Gamma' \subset \Gamma$ and an ergodic G-space X with finite invariant measure that is an extension $X \to M'$ of $M' = P/\Gamma'$ (and hence M) such that*

i. $\alpha_X : G \times X \to H$ is totally superrigid, and

ii. the G-action on the principal Γ'-bundle $P_X \to X$, the pullback to X of $P \to P/\Gamma'$, is engaging.

Proof. We apply Lemma 3.2 twice. First, we can replace H by the algebraic hull of α [Z3]. Let $c_1 = p \circ \alpha$ where $p : H \to H/H^0$ where H^0 is the connected component of the identity. Applying Lemma 3.2 allows us to assume c_1 is trivial; i.e., by passing to a finite ergodic extension of M and a finite index subgroup of Γ, we can assume the algebraic hull is connected. We can then apply superrigidity in characteristic 0 [Z3], [Z6] to deduce that our cocycle on this extension is superrigid. Second, we apply Lemma 3.2 again to pass to a further extension and a possibly smaller subgroup of finite index to obtain total superrigidity and engaging.

4 Specializations

In this section we develop some specialization theorems for finitely generated groups that we will use to reduce the proofs of our main results to the case of linear groups over $\overline{\mathbb{Q}}$. More precisely, suppose $\Gamma \subset GL(n,\mathbb{C})$ is a finitely generated linear group. Then there is a ring A which is a finitely generated $\mathbb{Q}$-algebra such that $\gamma \in GL(n,A)$ for all $\gamma \in \Gamma$. If $\psi : A \to \overline{\mathbb{Q}}$ is a $\mathbb{Q}$-algebra homomorphism then it induces a homomorphism $\psi : GL(n,A) \to GL(n,\overline{\mathbb{Q}})$ and in particular a homomorphism $\psi : \Gamma \to GL(n,\overline{\mathbb{Q}})$. Then ψ (or $(\psi(\Gamma))$ is called a specialization of Γ.

Definition 4.1. *Suppose H_i are algebraic k-groups, $i = 1,2$, and $H_i = L_i \ltimes U_i$ are Levi decompositions defined over k. We call H_1 and H_2 k-isotopic if there is a k-isomorphism $L_1 \to L_2$, such that under this isomorphism $\mathfrak{u}_i$(= Lie algebra of U_i) are k-isomorphic L_i modules.*

Our main result about specializations is the following.

Theorem 4.2. *Let $\Gamma \subset GL(\mathbb{R}^{\ltimes})$ be a finitely generated group. Suppose that for each irreducible component π of the semisimplification of this linear realization we have $tr(\pi(\gamma)) \in \overline{\mathbb{Q}}$ for all $\gamma \in \Gamma$. Let $\overline{\Gamma}$ denote the Zariski closure. Then (after a suitable choice of basis in $\mathbb{R}^{\ltimes}$) there is a specialization ψ of Γ such that:*

i) $\psi(\Gamma) \subset GL(n, \overline{\mathbb{Q}} \cap \mathbb{R})$;

ii) ψ *is faithful on* Γ;

iii) $\overline{\psi(\Gamma)}$ *and* $\overline{\Gamma}$ *are* $\mathbb{R}$-*isotopic.*

Furthermore if $\overline{\Gamma}$ is defined over $\overline{\mathbb{Q}}$, then we have

iv) $\overline{\psi(\Gamma)} = \overline{\Gamma}$.

We begin the proof of Theorem 4.2 by recalling the following result, which is well-known. (cf. [B, Section2] for example.)

Lemma 4.3. *If $\Lambda \subset GL(n,\mathbb{R})$ is finitely generated and irreducible, and $tr(\lambda) \in \overline{\mathbb{Q}}$, then Λ is conjugate over $\mathbb{R}$ to a subgroup of $GL(n, \overline{\mathbb{Q}} \cap \mathbb{R})$.*

¿From this lemma and the hypotheses of Theorem 4.2 we deduce the following. Let $\overline{\Gamma} = L \ltimes U$ be a Levi decomposition over $\mathbb{R}$. (More precisely, we are taking L and U to be real algebraic groups.) We can find a flag $0 \subset V_1 \subset \ldots \subset V_r = \mathbb{R}^n$ subspaces W_i such that $V_i \oplus W_{i+1} = V_{i+1}$ and a basis for $\mathbb{R}^\ltimes$ which is a union of bases for $W_i, i = 1, \ldots, r$, such that: each V_i is $\overline{\Gamma}$ invariant; U acts by the identity on V_{i+1}/V_i; each W_i is L-invariant; the action of Γ on each W_i (via projection to L) is irreducible; and finally, writing $\gamma \in \Gamma$ as $\gamma_s\gamma_u$, where $\gamma_s \in L$ and $\gamma_u \in U$, we have $\gamma_s \in GL(n, \overline{\mathbb{Q}} \cap \mathbb{R})$ with respect to the above basis.

Let $N = \Gamma \cap U$. (This may be trivial.) Then N is a group of unipotent matrices, but is itself not a priori finitely generated. We establish the next lemma to be able to apply results and techniques of Grunewald-Segal [GS]. We thank Shahar Mozes for his contribution to the proof of this lemma.

Lemma 4.4. *Let $Z(N)$ be the center of N. Then $Z(N)$ is an abelian group of finite rank; i.e. $Z(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$.*

Proof. Since Γ is finitely generated, we can find a number field F such that $\gamma_s \in GL(n, F)$ for all γ. Let $\{\gamma^1, \ldots, \gamma^r\}$ be a generating set for Γ. Let $\{x_1, \ldots, x_l\} \subset \mathbb{R}$ be the set of real numbers appearing as entries in the matrices for $\{\gamma_u^i\}$. A straightforward induction (and some matrix multiplication) establishes the following. Consider for any matrix with respect to the above basis the set of matrix entries corresponding to $\mathrm{Hom}(W_j, W_i), j \geq i$. Then for any word (of any length) in $\{\gamma^i\}$, such a matrix entry is a polynomial of degree at most $j - i$ in $\{x_1, \ldots, x_l\}$ with coefficients in F. (The induction is done on $j - i$.) This implies that each matrix entry for all γ_u is a polynomial in $\{x_1, \ldots, x_l, y_1, \ldots, y_m\}$ over $\mathbb{Q}$ of degree at most $\max\{n, m\}$ where $\{y_1, \ldots, y_m\}$ is a basis for F over $\mathbb{Q}$. This implies that any abelian subgroup of N is of finite rank, and in particular proves the lemma.

Now let V be the unipotent group of all matrices with respect to the basis of $\mathbb{R}^\ltimes$ chosen above so that $T \in V$ if and only if $T|W_i =$Id, and $T_{ij} = 0$ if $i > j$. Thus, $U \subset V$ and L normalizes V. Let $\mathfrak{u} \subset \mathfrak{v}$ be the corresponding Lie algebras, which are both L-modules. The map $\exp : \mathfrak{v} \to \mathfrak{V}$ is a bijection which is a $\mathbb{Q}$-regular map, as is the inverse which we denote by $\log$. We have $\exp|\mathfrak{u} : \mathfrak{u} \to \mathfrak{U}$ is also a bijection, although we recall that $\mathfrak{u}$ itself may or may not be defined over $\mathbb{Q}$ with respect to the standard $\mathbb{Q}$-structure on $\mathfrak{v}$. The matrix entries for the action of Γ_s on $\mathfrak{v}$ lie in $\overline{\mathbb{Q}} \cap \mathbb{R}$. The maps exp and log commute with the actions of L on $\mathfrak{v}$ and V.

Recall that for each $\gamma \in \Gamma$, we write $\gamma = \gamma_s \gamma_u$. Choose $\{\gamma^i\}_{i=1,\ldots,n} \subset \Gamma$ such that $X_i = \log \gamma_u^i$ is a basis for $\mathfrak{u}$. Extend this to a larger set $\{\gamma^i\}_{i=1\ldots n,\ldots,m}$ that generates Γ. Let c_{ij}^k be the structural constants for the Lie algebra $\mathfrak{u}$ with respect to the basis $\{X_i; i = 1,\ldots n\}$; i.e. write $[X_i, X_j] = \sum c_{ij}^k X_k$, where $c_{ij}^k \in \mathbb{R}$. For $n < j \leq m$, write $X_j = \sum_{i=1}^n a_{ij} X_i$. Let $X_i = \sum b_i^{jk} E_{jk}$ be the expression for X_i in terms of the standard $\mathbb{Q}$-basis for $\mathfrak{v}$. Finally, let A be the (finitely generated) $\mathbb{Q}$-algebra generated by $\{a_{ij}, c_{ij}^k, b_i^{jk}\}$. Let $\psi : A \to \overline{\mathbb{Q}}$ be a specialization. Let $Y_i = \psi(X_i) = \sum \psi(b_i^{jk}) E_{jk}$. Assume for the moment that $\{Y_i, i = 1,\ldots,n\}$ are linearly independent. Let $\mathfrak{u}'$ be the subspace spanned by Y_i. Since $c_{ij}^k \in A$, and $[Y_i, Y_j] = \sum \psi(c_{ij}^k) Y_k$, $\mathfrak{u}'$ is a Lie algebra defined over $\overline{\mathbb{Q}}$. Denote the natural action of $\mathfrak{L}$ on $\mathfrak{v}$ by Ad. Then if $g \in L_{\overline{\mathbb{Q}}}$, we have $\mathrm{Ad}(g) X_i$ has entries in A, and $\psi(\mathrm{Ad}(g)X_i) = \mathrm{Ad}(g)(\psi(X_i))$. Thus, $\mathfrak{u}$ and $\mathfrak{u}'$ are isomorphic L-modules. Letting $U' = \exp(\mathfrak{u}')$, we thus have $L \ltimes U$ and $L \ltimes U'$ are $\mathbb{R}$-isotopic Lie groups. We now claim that the specialization $\psi(\Gamma) \subset L \ltimes U'$. We have $\psi(X_j) = \sum_1^n \psi(a_{ij}) Y_i$, so $\psi(X_j) \in \mathfrak{u}'$ for $j = 1,\ldots,m$. Thus, $\exp(\psi(X_j)) = \psi(\exp(X_j)) = \psi(\gamma_u^j) \in U'$. Since $\psi(\gamma) = \gamma\psi(\gamma_u)$, it follows that $\psi(\gamma^j) \in L \ltimes U'$ for γ^j in a generating set, and hence $\psi(\Gamma) \subset L \ltimes U'$. It is then also clear that $\overline{\psi(\Gamma)} = L \ltimes U'$. If $\overline{\Gamma} = L \times U$ is itself defined over $\overline{\mathbb{Q}}$, then so is any specialization of $\mathfrak{u}$ over $\overline{\mathbb{Q}}$, and hence we would have $U' = U$.

Turning to the injectivity of ψ on Γ, we observe that $\psi(\gamma) = \psi(\gamma_s\gamma_u) = \gamma_s\psi(\gamma_u)$. Evidently, this can vanish only if γ_s is trivial. I.e., it suffices to see that $\psi|N$ (where $N = \Gamma \cap U$) is injective. In sum, we have shown that to prove Theorem 4.2, it suffices to prove the following lemma.

Lemma 4.5. *We can choose a specialization $\psi : A \to \overline{\mathbb{Q}} \cap \mathbb{R}$ such that:*

i) $\psi(X_i)$ are linearly independent (over $\mathbb{R}$).

ii) $\psi|N$ is injective.

Proof. Let F be quotient field of A, so $F = \mathbb{Q}(\underline{x})$ where $\underline{x} = (x_1,\ldots,x_l) \in \mathbb{R}^{<}$. Then $\underline{x}$ generates an absolutely irreducible affine variety $V = \mathrm{spec}(A)$ over $k = F \cap \overline{\mathbb{Q}}$. Noether's normalization theorem supplies $t_1,\ldots,t_r \in A$

which are algebraically independent over k such that A is an integral extension of $B = k[t_1, \dots, t_r]$. There is a natural map $\eta : \mathrm{spec}(A) \to \mathrm{spec}(B)$ which induces a surjective map from V onto $\mathbb{C}^r$, since A is integral over B. The map $\eta : V(\mathbb{C}) \to \mathbb{C}^r$ is continuous in the Zariski and complex topologies. As $k \subseteq \mathbb{R}, \eta(\mathbb{V}(\mathbb{R})) \subseteq \mathbb{R}^r$.

The proof of Theorem 3.1 (and Theorem 2.8) in [GS] implies that for every number field $K \supseteq k$, there exists an Hilbert set $H \subseteq K^r$ such that if $\psi \in \mathrm{spec}(A)$ and $\eta(\psi) \in H$, then ψ induces an injective map on N. (We apply this with $k = K$.) Moreover, the condition that $\{\psi(X_i)\}$ are linearly independent is an open condition. It is well known that the intersection of a Hilbert set with a Zariski-open set is still an Hilbert set (cf. [GS] and the references therein) so we can assume that both conditions (i) and (ii) are satisfied for every ψ such that $\eta(\psi) \in H$. To finish the proof we still need to ensure that the image of ψ is in $\mathbb{R}$. (Note that even if $\eta(\psi) \in k^r \subseteq \mathbb{R}^r$, this does not ensure that the image of ψ is in $\mathbb{R}$, but merely says that $\psi(B) \subseteq \mathbb{R}$).

To this end we prove part (c) of the following Lemma which was provided to us by Moshe Jarden:

Lemma 4.6. *Suppose $k \subseteq A \subseteq \mathbb{Q}((\underline{\curvearrowleft})) \subseteq \mathbb{R}$ as before, and K a number field with $k \subseteq K \subseteq \mathbb{R}$.*

- *(a) There is a real-open neighborhood u_1 of $\underline{x}$ in $V(\mathbb{R})$ and a real-open ball u_2 around $\underline{t} = (t_1, \dots, t_r) \in \mathbb{R}^r$ such that η maps u_1 homeomorphically onto u_2.*
- *(b) $u_2 \cap H \neq \emptyset$ for every Hilbert subset H of K^r.*
- *(c) For each Hilbert subset H of K^r there exists $\underline{x}' \in V(\mathbb{R})$ such that $\eta(\underline{x}') \in H$.*

Proof.

(a) From [GPR, Cor. 9.5] it follows that $\eta : V(\mathbb{R}) \to \mathbb{R}^r$ is a local homeomorphism in the neighborhood of $\underline{x}$. This is just a reformulation of (a). (Actually, [GPR, Cor. 9.5] deals with a Henselian field rather than with $\mathbb{R}$, but one can carry out an analogous proof for $\mathbb{R}$.)

(b) This follows from Lemma 4.1 of [J]. (That lemma deals with valuation, but again an analogous proof works for $\mathbb{R}$. One can also deduce (b) from a more general and more difficult theorem of Geyer [G, Lemma 3.4] in which this density result is proved simultaneously for several valuations and orderings.)

(c) is a consequence of (a) and (b).

5 Statements and proofs of the main results.

In this section, we state and prove sharper versions of Theorems A, B, C of the introduction.

Theorem 5.1. *Let G be a connected simple Lie group with $\mathbb{R}$-rank $(G) \geq 2$, and suppose G acts on a compact M, preserving a finite measure and engaging. Let $\sigma : \pi_1(M) \to GL(n, \mathbb{R})$ be any linear representation, with image $\Gamma = \sigma(\pi_1(M))$ an infinite group, and Zariski closure $\overline{\Gamma} \subset GL(n, \mathbb{R})$. Then Γ is $\mathfrak{s}$-arithmetic. More precisely, there is a real algebraic $\mathbb{Q}$-group H, an embedding $\Gamma \hookrightarrow H_{\mathbb{Q}}$ (and hence necessarily in $H_{\mathbb{Z}_S}$ for some finite set of primes S) and subgroups $\Gamma_\infty \subset \Gamma_0 \subset \Gamma$ such that:*

(i) H contains a group $\mathbb{R}$-isotopic to $\overline{\Gamma}$ (see Definition 4.1)

(ii) $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$.

(iii) $[\Gamma : \Gamma_0] < \infty$

(iv) Γ_∞ is profinitely dense in Γ_0

(v) Γ_∞ is commensurable with $H_{\mathbb{Z}}$ and is a lattice in H.

Furthermore, (perhaps by passing to a finite cover of G), there is a local embedding $G \to H$ such that $C \setminus H/\Gamma_\infty$ is a virtual arithmetic quotient of M. In particular, $C \setminus H/\Gamma_\infty \prec A(M)$, where $A(M)$ is the canonical maximal arithmetic quotient [LZ].

Theorem 5.2. *With the hypotheses of Theorem 5.1, and the additional hypothesis that the action is totally engaging, we may take $\Gamma_\infty = \Gamma_0$. In particular, Γ is arithmetic.*

In particular, we obtain

Corollary 5.3. *If G is a connected Lie group with $\mathbb{R}$-rank $(G) \geq 2$, and G has a totally engaging action on a compact manifold M, then for any representation $\sigma : \pi_1(M) \to GL(n, \mathbb{R})$ with infinite image, $\sigma(\pi_1(M))$ has a subgroup of finite index that is an arithmetic subgroup of a $\mathbb{Q}$-group H such that $\mathfrak{g} \hookrightarrow \mathfrak{h}$. Thus, if $\pi_1(M)$ is a linear group, it has a subgroup of finite index that is arithmetic in a group H with $\mathfrak{g} \hookrightarrow \mathfrak{h}$.*

In fact, our proof will work for any principal bundle with discrete fiber, not just $\widetilde{M}$. More precisely, we have:

Theorem 5.4. *Let π be a discrete group and $P \to M$ a principal π-bundle on which G acts preserving a finite ergodic measure on the compact space M. Let Γ be the image of π under any finite dimensional linear representation over $\mathbb{C}$. Assume Γ is infinite. If the action of G on $P \to M$ is engaging, the conclusions of Theorem 5.1 hold. If the action is totally engaging, those of 5.2 hold.*

We now turn to the proof of these theorems.

Our general approach to the proofs of these results will be to reduce to the case in which Γ has algebraic entries, and then to further develop the arguments of [Z8] using [Z1, Proposition 3.6] and the results in earlier sections. Rather than reproduce the arguments of [Z8] in full detail, we shall freely refer to that paper when it is convenient to do so. We now assume all hypotheses of Theorem 5.1.

Lemma 5.5. *Let k be a local field of positive characteristic. Let $\lambda : \Gamma \to GL(n, k)$ be a representation with discrete image. Then $\lambda(\Gamma)$ is finite.*

Proof. Let $\alpha : \widetilde{G} \times M \to \Gamma$ be defined by the action of $\widetilde{G}$ on the principal Γ-bundle P, where $P = \widetilde{M}/\ker(\sigma)$. By 3.1 and 1.12(i), $\lambda \circ \alpha$ is equivalent to a cocycle into a finite subgroup of $\lambda(\Gamma)$, which must be profinitely dense in $\lambda(\Gamma)$. Being linear, $\lambda(\Gamma)$ is residually finite, and since it has a profinitely dense finite subgroup, $\lambda(\Gamma)$ is itself finite.

Lemma 5.6. *Any representation of* Γ *over a local field with positive characteristic has finite image.*

Lemma 5.6 follows from Lemma 5.5 and

Lemma 5.7. *Let* Δ *be a finitely generated infinite linear group over a local field* F *with* $char(F) > 0$*. Then* Δ *has a linear representation over* F *with infinite discrete image.*

Remark. Lemma 5.7 is not true in the case of characteristic 0. For example, $SL_n(\mathbb{Z}[1/p])$ has no infinite discrete representation over $\mathbb{C}$.

We need a sublemma.

Sublemma 5.8. *If* Δ *is an infinite finitely generated linear group in char* $p > 0$*, then it has a representation with an infinite image over a global field of char* $p > 0$ *(i.e. of transcendence degree* $= 1$*.)*

Proof. Let H be the Zariski closure of Δ. If the unipotent radical is of finite index then Δ is virtually nilpotent, hence torsion, hence finite since it is finitely generated. So Δ has an infinite image in a reductive group. If the reductive group has a finite index infinite central torus, then Δ has a finite index central subgroup with an infinite abelian quotient. So Δ certainly would have an infinite representation over a global field. Thus we may assume Δ has an infinite representation into a semisimple group M. Let ϕ be a faithful irreducible representation of M, so it is also irreducible with respect to Δ. Let D be the ring generated by the traces of $\phi(\Delta)$, and $Q(D)$ be the quotient field. If the transcendence degree of $Q(D)$ is 0, then by [B, Corollary 2.5], $\phi(\Delta)$ is conjugate to a group with entries in a finite extension of $Q(D)$. This is impossible since $\phi(\Delta)$ is infinite and we are in positive characteristic.

So for some $\gamma_0 \in \Delta$, $\mathrm{tr}(\gamma_0)$ is not algebraic. Take now a specialization of D into a global field such that this element of D, (i.e. $\mathrm{tr}(\gamma_0)$) is not algebraic. This ensures a representation with infinite image of Δ, and proves the sublemma.

Proof of Lemma 5.7. We can, by the sublemma, assume that Δ has an infinite representation into a global field. So assume $\Delta \subset GL_n(K), K$ a global field.

By choosing a finite set of primes S_0, Δ is discrete in $\prod_{r \in S_0} GL_n(K_r)$. But each one of the K_r's is a finite extension of $\mathbb{F}_{l}((\approx))$. So altogether we get a faithful discrete representation over $\mathbb{F}_{l}((\approx))$, and hence over F which is $\mathbb{F}_{ll}((\approx))$ for q equal some power of p.

Lemma 5.9. *For any linear representation π of Γ, $tr(\pi(\gamma)) \in \overline{\mathbb{Q}}$ for all $\gamma \in \Gamma$.*

Proof. Since Γ is finitely generated, there is a finitely generated ring A with $\pi(\Gamma) \subset GL(n, A)$. For any transcendental $a \in A$, there is a ring homomorphism $\psi : A \to F$ where F is a local field of positive characteristic with $\psi(a)$ still transcendental. If some $\gamma \in \Gamma$ had $\mathrm{tr}(\pi(\gamma)) \notin \overline{\mathbb{Q}}$, then ψ would define a representation π_ψ over F with $\mathrm{tr}(\pi_\psi(\gamma))$ transcendental. This implies that $\pi_\psi(\Gamma)$ is infinite, contradicting Lemma 5.6.

Now apply Theorem 4.2 to Γ. We identify Γ with $\psi(\Gamma)$, but still denote by $\overline{\Gamma}$ the Zariski closure in the original representation. We can use restriction of scalars to find an algebraic $\mathbb{Q}$-group H in which Γ is embedded as a Zariski dense subgroup with $\Gamma \subset H_{\mathbb{Q}}$. Furthermore, it is easy to check that H must contain a subgroup $\mathbb{R}$-isotopic to $\overline{\Gamma}$.

By applying Corollary 3.3 we can pass to a finite index subgroup Γ_0 and an ergodic G-space X with finite invariant measure such that, letting $\alpha : G \times X \to \Gamma_0$ be the cocycle defined by the engaging action of P_X and $\alpha_{\mathbb{R}}$ the composition of α with the embedding of $\Gamma_0 \subset H_{\mathbb{R}}$, we have that $\alpha_{\mathbb{R}}$ is totally superrigid, defining a homomorphism $\sigma : G \to H_{\mathbb{R}}$. We also note that we may assume, perhaps by passing to a further subgroup of finite index, that $\Gamma_0 \subset H^0_{\mathbb{R}} \subset H_{\mathbb{R}}$ is Zariski dense in $H_{\mathbb{R}}$, (replacing the latter by the Zariski closure of Γ_0 if necessary.) Now choose a finite set of primes S such that $\Gamma_0 \subset H_{\mathbb{Z}_{\mathbb{S}}}$. Then Γ_0 is discrete in its diagonal embedding $d : \Gamma_0 \to H_{\mathbb{R}} \times H_f$ where $H_f = \Pi_{p \in S} H_{\mathbb{Q}_l}$. By p-adic superrigidity for cocycles, α_f, the projection of $d \circ \alpha$ onto H_f, is equivalent to a cocycle into a compact subgroup $K \subset H_f$, which we can assume is open. This implies $d \circ \alpha$ is equivalent to a cocycle taking values in $H_{\mathbb{R}} \times K$ which is open in $H_{\mathbb{R}} \times H_f$. By Proposition 1.12(i), this implies that $d \circ \alpha$ is equivalent to a cocycle into $d(\Gamma_0) \cap (H_{\mathbb{R}} \times K')$ for some conjugate K' of K. Let Γ_∞ be the projection of $d(\Gamma_0) \cap (H_{\mathbb{R}} \times K')$ into $H_{\mathbb{R}}$. We then have $\Gamma_\infty \subset \Gamma_0 \subset H^0_{\mathbb{R}}$ with Γ_∞ discrete, and α is equivalent to a cocycle into Γ_∞. By Proposition 1.4, Γ_∞ is profinitely dense in Γ_0, hence Zariski dense in $H_{\mathbb{R}}$. As in the argument of [Z8], the fact that $\alpha_{\mathbb{R}}$ is totally superrigid shows that there is a G-equivariant measurable map $\phi : X \to H^0_{\mathbb{R}}/\Gamma_\infty$ and, applying Ratner's theorem exactly as in [Z8], almost all $\phi(X)$ lie in a L-orbit where L is a connected Lie group, say with stabilizer $L \cap h\Gamma_\infty h^{-1}$ that is a lattice in L. The argument of [Z1, Proposition 3.6] now implies that α is equivalent to a cocycle β into $\Lambda = \Gamma_\infty \cap h^{-1}Lh$. By Proposition 3.4 again, Λ is Zariski dense in $H_{\mathbb{R}}$.

Let $J = h^{-1}Lh$. Let A be the image of Λ in $J/[J, J]$. The projection of β must be equivalent to a cocycle into a finite subgroup since A is abelian and G has Kazhdan's property [Z3, Proposition 9.11]. Therefore β is equivalent to a cocycle into a subgroup $\Delta \subset \Lambda$ such that $\Delta_0 = \Delta \cap [J, J]$ is of finite index in Δ. Since the Lie subalgebra $[\mathfrak{j}, \mathfrak{j}]$ is algebraic, $[J, J]$ is of finite index in its Zariski closure. However, the Zariski closure of Δ is $H_{\mathbb{R}}$ and since $\Delta_0 \subset \Delta$ is of finite index and $H_{\mathbb{R}}$ is algebraically connected, it follows that $[J, J] = H^0_{\mathbb{R}}$. (Recall J is connected.) Therefore, we deduce $L = H^0_{\mathbb{R}}$.

We now have $\Lambda = \Gamma_\infty \subset H^0_{\mathbb{R}}$ and Γ_∞ is a lattice in $H^0_{\mathbb{R}}$. Since $\Gamma_\infty \subset H_{\mathbb{Z}_{\mathbb{S}}}$ and its projection to H_f has compact closure, it follows that $\Gamma_\infty \cap H_{\mathbb{Z}}$ is of finite index in Γ_∞. Since it is a lattice, we deduce that Γ_∞ and $H_{\mathbb{Z}}$ are commensurable. This completes the proof of those parts of Theorem 5.1 that are not explicity stated in [Z8]. For the remainder of the conclusion, one can see [Z8] or easily deduce them from the structure described above.

Proof of Theorem 5.2. We need the following two general lemmas concerning totally engaging actions.

Lemma 5.10. *Suppose the G-action on a Γ-bundle $P \to M$ is totally engaging, where G acts on M ergodically with finite invariant measure. (We assume Γ is infinite.) Let Γ_0 be a finite index subgroup. Then the G action on the Γ_0 bundle $P \to P/\Gamma_0$ is totally engaging.*

Lemma 5.11. *Let P, M, Γ, G as in Lemma 5.10. Let $X \to M$ be an ergodic extension with finite invariant measure. Let $P_X \to X$ be the pull-back of P to X. Assume the G action on $P_X \to X$ is engaging. Then it is totally engaging.*

Proof of Lemmas. We first prove 5.11.

Suppose there is a subgroup $\Delta \subset \Gamma$ such that there is a G-invariant section $s : X \to P_X/\Delta$. Decompose the $G-$ invariant measure μ on X over M, say $\mu = \int^{\oplus} \mu_m dm$, where μ_m is supported on the fiber in X over $m \in M$. Then for each $m, s_*(\mu_m)$ is a finite measure on the discrete set P_m/Δ. For some $\epsilon > 0$, the set

$$A_m = \{x \in P_m \mid s_*(\mu_m)(\{x\}) \geq \epsilon\}$$

will be non-empty (and obviously finite) for a set of m of positive measure, and by invariance of μ and ergodicity of G on M, this will be non-empty for a.e. m. Thus, $m \mapsto A_m$ defines a G-invariant section of the bundle $(P/\Delta)^* \to M$ whose fiber consists of finite subsets of P_m/Δ. Since Γ is discrete, the cocycle reduction lemma [9] implies that there is a G-invariant reduction of P to a group $\Delta' \subset \Gamma$ where Δ' stabilizes a finite subset of Γ/Δ. Since the G-action of $P \to M$ is totally engaging, $\Delta' = \Gamma$, and hence $\Delta \subset \Gamma$ is of finite index. Since the action on $P_X \to X$ is engaging, $\Delta = \Gamma$, completing the proof of Lemma 5.11.

The proof of Lemma 5.10 is similar. A section of $P/\Delta \to P/\Gamma_0$ for some $\Delta \subset \Gamma_0$ in a manner similar to proof of 5.11 yields a reduction of $P \to M$ to a subgroup $\Delta' \subset \Gamma$ that leaves a finite subset in Γ/Δ invariant. As above, this implies that $\Delta \subset \Gamma$ is of finite index, and since totally engaging implies engaging, this is impossible.

To complete the proof of Theorem 5.2, we need now only observe that in the proof of Theorem 5.1, we showed that α is equivalent to a cocycle into $\Gamma_\infty \subset \Gamma_0$. This, with the hypotheses of Theorem 5.2, Lemmas 5.10, 5.11 imply that $\Gamma_\infty = \Gamma_0$, verifying the theorem.

6 On the relationship of arithmeticity and totally engaging actions.

In the section we further clarify the relationship of arithmeticity to the engaging conditions.

Theorem 6.1 *Let G be as in Theorem 5.1, and suppose G acts on $P \to M$, a principal Γ-bundle, with M compact and a finite invariant ergodic measure on M. Suppose Γ is arithmetic. If the G action is engaging, it is totally engaging.*

Proof: Let $\alpha : G \times M \to \Gamma$ be the cocycle defined by the action on P. If the action is not totally engaging, there is some $\Lambda \subset \Gamma$ of infinite index such that $\alpha \sim \beta$ with $\beta(G \times M) \subset \Lambda$. Since Λ is a discrete linear group, we can apply the argument of [Z8] and deduce that $\Lambda \supset \Delta$, where Δ is a lattice in some algebraic group and $\beta \sim \delta$ with $\delta(G \times M) \subset \Delta$. Since the action is engaging, Δ must be profinitely dense in Γ, and hence Zariski dense. Since Δ is a lattice, this implies Δ is of finite index in Γ, which is a contradiction.

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