CENTRAL KÄHLER METRICS

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ABSTRACT. This paper investigates a new type of distinguished Kähler metric, generalizing the Kähler-Einstein type. It is defined using a function we term the central curvature, which is a ratio of the determinants of the Ricci and Kähler forms. This notion is treated, as far as possible, in analogy with the scalar curvature, and the metrics are compared with extremal metrics. An analog of the Futaki invariant [FMo1] is employed for this purpose. It is shown that such metrics realize the minimum of an L^2 functional defined on the space of Kähler metrics in a given Kähler class. For the special case of constant central curvature, various results are obtained regarding existence, uniqueness and a classification in complex dimension two. Some of these rely on the observation that existence of such metrics constrains the asymptotic growth of the dimensions of sheaf cohomology groups with values in sections of (multiples of) the anticanonical line bundle.

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1. Introduction

Of the many curvature notions in higher dimensional Riemannian geometry, scalar curvature holds one claim to simplicity by virtue of being, like the Gauss curvature, merely a function on the given space. This work investigates another geometric function that we term the central curvature. Although this notion belongs in the general Riemannian arena, we confine ourselves to Kähler geometry, in the context of the search for distinguished metrics on compact manifolds.

In this realm, Kähler-Einstein metrics serve as a basic prototype. The associated closed 2-forms of such metrics, unless Ricci-flat, can live only in specific cohomology classes, namely multiples of the first Chern class. The existence question for these metrics is thus limited to complex manifolds for which the first Chern class, if it is not the zero class, admits forms of strict definiteness. There are thus restrictions both on the types of manifolds, as well as on the range of (Kähler) cohomology classes on a given manifold, in which such metrics can exist.

A distinguished metric notion for which these limitations are partially relaxed was developed by Calabi in his work on extremal Kähler metrics [C1, C2]. Although generalizing Kähler-Einstein metrics, such metrics still do not exist on all Kähler manifolds [Lv, BdB]. Furthermore, it is not known whether their existence in one Kähler class implies their existence throughout the cohomology cone of Kähler classes. Examples are known for which this either does not hold, or else uniqueness within a particular Kähler class fails [Tf].

Using the central curvature we define central metrics, a new candidate for the notion of a distinguished Kähler metric, which we regard as a close cousin of the extremal one. It serves as a more uniform generalization of the concept of a Kähler-Einstein metric. In particular we describe cases where the Kähler cone is completely filled with class-unique central representatives.

To better describe how these metrics are related with extremal metrics, we recall some themes in the development of the latter. Originally extremal metrics were defined using a variational principle, whose Euler-Lagrange equations lead to a holomorphicity requirement on a distinguished vector field, defined via the scalar curvature. When this extremal vector field is just the zero vector field, the metric has constant scalar curvature, an important special case. In this context came the introduction of the Futaki invariant [Ft1, Ft3]. This is a linear functional on holomorphic vector fields, attached to a given Kähler class, which provides a criteria for distinguishing those Kähler classes in which extremal metrics must have constant scalar curvature. Eventually a duality relation was understood [C2, FMa, H2], pairing the Futaki invariant with the extremal vector field [C2, FMa, H2]. This pairing is achieved using a non-degenerate bilinear form on (a subalgebra of) the holomorphic vector fields, also attached to the Kähler class.

Thus one can approach extremal Kähler metrics by starting with the Futaki invariant, passing to the corresponding vector field via duality, and ending with the scalar curvature as the potential from which this vector field is constructed. Analogous concepts of distinguished metrics can be found using a similar procedure, but starting with different invariant linear functionals. This is one way to arrive at the concept of a central metric. However, one can consider two different bilinear forms that achieve a pairing as above, and thus we actually get two separate types of central metrics originating from a single invariant linear functional. Metrics of constant central curvature belong to both types.

A more direct way to understand the analogy with scalar curvature is to recall that in Kähler geometry, the scalar curvature s is a ratio of two top-dimensional forms, each built out of the Ricci form and the Kähler form: $s\omega^{\wedge n} = 2n\rho \wedge \omega^{\wedge n-1}$, where ω is the Kähler form, ρ the Ricci form and n the complex dimension. Likewise, the central curvature C is defined using a similar, perhaps simpler equation: $\rho^{\wedge n} = C\omega^{\wedge n}$. Then one can consider metrics for which C is constant, or, more generally, ones for which C gives rise to a holomorphic vector field. This is the type II central metric in the previous paragraph. In the case of type I, C is replaced by its Laplacian.[1] Most of our consideration is given to examining this type.

We now describe the content of the various sections. In Section 2 we review the definition and basic properties of the Futaki invariant, and in Section 3 we give a summary of Holomorphic Equivariant Cohomology in the sense of [Lu]. This is used in Section 4 to reinterpret the Futaki invariant as an integral of a (holomorphic) equivariantly closed form. One arrives at a simple formula for it, which involves the equivariant extensions of the Kähler and Ricci forms. A by-product of this formula is a simple derivation of the localization formula for the Futaki invariant, already found in [T]. However, the formula is also of use in the current context since different combinations of the above two equivariant extensions give rise to $n+1$ different invariant linear functionals. Of the resulting $n+1$ dual vector fields, only two can be derived

[1] After completing a draft of this paper I learned that the function C was considered in a distinct context related to volume decreasing holomorphic maps in [Kb3].

from simple yet non-trivial potential functions: the scalar and central curvatures. Section 4 also gives a description of the above-mentioned bilinear pairings.

The definitions of extremal and central metrics are given in Section 5. This section

and the next one deal mainly with central metrics of type I. Analogs are developed for standard results about extremal metrics. Examples are not given here, but a lower bound is derived, for metrics in a given Kähler class, of an L^2 quantity related to the central curvature. This bound is achieved exactly by central metrics of type I. Section 6 employs a result of Futaki and Morita [FMo1, FMo2] that relate the two invariant linear functionals in a special case. This is used to deduce connections between the resulting dual vector fields.

Section 7 gives results valid for metrics of constant central curvature. The first striking fact is that Yau's solution to the Calabi conjecture implies such metrics exist in any Kähler class, provided they exist in one such class. Also, unlike the situation for extremal metrics, the various central curvatures of central metrics on a given Kähler manifold are either all, or all not, constants. Next, because such metrics have Ricci forms of constant rank and signature, vanishing theorems of Demailly [Dm] and others allow us to deduce, from their existence, prescribed asymptotic growth rates for the dimensions of cohomology groups with values in multiples of the anticanonical line bundle. This leads to a class-uniqueness theorem for such metrics on manifolds admitting Kähler-Einstein metrics, and also to a partial classification in complex dimension two, featuring examples in all Kodaira dimensions.

Acknowlegements

The notion of a central metric originated in my thesis, and this work is in part a summary, in part a continuation, of the former. I would like to thank my advisor, Claude LeBrun, for years of instruction, in direct form, by example and through demonstration. In particular I thank him for suggesting the classification above for Kähler surfaces. I would also like to thank Yael Karshon and Maxim Braverman for introducing me to recent vanishing theorems in many enlightening conversations, and Marina Ville for referring me to reference [Kb3]. Finally, I wish to thank Christina Wiis Tønnesen-Friedman for many clarifications regarding explicit constructions of extremal metrics.

2. The Futaki Invariant

Let (M, g) be a compact Kähler manifold, with Kähler form ω and Ricci form ρ. Let ρ_H denote the harmonic part of ρ. Since ρ and ρ_H belong to the same cohomology class, there exists (cf. [GH, Chapter 1, Section 2]) a smooth real valued function F, called the **Ricci**

potential, such that

$$\rho - \rho_H = i\partial\bar{\partial}F. \tag{1}$$

Unless otherwise stated, we normalize F to be perpendicular to the constants. Taking traces above, with respect to ω, we get (via the hard Lefschetz theorem)

$$\Delta F = -\frac{1}{2}(s - s_0), \tag{2}$$

where $\Delta := \Delta_{\bar{\partial}}$ denotes the $\bar{\partial}$-Laplacian: $\Delta := \bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*$, s is the scalar curvature of g and s_0 its average value over M,

$$s_0 = \frac{\int_M s\frac{\omega^{\wedge n}}{n!}}{\int_M \frac{\omega^{\wedge n}}{n!}}.$$

We recall the definition of the Futaki invariant.

Definition. Let (M_n, ω) be a compact Kähler manifold with Ricci potential F. The **Futaki character** is the map $\mathcal{F}_{[\omega]} : h(M) \to \mathbb{C}$, where $h(M)$ denotes the Lie algebra of holomorphic vector fields on M, given by

$$\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi F \frac{\omega^{\wedge n}}{n!}. \tag{3}$$

Remark 2.1. *The values of this character do not depend on the choice of metric in the Kähler class* $[\omega]$ *(see* [B, C2, Ft3]*), i.e., it is a Kähler class invariant. One way to see this involves moment maps and holomorphic equivariant cohomology, which will be reviewed in the next section.*

Remark 2.2. *This invariance implies that* $\mathcal{F}_{[\omega]}$ *really is a Lie algebra character (cf.* [C2]*).*

Remark 2.3. $\mathcal{F}_{[\omega]}$ *is completely determined by its values on a subalgebra of* $h(M)$*: by the Hodge decomposition, the* $(0,1)$*-form* α *which is metrically dual to a given holomorphic vector field* Ξ *decomposes as* $\alpha = \alpha_H + \bar{\partial}f$*, with* α_H *harmonic and* f *a complex valued function (the term involving* $\bar{\partial}^*$ *vanishes by the holomorphicity of* Ξ *and the local implications of* g *being Kähler). Then*

$$\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi F \frac{\omega^{\wedge n}}{n!} = \int_M (\alpha_H, \bar{\partial}F)\frac{\omega^{\wedge n}}{n!} + \int_M (\bar{\partial}f, \bar{\partial}F)\frac{\omega^{\wedge n}}{n!},$$

where $(\cdot,\cdot)$ *denotes the pointwise inner product induced on* $(0,1)$*-forms by the Kähler metric. But the first term on the right hand side vanishes since after integrating by parts we see that it equals* $\int_M \bar{\partial}^*\alpha_H F\frac{\omega^{\wedge n}}{n!}$ *and* α_H*, being harmonic, is co-closed. It follows that it suffice to consider* *holomorphic vector fields* $\Xi = \Xi_f$ *whose dual* $(0,1)$*-form is of the form* $\bar\partial f$. *We call* Ξ_f *a gradient vector field, and* $f = f_\Xi$ *a holomorphy potential. On compact Kähler manifolds the gradient vector fields form a Lie subalgebra. Another way to characterize them is as those holomorphic vector fields having a non-empty zero set (cf.* [Kb, Part II, Corollary 4.6]*). The holomorphy potential satisfies a system of partial differential equations given in a complex coordinates system* $\{z^\alpha\}$ *by* $f^{,\alpha}{}_{,\bar\beta} = 0$. *Here* $\_{,\bar\beta} = \frac{\partial}{\partial \bar z^\beta}\_$, $\_^{,\alpha} = g^{\alpha\bar\gamma}\frac{\partial}{\partial \bar z^\gamma}\_$, *using the summation convention.*

3. Moment Maps and Holomorphic Equivariant Cohomology

We first recall some important notions from symplectic geometry. Let (M^{2n},ω) be a symplectic manifold with a Hamiltonian T-action, where T is a torus with Lie algebra t whose dual is t^*. Denote by X_ξ the vector field corresponding to a Lie algebra element $\xi \in t$.

Definition. In this setting, a **moment map** is a smooth map $\Phi : M \to t^*$ satisfying

1. $\imath_{X_\xi}\omega = d < \Phi, \xi >$ for all $\xi \in t$,
2. $\Phi \circ g = g \circ \Phi \circ g^{-1}$ for all $g \in T$.

An interesting case occurs when $T = S^1$. Then Φ reduces to a single smooth function $f : M \to \mathbb{R} \cong t^*$, condition i) becomes the equation

$$\imath_{X_\xi}\omega = df, \tag{4}$$

and the flow ϕ_t of $X_\xi := X$, is periodic: $\phi_1 = id$. In the corresponding equivariant cohomology one has a $\mathbb{Z}_2$-graded complex of even/odd invariant forms, with operator $d_{X_\xi} = d - \imath_{X_\xi}$ with respect to which $w + f$ is equivariantly closed. We are interested in a generalization of this situation, where M is Kähler, X gives rise to a holomorphic vector field $\Xi = X + iJX$, and f may be complex valued. We thus give a concise summary of this notion, called holomorphic equivariant cohomology, following [Lu]. This approach has its origins in [W], and in early works of Bott.

For a compact complex manifold (M_n, J), let

$$\Omega^{(r)}(M) = \bigoplus_{q-p=r} \Omega^{p,q}(M),$$

where $\Omega^{p,q}(M)$ denotes the smooth (p,q)-forms. Given a holomorphic vector field Ξ on M, define the differential operator

$$\bar\partial_\Xi = \bar\partial - \imath_\Xi.$$

Since now

$$\bar{\partial}_\Xi^2 = -(\bar{\partial}\imath_\Xi + \imath_\Xi\bar{\partial}) = 0,$$

we see that $(\Omega^{(*)}(M), \bar{\partial}_\Xi)$ constitutes a differential complex, where $^{(*)}$ denotes the range of $r = -n, -n+1, ...n-1, n$. We denote the resulting cohomology by $H_\Xi^{(r)}(M)$, if r is non-zero, and $H_\Xi(M)$ otherwise. The relevant localization formulas are valid for the latter. To state these, we first carefully describe the degeneracy behavior of Ξ at its zero locus.

Assume M_0 is a (disconnected) complex submanifold of zeros of Ξ (In the case of a circle action on a symplectic manifold, the zero set is a symplectic submanifold. In our context the above is an explicit assumption, not necessarily the weakest possible). Let $N = T^{(1,0)}M/T^{(1,0)}M_0$ be its holomorphic normal bundle, with locally constant *complex* rank denoted by $\mathrm{rk}(N)$.

Let $L_\Xi \in \Gamma(End(N))$ be the induced (complex Lie derivative) action of Ξ on N. Another major difference from the symplectic case is that L_Ξ is now explicitly *assumed* to be an invertible complex automorphism of N. In this case we call Ξ a non-degenerate vector field.

As before, integration over M is well-defined as a map

$$\int_M : H_\Xi(M) \to H_\Xi(point).$$

We have:

Theorem 3.1 (Holomorphic Localization [Lu])**.** *Let M, M_0 and N be as above, with Ξ non-degenerate. Given α with $[\alpha] \in H_\Xi(M)$,*

$$\int_M \alpha = (-2\pi)^{\mathrm{rk}(N)} \int_{M_0} \frac{\alpha}{\det(L_\Xi + \Omega)}. \tag{5}$$

where Ω is the (complex endomorphism valued) curvature 2-form of an L_Ξ-invariant connection on N, induced from any Hermitian metric on M, and the determinant is complex, taken with respect to the complex structure on N.

Generalizations abound [Lu]. For example, when the zero set of the vector field does not form a complex manifold, or for degenerate holomorphic vector fields with the formula involving the Grothendieck residue. Even more strikingly, a corresponding formula holds for meromorphic vector fields. This extends the utility of such results to a much larger class of complex manifolds.

As before, for a holomorphy potential f as above, $\omega + f$, for f as above, is a holomorphically equivariantly *closed* form. It is closed exactly because the variant of equation (4)),

$$\imath_\Xi\omega = \bar{\partial}f, \tag{6}$$

holds between a gradient vector field and its holomorphy potential. Note that for this to hold, the non-degeneracy of the symplectic form is immaterial. This allows us to consider equivariantly closed extensions of the Ricci form.

Proposition 3.2. *Let M be a complex manifold of Kähler type and Ξ a gradient holomorphic vector field on it. Suppose g is a Kähler metric on M with Kähler form ω and Ricci form ρ. Then if f is a smooth complex valued function on M satisfying*

$$\imath_\Xi \omega = \bar{\partial} f, \tag{7}$$

we have

$$\imath_\Xi \rho = \bar{\partial}(\Delta f). \tag{8}$$

Moreover, if M is compact, the second equation implies the first.

Remark 3.3. *Equivalently, using the $\bar{\partial}$-Laplacian on 1-forms, one can write:*

$$\imath_\Xi \rho = \Delta(\imath_\Xi \omega).$$

Proof of Proposition 3.2 This proposition is well known and goes back to Bochner [Bc2] and Yano [Yn]. For a convenient proof of a more general statement involving *any* holomorphic vector field, see [Kb, Theorem 4.2]. □

4. The Futaki Character as a Linear Holomorphic Equivariant Invariant

In the previous section we have seen that given a gradient holomorphic vector field Ξ with holomorphy potential f, any Kähler metric g gives rise to two distinguished holomorphic equivariantly closed forms: $\omega + f$ and $\rho + \Delta f$. Using them alone, we can recover the Futaki Invariant.

Proposition 4.1. *Let (M_n, g) be a compact Kähler manifold, and Ξ_f a gradient holomorphic vector field with holomorphy potential f.*

Then:

$$\mathcal{F}_{[\omega]}(\Xi) = \frac{1}{2(n+1)!} s_0 \int_M (\omega + f)^{\wedge n+1} - \frac{1}{n!} \int_M (\rho + \Delta f) \wedge (\omega + f)^{\wedge n}, \tag{9}$$

where s_0 denotes the average of the scalar curvature s of g.

Proof. For such a vector field, we have

$$\mathcal{F}_{[\omega]}(\Xi_f) = \int_M \Xi_f F \frac{\omega^{\wedge n}}{n!} =$$

$$\int_M (\partial F, \partial f) \frac{\omega^{\wedge n}}{n!} = \int_M \Delta_\partial F f \frac{\omega^{\wedge n}}{n!} =$$

$$\int_M f \Delta F \frac{\omega^{\wedge n}}{n!} = -\frac{1}{2} \int_M f(s - s_0) \frac{\omega^{\wedge n}}{n!} \quad . \tag{10}$$

The last equality in equation (10) follows from relation (2). Recall that by the Hard Lefschetz theorem (or Hodge theory),

$$s\omega^n = 2n\rho \wedge \omega^{n-1}, \tag{11}$$

so the Futaki invariant equals

$$\frac{1}{2n!} s_0 \int_M f\omega^{\wedge n} - \frac{1}{(n-1)!} \int_M f\rho \wedge \omega^{\wedge n-1}. \tag{12}$$

The right hand side of expression (9), on the other hand, evaluates to
$\frac{1}{2(n+1)!} s_0 (n+1) \int_M f\omega^n - \frac{1}{n!} \int_M \rho \wedge (\sum_{k=0}^n \binom{n}{k} f^{n-k} \omega^k) -$
$\frac{1}{n!} \int_M \Delta f (\sum_{k=0}^n \binom{n}{k} f^{n-k} \omega^k) =$
$\frac{1}{2n!} s_0 \int_M f\omega^n - \frac{1}{n!} n \int_M f\rho \wedge \omega^{n-1} - \frac{1}{n!} \int_M \Delta f \omega^{\wedge n} =$
$\frac{1}{2n!} s_0 \int_M f\omega^n - \frac{1}{(n-1)!} \int_M f\rho \wedge \omega^{n-1}$,
where we have used the Divergence Theorem in the last step. Comparing with (12), we are done. □

Combining this proposition with the Holomorphic Localization Theorem allows one to arrive at a simple derivation of the localization formula for the Futaki invariant, already given in [T]:

Theorem 4.2 (Tian)**.** *Let (M_n, J) be a Kähler manifold, Ξ a nondegenerate gradient holomorphic vector field and ω a Kähler form of an arbitrary Kähler metric g on M. Denote by ρ the Ricci form of g, and f the holomorphy potential of Ξ with respect to the metric. If M_0, the zero set of Ξ, forms a complex submanifold, then*

$$\mathcal{F}_{[\omega]}(\Xi) = (-2\pi)^{\mathrm{rk}(N)} \{ \frac{1}{2(n+1)!} s_0 \int_{M_0} \frac{(\omega+f)^{\wedge n+1}}{\det(L_\Xi + \Omega)} - \frac{1}{n!} \int_{M_0} \frac{(\rho + \Delta f) \wedge (\omega + f)^{\wedge n}}{\det(L_\Xi + \Omega)} \},$$

where rk, det, L_Ξ, Ω *are as in Theorem 3.1.*

An explicit 4-dimensional formula derived from the above can be found in [Ms].

Our main interest lies, however, in formula (9) itself. Since the holomorphy potential f is only determined up to an additive constant, normalizing f by setting

$$\frac{1}{(n+1)!}\int_M (\omega+f)^{\wedge n+1} = \int_M f\frac{\omega^{\wedge n}}{n!} = 0, \tag{13}$$

gives the simplified formula

$$\mathcal{F}_{[\omega]}(\Xi_f) = -\frac{1}{n!}\int_M (\rho+\Delta f)\wedge(\omega+f)^{\wedge n}. \tag{14}$$

Its suggestiveness allows one to devise other similar invariants *linear* in Ξ. Consider the family of $n+1$ expressions

$$\mathcal{A}_k(\Xi_f) = -\frac{1}{n!}\int_M (\rho+\Delta f)^{\wedge k}\wedge(\omega+f)^{\wedge n+1-k}, k=0...n+1. \tag{15}$$

They all constitute Kähler class invariants: a change of Kähler representative $\omega \to \omega + i\partial\bar\partial\phi$ results in a change $\omega+f \to \omega+f+\bar\partial_{\Xi_f}(-i(\partial\phi)+K$, where K is some constant which can be ignored by condition (13). Similarly, if the Ricci form transforms as $\rho \to \rho + \bar\partial\beta$, for a $(1,0)$-form β, then $\rho+\Delta f \to \rho + Def + \bar\partial_{\Xi_f}\beta$ (here there is no extra constant since the value of the Laplacian at the maximum of f has an invariant meaning, see [T]). Thus both form changed only by an equivariantly exact form, giving the invariance.

Of these, besides the Futaki invariant $\mathcal{F}_{[\omega]} = \mathcal{A}_1$ we will also be interested in $\mathcal{A}_{n+1}$, which we denote by $\mathcal{B}$, or $\mathcal{B}_{c_1}$:

$$\mathcal{B}(\Xi_f) = -\frac{1}{n!}\int_M (\rho+\Delta f)^{\wedge n+1}. \tag{16}$$

$\mathcal{B}$ is manifestly independent of the normalization for f, and in fact it does not even depend on the choice of Kähler class, but only on the complex structure of M, and can even be computed via Hermitian metrics [FMo1, FMo2]. Like the Futaki invariant, $\mathcal{B}$ can be defined for any holomorphic vector field by

$$\mathcal{B}(\Xi) = -\frac{1}{n!}\int_M \mathrm{div}(\Xi)\frac{\rho^{\wedge n}}{n!},$$

where div denotes the complex divergence, but, again, it is completely determined by its values on gradient holomorphic vector fields.

We will also be using two bilinear forms on gradient holomorphic vector fields. These will be:

$$\mathcal{K}^I_g(\Xi_1,\Xi_2) = \int_M f_{\Xi_1} f_{\Xi_2}\frac{\omega^{\wedge n}}{n!}$$

and

$$\mathcal{K}^{II}_g(\Xi_1, \Xi_2) = \int_M f_{\Xi_1} \Delta f_{\Xi_2} \frac{\omega^{\wedge n}}{n!},$$

where the holomorphy potentials f_{Ξ_i} are normalized to be orthogonal to the constants, as in (13) (This is only important for $\mathcal{K}^I_g$). Note that at least $\mathcal{K}^I_g$ can be understood via equivariant cohomology, as it can be written as an integral of equivariantly closed forms:

$$\mathcal{K}^I_g(\Xi_1, \Xi_2) = \frac{\left\{\int_M (\omega + f_{\Xi_1}) \wedge (\omega + f_{\Xi_2})^{\wedge n+1} - \binom{n+1}{2}/\binom{n+2}{1} \int_M (\omega + f_{\Xi_2})^{\wedge n+2}\right\}}{\binom{n+1}{1}}.$$

It follows that it is a also a Kähler class invariant, which we will denote by $\mathcal{K}^I_{[\omega]}$. This was shown via different methods in [FMa].

5. $\mathcal{F}_{[\omega]}, \mathcal{B}$ and Distinguished Kähler Metrics

In [C1] Calabi defined a new notion of a distinguished Kähler metric.

Definition. A Kähler metric g with Kähler form ω on a complex manifold M will be called an **extremal** Kähler metric if it is critical point of the functional

$$g \longrightarrow \int_M s_g^2 \frac{\omega^{\wedge n}}{n!} \tag{17}$$

among Kähler metrics in the class $[\omega]$. Here s_g denotes the scalar curvature of g.

Specializing to compact manifolds, one has:

Proposition 5.1 (Calabi [C1])**.** *For M compact a Kähler metric is extremal if and only if its scalar curvature is a holomorphy potential, i.e.* $s^{,\alpha}{}_{,\bar\beta} = 0$.

In particular, if g is a Kähler-Einstein metric, or, more generally, any Kähler metric of constant scalar curvature, it is extremal.

The basic relation between extremal Kähler metrics and the Futaki invariant is that the latter is $\mathcal{K}^I_{[\omega]}$-dual to the extremal vector field $\Xi_s = \Xi_{s-s_0}$, where s_0 denotes as usual the average scalar curvature:

$$-\mathcal{F}_{[\omega]}(\Xi_f) = -\frac{1}{2} \int_M f_\Xi (s - s_0) \frac{\omega^{\wedge n}}{n!} = \mathcal{K}_{[\omega]}(\Xi_f, \Xi_{s-s_0})), \tag{18}$$

by relation (10).

In [FMa] further information was drawn concerning other Kähler metrics in the class. For (M, g) Kähler, denote:

$C_0^\infty(M, \mathbb{C}, g)$ **:** the space of all smooth functions on M g-perpendicular to the constants,

$\Gamma_0 := \Gamma_{0,g}$ **:** the subspace of all g-holomorphy potentials also perpendicular to the constants, and

$\pi_g : C_0^\infty(M, \mathbb{C}, g) \to \Gamma_0$ **:** the orthogonal projection with respect to the inner product

$$< f, h >= \int_M f\bar{h}\frac{\omega^{\wedge n}}{n!}.$$

Theorem 5.2 (Futaki-Mabuchi [FMa]). *Let M be a compact Kähler manifold. For any Kähler metric g in a Kähler class $[\omega]$, the number $\mathcal{F}_{[\omega]}(\Xi_{\pi_g(s-s_0)})$ depends only on the Kähler class.*

$\Xi_{\pi_g(s-s_0)}$ is called the **extremal vector field** of g.

This was used in [H2] to give:

Theorem 5.3 (Hwang [H1, H2], see also [Sm]). *Let M be a compact Kähler manifold, g a Kähler metric on it with Kähler form ω. Then:*

$$\int_M (s - s_0)^2 \frac{\omega^{\wedge n}}{n!} \geq -\mathcal{F}_{[\omega]}(\Xi_{\pi_g(s-s_0)}), \tag{19}$$

where the notations are as in Theorem 5.2. The right hand side of inequality (19) is real and non-negative, and equality occurs exactly when g is extremal.

Remark 5.4. *Since by Theorem 5.2 the real number on the right hand side of inequality (19) depends only on the Kähler class $[\omega]$, it gives, by the above, a lower bound for the left hand side over all Kähler metrics in the class.*

The following two corollaries have been found earlier and independently:

Corollary 5.5 (Calabi [C2]). *Under the conditions of Theorem 5.3, if the right hand side of inequality (19) vanishes (in particular if $\mathcal{F}_{[\omega]}$ is identically zero), an extremal Kähler metric in the class $[\omega]$ has constant Ricci potential (equivalently, constant scalar curvature).*

Corollary 5.6 (Futaki [Ft3]). *Under the conditions of Theorem 5.3, if there exists a Kähler metric of constant Ricci potential (or constant scalar curvature) in the class $[\omega]$, then $\mathcal{F}_{[\omega]}(\cdot) \equiv 0$.*

We introduce now a new definition of a distinguished Kähler metric, which broadens the notion of a Kähler-Einstein metric in a manner similar to, but more uniform, than that of an extremal Kähler metric. We actually employ two variants for this notion, and for one of these we give parallels of the results above.

Definition. Let (M,g) a compact Kähler manifold, with Kähler form ω and Ricci form ρ. Calling

$$C := \frac{\det \rho}{\det \omega} \tag{20}$$

the **central curvature** of g, g will be called **central** of **type I (II)** if and only if ΔC (C) is a holomorphy potential, i.e. $(\Delta C)^{,\alpha}{}_{,\bar\beta} = 0$ $(C^{,\alpha}{}_{,\bar\beta} = 0)$.

Remark 5.7. *Note that* det *refers to the complex determinant, and that C is well defined because ω is non-degenerate. Also note that if C is constant, it is central of* both *types, and corresponds to the zero vector field.*

Remark 5.8. *For a Fano manifold (see Section 6), one can regard c_1 as a central element in the Kähler cone of $H^{1,1}(M)$. The terminology in the definition is thus meant to suggest that such a metric (of either type) is determined to a large degree by corresponding Kähler metrics in c_1. Or, for more general spaces, by the behaviour of the Ricci forms in c_1.*

Remark 5.9. *Of all the linear invariants $\mathcal{A}_k$, $\mathcal{A}_1$ and $\mathcal{A}_{n+1}$ are distinguished, since in the duality schemes we employ they are the only ones that determine simple potential functions, i.e. one that are given not by a sum of two functions (actually, neither does $\mathcal{A}_0$, but it is* too *simple, as it determines a constant function).*

The following are parallels of the above results for type I central metrics.

Theorem 5.10. *Let M be a compact Kähler manifold. Maintaining the notations of Theorem 5.2, for any Kähler metric g having Kähler class $[\omega]$, the number $\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)})$ depends only on the Kähler class.*

We will call $\Xi_{\pi_g(\Delta C)}$ the **(type I) central vector field** of g.

Theorem 5.11. *Let M be a compact Kähler manifold with a Kähler metric g having Kähler form ω and central curvature C. Then*

$$\int_M (\Delta C)^2 \frac{\omega^{\wedge n}}{n!} \geq -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \tag{21}$$

where $\pi_g : C_0^\infty(M,\mathbb{C},g) \to \Gamma_0$ is as in Theorem 5.3. The right hand side of inequality (21) is real non-negative, and equality occurs exactly when g is central of type I.

Note that although Calabi has shown in [C1] that many functionals on the space of Kähler metrics in a given cohomology class, are either uninteresting, or else provide equivalent information to that deduced from the L^2 norm of the scalar curvature, this is not the case for the functional on the left hand side of inequality (21).

The following corollaries of Theorem 5.11 are the analogs of Corollaries 5.5 and 5.6.

Corollary 5.12. *Under the conditions of Theorem 5.11, if the right hand side of inequality (21) vanishes (in particular if $\mathcal{B}_{c_1}$ is identically zero), a central Kähler metric of type I on M has constant central curvature.*

Remark 5.13. *A variant of this corollary holds also for central metrics of type II, since for such a metric $\mathcal{B}_{c_1}(\Xi_C) = 0$ means*

$$0 = \int_M \Delta C \frac{\rho^{\wedge n}}{n!} = \int_M (\Delta C) C \frac{\omega^{\wedge n}}{n!} = \int_M ||\bar{\partial} C||^2 \frac{\omega^{\wedge n}}{n!},$$

and so C is holomorphic, and therefore constant.

Corollary 5.14. *Under the conditions of Theorem 5.11, if there exists on M a Kähler metric with constant central curvature, then $\mathcal{B}_{c_1}(\cdot) \equiv 0$.*

The proofs of the above results are very similar to those of the first set of statements. We review the most important steps. The pertinent Lie theoretic background here is that the Lie algebra of gradient vector fields on a compact Kähler manifold is the Lie algebra of an algebraic group, and as such splits into a sum of a reductive subalgebra and a nilpotent radical [Fj]. Both the former's embedding and, consequently, the splitting, are non-canonical. See the relevant papers quoted below for further information.

Proof of Theorem 5.10. Let $f_{\Xi_1}, f_{\Xi_2} \in \Gamma_{0,g}$. Following Hwang [H2] we extend the invariant $\mathbb{C}$-bilinear form on

$$\mathcal{K}^I_{[\omega]}(\Xi_1, \Xi_2) = \int_M f_{\Xi_1} f_{\Xi_2} \frac{\omega^{\wedge n}}{n!},$$

from gradient vector fields to all holomorphic vector fields by fixing an arbitrary (non-degenerate) bilinear form on nowhere vanishing vector fields, and declaring the latter orthogonal to the former. We denote the resulting bilinear form $\mathcal{K}_{[\omega]}$. Let ρ be the Ricci form of g, note that C is real valued and satisfies $\rho^{\wedge n} = C\omega^{\wedge n}$. We now have, for any gradient vector field Ξ_f:

$$\begin{aligned} -\mathcal{B}_{c_1}(\Xi_f) &= \int_M (\Delta f_\Xi)\frac{\rho^{\wedge n}}{n!} = \\ \int_M (\Delta f_\Xi) C\frac{\omega^{\wedge n}}{n!} &= \int_M f_\Xi \Delta C \frac{\omega^{\wedge n}}{n!} = \\ \int_M f_\Xi \, \pi_g(\Delta C)\frac{\omega^{\wedge n}}{n!} &= \mathcal{K}_{[\omega]}(\Xi_f, \Xi_{\pi_g(\Delta C)}), \end{aligned} \tag{22}$$

where we have used integration by parts, and then in the penultimate step, the orthogonality of the projection π_g. (By the construction of $\mathcal{K}_{[\omega]}$, and because (as we will see) $\mathcal{B}_{c_1}$ vanishes on nowhere vanishing vector fields, both sides of the above equation are zero for such vector fields, and so these can be safely ignored).

Therefore, because for any g belonging to the Kähler class, $\Xi_{\pi_g(\Delta C)}$ is $\mathcal{K}_{[\omega]}$-dual to the fixed functional $-\mathcal{B}_{c_1}$, as g varies in $[\omega]$ this vector field can only vary in the subspace of $h(M)$ on which $\mathcal{K}_{[\omega]}$ degenerates, i.e. it is fixed up to an additive element of the nilpotent radical of $h(M)$: in [FMa] it is shown that $\mathcal{K}_{[\omega]}$ has this degeneracy subspace for metrics having a maximal compact group of isometries, and since this subspace, as well as $\mathcal{K}_{[\omega]}$, are metric independent, this follows for any metric in the class. But $-\mathcal{B}_{c_1}$ vanishes on such elements – as in [Mb1], or, alternatively, since the invariance of this character implies that it is invariant under the adjoint action of the identity component of the group of biholomorphisms (as in, e.g., [C2]). Its codimension one kernel is therefore also invariant under the adjoint action, and so induces the zero functional after dividing by the kernel of this action, i.e. the center of the reductive part of the Lie algebra. The quotient includes the nilpotent radical (and also the nowhere vanishing vector fields). Thus although $\Xi_{\pi_g(\Delta C)}$ may vary with the metric, $-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)})$ does not, and we are done. □

Remark 5.15. *By (22), for type I central metrics, $\mathcal{B}$ is $K^I_{[\omega]}$-dual to $\Xi_{\Delta C}$, like $\mathcal{F}_{[\omega]}$ is with Ξ_{s-s_0} for extremal metrics. For type II central metrics, $\mathcal{B}$ is K^{II}_g dual to Ξ_C. The latter are analogs of extremal metrics also because in both the distinguished holomorphy potential is ratio of (n,n)-forms built out of the Kähler form and the Ricci form ($s\omega^n = 2n\rho \wedge \omega^{n+1}$, $C\omega^n = \rho^n$).*

In what follows we will ignore the ambiguity in the definitions of the extremal and type I central vector fields, as we have just shown these do not cause an ambiguity in the value of the above lower bounds.

Proof of Theorem 5.11. We write the L^2_g-orthogonal decomposition of ΔC as

$$\Delta C = \pi_g(\Delta C) + T. \tag{23}$$

Then,

$$\begin{aligned} 0 \leq < \pi_g(\Delta C), \pi_g(\Delta C) > &= < \pi_g(\Delta C), \Delta C - T >= \\ < \pi_g(\Delta C), \Delta C > &= < \Delta(\pi_g(\Delta C)), C >= \\ -\int_M \mathrm{div}_g \Xi_{\pi_g(\Delta C)} C \frac{\omega^{\wedge n}}{n!} = -\int_M \mathrm{div}_g \Xi_{\pi_g(\Delta C)} \frac{\rho^{\wedge n}}{n!} &= -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \end{aligned}$$

where first we have used the orthogonality of the relation (23), then the fact that ΔC was real-valued meant that conjugation was unnecessary in the integral, and finally integration by parts. Combining this again with the orthogonality of condition (23), we have

$$\begin{aligned} ||\Delta C||_g^2 &= ||\pi_g(\Delta C)||_g^2 + ||T||_g^2 = \\ -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) + ||T||_g^2 &\geq -\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}), \end{aligned}$$

as required. Equality here exactly means $T = 0$, or $\Delta C = \pi_g(\Delta C)$, so g is central. □

Remark 5.16. *In the next section we will show that there are indeed cases where the bound on $||\Delta C||_g^2$ given in Theorem 5.11 is strictly positive.*

6. Extremal and Type I Central Vector Fields

In this section we employ known results about extremal metrics to show that non trivial type I central vector fields do exist, and to relate them to extremal vector fields. To this end, we first recall the relationship between the Futaki invariant and $\mathcal{B}$.

Recall that a complex manifold (M, J) is called a *Fano manifold*, when the class c_1 contains positive definite $(1,1)$-forms.

Proposition 6.1 (Futaki-Morita [FMo2])**.** *Let (M, J) be a compact Fano manifold. Let g be a Kähler metric with Kähler form ω having Kähler class c_1. If F denotes the Ricci potential of g, then for any holomorphic vector field Ξ,*

$$\mathcal{F}_{c_1}(\Xi) = \int_M \Xi(F) \frac{\omega^{\wedge n}}{n!} = \int_M \mathrm{div}_g \Xi \frac{\rho^{\wedge n}}{n!} = \mathcal{B}_{c_1}(\Xi). \tag{24}$$

From the point of view of holomorphic equivariant cohomology, the proposition follows since both sides of equation (24) equal the integral of another equivariantly closed form, namely $(\rho_H + \Delta_F f)^{\wedge n+1}$, where $\Delta_F f = \Delta f - (\partial F, \partial f)$ (see [Ft2]) . Unlike the original argument, this does not make any use of the Calabi-Yau theorem.

To obtain more symmetric expressions in what follows, it is convenient to define the **extremal potential** E to be twice the Ricci potential F. Since for type I central metrics ΔC is the analog of $s - s_0$ for extremal metrics, we consider the central curvature C to be the analog of E in this section.

Lemma 6.2. *The following relation holds between the extremal and the type I central vector fields, of any Kähler metric:*

$$\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = \mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta E)}). \tag{25}$$

Proof. We use orthogonality, reality, and integration by parts, as before.

$$\begin{aligned}
\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) &= \\
-\int_M \pi_g(\Delta C)\Delta E\frac{\omega^{\wedge n}}{n!} &= -\int_M \pi_g(\Delta C)\pi_g(\Delta E)\frac{\omega^{\wedge n}}{n!} = \\
-\int_M \pi_g(\Delta E)\Delta C\frac{\omega^{\wedge n}}{n!} &= -\int_M \Delta(\pi_g(\Delta E))C\frac{\omega^{\wedge n}}{n!} = \\
-\int_M \Delta(\pi_g(\Delta E))\frac{\rho^{\wedge n}}{n!} &= \mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta E)}).
\end{aligned}$$

□

Combining this with Proposition 6.1, we get:

Corollary 6.3. *For any Kähler metric in a class $[\omega]$ on a Fano manifold M,*

$$\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = \mathcal{F}_{c_1}(\Xi_{\pi_g(\Delta E)}). \tag{26}$$

Proposition 6.4. *There exist manifolds with Kähler metrics satisfying*

$$-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) > 0.$$

Proof. Let M be a Fano manifold having an extremal Kähler metric of non-constant scalar curvature in the class c_1. An example of such a manifold would be the one point blow-up of CP^2 (cf. [C1]). Taking $[\omega] = c_1$, Corollary 6.3 gives

$$-\mathcal{B}_{c_1}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{c_1}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{[\omega]}(\Xi_{\pi_g(\Delta C)}) = -\mathcal{F}_{c_1}(\Xi_{\Delta E}) > 0,$$

by Corollary 5.5. □

Corollary 6.3 also implies:

Theorem 6.5. *Let g be a Kähler metric with Kähler form in c_1. Then, if g is type I central and extremal, its type I central and extremal vector fields coincide.*

Proof. Let E and C be the extremal potential and central curvature of g, respectively. By (26),

$$\mathcal{F}_{c_1}(\Xi_{\Delta C} - \Xi_{\Delta E}) = \mathcal{F}_{c_1}(\Xi_{\Delta C - \Delta E}) = 0. \tag{27}$$

Suppose $\Xi_{\Delta E}$ and $\Xi_{\Delta C}$ span a real 2-dimensional subspace in the space of gradient (holomorphic) Killing vector fields. The restriction of the bilinear form $\mathcal{K}_{c_1}(\cdot,\cdot)$ to this subspace is a positive definite inner product. Now $\Xi_{\Delta C - \Delta E}$ is also a nonzero vector in this subspace, and (minus) the relation (27) can be rewritten in two ways:

$$\mathcal{K}^I_{c_1}(\Xi_{\Delta C - \Delta E}, \Xi_{\Delta C}) = \mathcal{K}^I_{c_1}(\Xi_{\Delta C - \Delta E}, \Xi_{\Delta E}) = 0. \tag{28}$$

Since a nonzero vector cannot be orthogonal to all vectors of a basis, $\Xi_{\Delta C}$ and $\Xi_{\Delta E}$ coincide up to constant multiple. But then relations (28) force them to coincide exactly (even if one of them was zero). □

Remark 6.6. *In* [Ms] *it is shown that even if the metric is just type I central, its extremal vector field $\Xi_{\pi_g(\Delta E)}$ equals its type I central vector field. Also such results, hold not only for metrics in c_1, but also in other Kähler classes $[\omega]$ for which $\mathcal{F}_{[\omega]}(\cdot) \equiv \mathcal{B}(\cdot)$, since together with Lemma 6.2 one has:*

$$\mathcal{F}_{[\omega]}(\Xi_{\Delta C}) = \mathcal{B}_{c_1}(\Xi_{\Delta E}) = \mathcal{F}_{[\omega]}(\Xi_{\Delta E}),$$

and so the proof above holds with $\mathcal{F}_{[\omega]}$, $\mathcal{K}^I_{[\omega]}$ replacing $\mathcal{F}_{c_1}$, $\mathcal{K}^I_{c_1}$, respectively.

7. (Class) Existence and Uniqueness - Constant Central Curvature

Consideration of extremal Kähler metrics can be divided into two main cases: constant and non-constant scalar curvature. Existence questions in both are challenging, and not completely solved. In particular, the behaviour throughout the Kähler cone is not fully understood. In contrast, for central Kähler metrics having constant central curvature, this question has been *implicitly* understood for some time. For this recall the Calabi-Yau Theorem.

Theorem 7.1 (Calabi-Yau [Yu]). *Let M be a compact Kähler manifold. If $\bar{\rho}$ is a real closed $(1,1)$-form representing $c_1(M)$ (if Ψ is a real non-degenerate (n,n)-form), then in every Kähler class there exists a* *unique Kähler form ω, whose Ricci form equals $\bar{\rho}$ (whose volume form equals a multiple of Ψ).*

This leads to the following consequence.

Theorem 7.2. *Let M be a complex manifold admitting a central Kähler metric of constant central curvature. Then M admits such metrics in every Kähler class.*

Proof. Let g be a central metric with Kähler form ω, Ricci form ρ and constant central curvature C. Given a fixed Kähler class, let $\tilde{g}$ be the unique Calabi-Yau metric in it having Ricci form $\tilde{\rho}$ equal to ρ. Then in particular $\det\tilde{\rho} = \det\rho$, and, again by the Calabi-Yau Theorem, $\det\tilde{\omega} = A\det\omega$ for some positive constant A. Therefore, the central curvature $\tilde{C}$ of $\tilde{g}$ satisfies $\tilde{C} = \frac{1}{A}C$, and so is constant. □

Note also that the sign of $\tilde{C}$ (if non-zero) is the same as that of C.

Thus, for example, the Kähler cone of a manifold admitting a Kähler-Einstein metric (or having $\mathcal{B} \equiv 0$ and admitting one central metric) is completely filled with central Kähler representatives, but none have non-constant central curvature, by Corollary 5.12. $\mathcal{B}$ not identically zero, on the other hand, implies that every central metric must have non-constant central curvature, by Corollary 5.14. This should be contrasted with the behavior of extremal Kähler metrics.

Note that there are manifolds that do not admit Kähler-Einstein metrics but still admit metrics of constant central curvature. For example, generalized Kähler-Einstein metrics in the sense of [Mt] (i.e. metrics with eigenvalues of the Ricci tensor constant with respect to the metric) have constant central curvature.

To give another characterization, every Kähler metric of constant non-zero central curvature has a symplectic Ricci form. Actually, a little more can be said.

Proposition 7.3. *A central Kähler metric g having constant non-zero central curvature C has an Einstein-symplectic Ricci form ρ (for this terminology see* [Mb1]*, or the proof).*

Proof. For such g, its Ricci form ρ is symplectic, so $\log\det(\rho)$ is well defined. We have:

$$0 = i\partial\bar{\partial}\log C = i\partial\bar{\partial}\log\det(\rho) - i\partial\bar{\partial}\log\det(\omega) = i\partial\bar{\partial}\log\det(\rho) + \rho.$$

The above equation precisely says that ρ is Einstein-symplectic. □

There are cases where central metrics are also extremal. For this, and for issues relating to uniqueness of central metrics, we need the following asymptotic estimates.

Theorem 7.4 (Demailly [Dm]). *Let M_n be a compact complex manifold, and L a holomorphic line bundle with a smooth hermitian metric having curvature ρ_L. Denote*

$$M_{\rho_L}(q) := \left\{ m\epsilon M \mid (\rho_L)_m \begin{array}{l} \text{has } q \text{ negative eigenvalues} \\ \text{and } n-q \text{ positive eigenvalues} \end{array} \right\},$$
$$h^q(L) := \dim_{\mathbb{C}} H^q(M, \Theta(L)).$$

The cohomology groups with values in L^k satisfy, for large enough k,

$$h^q(L^k) \leq \frac{k^n}{n!} \int_{M_{\rho_L}(q)} (-1)^q (\frac{i}{2\pi}\rho_L)^n + \circ(k^n), \quad \textit{and} \tag{29}$$

$$\chi(L^k) = \frac{k^n}{n!} \int_M (\frac{i}{2\pi}\rho_L)^n + \circ(k^n). \tag{30}$$

Corollary 7.5. *Given the above, if $c_1(L)$ admits a positive definite $(1,1)$-form, it does not admit a non-degenerate $(1,1)$-form of any other signature.*

Proof of Corollary 7.5 Using the positive definite form ρ_L, for any $q > 1$, $h^q(L^k)$ has an asymptotic growth rate strictly smaller than k^n, by (29) (since $M_{\rho_L}(q) = \emptyset$). On the other hand, using the equality (30), The Euler characteristic $\chi(L^k)$ has asymptotic growth equal to n^k. It follows that $h^0(L^k)$ has asymptotic growth rate equal to n^k (for this part one can also use the proof of the Kodaira Embedding Theorem). Now if there had also existed a non-degenerate form $\tilde{\rho}_L$ in $c_1(L)$, having q_0 negative eigenvalues, with $q_0 > 0$, then by using inequality (29) again with $\tilde{\rho}_L$, one would get that $h^0(L^k)$ has asymptotic growth strictly less than n^k (since $M_{\tilde{\rho}_L}(0) = \emptyset$). This is a contradiction. □

This corollary can be considerably strengthened, at least for projective manifolds, where one can show that if a $(1,1)$-class admits a curvature form of constant rank (and, therefore, constant signature), then any other such form in this class will have the same rank and signature [Br]. Using another vanishing theorem we will prove a relevant special case.

Theorem 7.6. *If a hermitian line bundle L over a compact complex manifold has a curvature form which is negative in one direction at each point, than L admits no nonzero holomorphic sections.*

For the proof see [Kb2], Chapter *III*, before Corollary 1.16.

Corollary 7.7. *If L is a line bundle over a compact Kähler manifold M has $c_1(L) = 0$ then it does not admit a hermitian metric with non-zero curvature of constant rank.*

Proof. Assuming the contrary, either $c_1(L)$ or $c_1(L^*)$ admit a curvature 2-form with at least one negative-direction at each point, and by assumption they both admit a flat connection. Since one can sum the curvatures of the first connection with that of a flat one, producing no change, $c_1(L \otimes L^*)$ also admits a curvature 2-form with one negative eigen-direction, and so has no nontrivial holomorphic sections, by Theorem 7.6. But the trivial bundle $L \otimes L^*$ does admit such sections, and a contradiction is thus reached. □

Proposition 7.8. *Let M be a Fano manifold. If an extremal Kähler metric on M, having Kähler class c_1, has constant scalar curvature, then it is central. If a central Kähler metric, having Kähler class c_1, has constant central curvature, then it is extremal. Both metrics above are actually Kähler-Einstein.*

Proof. It is well known that a Kähler metric of constant scalar curvature having Kähler class in c_1 is Kähler-Einstein (see [Ft2, Lemma 2.2.3], or [Ft4]), and this is the first claim. For the second one, let g have constant central curvature C with Kähler form and Ricci form $\omega, \rho\epsilon c_1$, respectively. Then $\rho^n = C\omega^n$, with C a positive constant (since $c_1^n > 0$, which also implies $C = 1$). So ρ cannot have degeneracies, or else ρ^n, and therefore ω^n, would have zeros. But then by Corollary 7.5, ρ has to be positive definite, and by the uniqueness part of the Calabi-Yau theorem, it actually equals ω. □

Remark 7.9. *When $c_1 < 0$, a similar proof using $-\rho$ instead of ρ, shows that every Kähler metric of constant central curvature in $-c_1$ (or any negative multiple of c_1) is Kähler-Einstein. In the third case, $c_1 = 0$, we use corollary 7.7 to deduce at least that any metric of constant (necessarily zero) central curvature, for which ρ has constant rank, has to be Ricci flat.*

This has important consequences regarding the uniqueness question.

Theorem 7.10. *Let (M, J) compact complex manifold of Kähler type, for which either $c_1 > 0$ or $c_1 < 0$. If M admits Kähler metrics of constant central curvature, then every Kähler class admits exactly one G_0-orbit of such metrics, where G_0 is the identity component of $Aut(M)$.*

This holds also for the case $c_1=0$ at least if every central metric has a Ricci form of constant rank.

Note that for $c_1 \leq 0$ existence is guaranteed since the Kähler-Einstein existence problem is solved in that case [Yu, An].

Proof. By the end of the above remark, the case $c_1 = 0$ is really the Ricci flat case, which is known by Theorem 7.1. For the other cases, by Theorem 7.2 c_1 (-c_1) admits such metrics, and they are all Kähler-Einstein by (the remark after) Proposition 7.8. The theorem is then proved for this class by the uniqueness result of Bando and Mabuci [BM]. Given another Kähler class, by the proof of Theorem 7.2, the uniqueness part of the Calabi conjecture (Theorem 7.1) and Proposition 7.8 once more, any such metric has the Ricci form of a *unique* Kähler-Einstein metric, and so the number of G_0-orbits in this class has to be the same as the number in c_1 $(-c_1)$. □

Note that when $c_1 < 0$, since there are no automorphisms, it follows that there is a unique metric of constant central curvature in every Kähler class.

It appears that for Kähler manifolds admitting metrics of constant central curvature, a (very rough) relation exists between the degeneracy behaviour of the Ricci form of such a metric and the Kodaira dimension of the space. It seems that examples exist in every Kodaira dimension. The following partial classification for Kähler surfaces, which was suggested by Claude LeBrun, serves as an illustration.

For a Kähler surface, the hermitian matrix locally representing the Ricci form ρ has two eigenvalues at each point. When the central curvature is constant, the sign of $\det \rho$ is constant throughout the manifold. If it is positive, either both eigenvalues are positive, so $c_1 > 0$, or both are negative and $c_1 < 0$. If $c_1 > 0$ and such a metric exists, there exists another such metric with Kähler form in c_1, by Theorem 7.2. The latter is Kähler-Einstein by Proposition 7.8. The Del-Pezzo surfaces which admit such metrics are known. The Kodaira dimension is $-\infty$. If $c_1 < 0$, the same argument, or Yau's existence theorem for the complex Monge-Ampère equation imply again that the space admits a Kähler-Einstein metric and is of general type. The Kodaira dimension is 2.

The remaining non-degenerate case occurs when $\det \rho < 0$, i.e. the eigenvalues of ρ differ in sign. Such a surface must have Kodaira dimension $-\infty$, by Theorem 7.6. the eigenvalue structure of ρ shows the surface admits a semi-definite Kähler metric, and, in the context of a classification of semi-definite Kähler-Einstein metrics, examples are given in [Pt] which are minimal ruled surfaces over a base of genus greater than one. These ruled surfaces are all projectivizations of quasi-stable bundles.

Finally the determinant could be zero, which implies that $c_1^2 = 0$, and so the surface is not Fano, nor of general type. As examples having Kodaira dimension $-\infty$ one can take ruled surfaces over a base of genus 1. For such an example the non-zero eigenvalue is positive, so for $-\rho$ it is negative, and this Kodaira dimension can again be deduced from Theorem 7.6. If, on the other hand, the Kodaira dimension is 0 or 1, the above implies it is necessarily minimal. By Corollary 7.7, if the rank of ρ remains constant, Kodaira dimension zero corresponds to both eigenvalues being zero, i.e the Ricci-flat case including tori, K3 surfaces and their quotients, while Kodaira dimension one corresponds to exactly one eigenvalue being zero, and the surface being elliptic. Here the non-zero eigenvalue is negative.

Note that only in Kodaira dimension $-\infty$ there exist central metrics with differing eigenvalue behaviours of ρ (again, at least if the rank of ρ remains constant).

The following is a related conclusion from Demailly's theorem:

Theorem 7.11. *Let M_n be a compact complex manifold with anti-canonical line bundle $K^* = \Lambda^n T^{1,0}$. If M admits a Kähler metric of constant central curvature, having a non-degenerate Ricci form with q negative eigenvalues, then $h^q((K^*)^{\otimes k})$ has asymptotic growth n^k, and all other cohomology groups of K^* have slower growth rates.*

The proof is similar to that of Corollary 7.5. The proof of another vanishing theorem [AG, BU] shows that the other cohomology groups actually vanish asymptotically.

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