Inversion and characterization of the hemispherical transform[1]

BORIS RUBIN[2]

Preprint No. 14

1997/98

Abstract. Explicit inversion formulas are obtained for the hemispherical transform $(F\mu)(x) = \mu\{y \in S^n : x \cdot y \geq 0\}$, where S^n is the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, $n \geq 2$, μ is a finite Borel measure on S^n. If μ is absolutely continuous with respect to the Lebesgue measure dx on S^n, i.e. $d\mu(x) = f(x)dx$, we write $(Ff)(x) = \int_{x\cdot y>0} f(y)dy$ and consider the following cases: (a) $f \in C^\infty(S^n)$, (b) $f \in L^p(S^n)$, $1 \leq p < \infty$, and (c) $f \in C(S^n)$. In the case (a) our inversion formulas involve a certain polynomial of the Beltrami-Laplace operator. In the rest of the cases the relevant wavelet transforms are employed. The range of F is characterized and the action in the scale of Sobolev spaces $L^\gamma_p(S^n)$ is studied. For zonal $f \in L^1(S^2)$, the hemispherical transform Ff was inverted explicitly by P. Funk (1916). We reproduce his argument in higher dimensions.

---

[1]1991 *Mathematics Subject Classification*: Primary 42C10; Secondary 44A12.

Key words and Phrases: the hemispherical transform, spherical fractional integrals, wavelet transforms, Sobolev spaces.

[2]Partially supported by the Edmund Landau Center for research in Mathematical Analysis and Related Areas, sponsored by the the Minerva Foundation (Germany).

1. Introduction

The present article is a continuation of the series of papers [BR], [BrR], [Ru2-Ru8] devoted to applications of fractional integrals and wavelet transforms in integral geometry. In our treatment of the operator F we discriminate between the following problems:

(1) uniqueness of the solution of the homogeneous equation $F\mu = 0$;

(2) explicit determination of μ, if $(F\mu)(x)$ is known for all (or almost all) $x \in S^n$;

(3) characterization of the range of F.

Let us describe these problems shortly. After Funk [F], the problem (1) and similar problems for spherical caps of arbitrary radius were studied by Nakajima [N], Ungar [U], Schneider [Sch1, Sch2], Berenstein and Zalcman [BZ]; see also [Ga, p. 260], [Gr1, p. 1283], [Gr3] and references therein. Since the operators under consideration are $SO(n)$-invariant, the problem can be reduced (using the Funk-Hecke formula and table integrals) to investigation of zeros of the corresponding Gegenbauer polynomials. For the hemispherical transform F a final result then follows easily. If the radius r of the spherical cap differs from $\pi/2$, the situation becomes much more complicated. Here we only note that the kernel of the spherical cap transform is finite-dimensional if $r/\pi (\neq 1/2)$ is a rational number. This statement dramatically differs from the similar one for the hemispherical transform (the case $r/\pi = 1/2$), the kernel of which is infinitely dimensional and has a simple structure (see Lemma 2.3 below). The work on this problem, leading to diophantine approximations and small denominators, is now in progress, and further details will be published elsewhere.

The problem (2) arises in reconstructing a star-shaped body from its "half-volumes". Funk obtained an explicit inversion formula for Ff in the case of f zonal and $n = 2$, by reducing the problem to the Abel integral equation (for us it is an indication that certain multidimensional fractional integrals will arise in the general situation). In the nonzonal case, Funk suggested an averaging procedure, which enables one to reconstruct f, but cannot be regarded as an explicit inversion formula. The case of arbitrary $f \in L^2(S^2)$ was studied by Campi [C]. He has shown that each function φ, belonging to the Sobolev space $H^{3/2}(S^2)$ and such that $\varphi(x) + \varphi(-x) \equiv c = \text{const}$, can be represented in the form $\varphi = F[f + k]$ with $k = c/4\pi$. In the last formula $f \ \big(\in L^2(S^2)\big)$ is an odd function, for which a decomposition in spherical harmonics has been obtained. The function f is unique modulo even functions with the mean value 0. It is surprising, that after Funk (1916), as far as I know, no explict inversion formulas for the hemispherical transform were obtained in the closed form (even for smooth f). We complete this gap.

The problem (3), which is fairly elementary in the L^2-setting, becomes highly non-trivial if we want to characterize the range $F(L^p(S^n))$ for $p \neq 2$. It turns out that such a characterization is impossible in terms of Sobolev spaces because of oscillation of the corresponding Fourier-Laplace multiplier (see Theorem C below). In this situation the relevant wavelet transforms are helpful.

A bibliography at the end of the paper is not complete. References, related to the spherical Radon transform, which is an even counterpart of F, can be found in [Ga], [Gr3], [H1], [H2], [Ru2]. See also [BS], [BZ], [Z] about the Pompeiu problem, which is close to our topic.

NOTATION. In the following we use a standard notation for function spaces (e.g., $L^p = L^p(S^n),\ C = C(S^n),\ C^\infty = C^\infty(S^n)$). Similar spaces of odd functions are denoted by $L^p_{\text{odd}},\ C_{\text{odd}},\ C^\infty_{\text{odd}}$; $\mathcal{M} = \mathcal{M}(S^n)$ is the space of finite Borel measures μ on S^n; $(\mu, \omega) = \int_{S^n} \omega(x) d\mu(x),\ \omega \in C$. The abbreviation "$\lesssim$" indicates "$\le$" if the latter holds up to a constant multiple; $\mathbb{R}_+ = [0, \infty)$. We denote by $\{Y_{j,k}(x)\}$ an orthonormal basis of spherical harmonics on S^n. Here $j \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$; $k = 1, 2, \dots, d_n(j)$ where $d_n(j)$ is the dimension of the subspace of spherical harmonics of degree j. It is known ([M], p. 4) that

$$d_n(j) = (n + 2j - 1)\frac{(n+j-2)!}{j!\ (n-1)!}.$$

The Fourier-Laplace decomposition of $f \in C^\infty$ is written as $f = \sum_{j,k} f_{j,k} Y_{j,k}$; Δ designates the Beltrami-Laplace operator on S^n. We denote by $\mathcal{S}' = \mathcal{S}'(S^n)$ the dual of $C^\infty(S^n)$ (i.e., the space of distributions on S^n). For more information about analysis on S^n see, e.g., [BBP], [M], [Ru1], [S1], [S2] and references therein.

MAIN RESULTS. We first note that it suffices to handle odd functions f when dealing with Ff (see Section 2.2).

Theorem A. *Let $\varphi = Ff,\ f \in C^\infty_{\text{odd}}$. If n is odd, then*

$$f = P(\Delta) F\varphi, \quad P(\Delta) = 2^{-n-1}\pi^{1-n} \prod_{k=1}^{(n+1)/2} [-\Delta + (2k-2)(n+1-2k)]. \tag{1.1}$$

If n is even then

$$f = Q(\Delta)V^0\varphi, \quad Q(\Delta) = 2^{-n}\pi^{(1-n)/2}\prod_{k=1}^{n/2}[-\Delta + (2k-1)(n-2k)], \tag{1.2}$$

$$(V^0\varphi)(x) = \frac{1}{2\pi^{(n+1)/2}}\,\text{p.v.}\int\limits_{S^n}\frac{\varphi(y)}{x\cdot y}dy. \tag{1.3}$$

The above formulas are reminiscent of those for the spherical Radon transform (see [H1] for n odd, and [Ru4] for all $n \geq 2$). In fact, they are representatives of a **family** of formulas of such a type. For example, apart from (1.2), for n even one can get

$$f = \tilde{Q}(\Delta)\int\limits_{S^n}\varphi(y)(x\cdot y)\log\frac{1}{|x\cdot y|}dy + \frac{\Gamma(1+n/2)\Gamma((n+3)/2)}{\pi^{n+1/2}}\int\limits_{S^n}\varphi(y)(x\cdot y)dy, \tag{1.4}$$

$$\tilde{Q}(\Delta) = 2^{-n-1}\pi^{-n}\prod_{k=1}^{1+n/2}[-\Delta + (2k-3)(n+2-2k)], \qquad \varphi = Ff.$$

Now we pass to the "nonsmooth case". Let $\mathcal{M}_{\text{odd}} = \{\mu \in \mathcal{M}\colon\ (\mu,\omega) = -(\mu,\omega^-),\ \forall\omega \in C\}$, where $\omega^-(x) = \omega(-x)$. In order to invert $F\mu$ for $\mu \in \mathcal{M}_{\text{odd}}$, we introduce a wavelet transform

$$(W\varphi)(x,t) = \frac{1}{t}\int\limits_{S^n}\varphi(y)w\left(\frac{|x\cdot y|}{t}\right)sgn(x\cdot y)dy, \qquad x \in S^n,\ t > 0. \tag{1.5}$$

Here $w : \mathbb{R}_+ \to \mathbb{C}$ is an integrable function such that

$$\int\limits_0^\infty s^j w(s)ds = 0 \ \text{ for all } \ j = 1,3,\dots,\begin{cases} n & \text{if } n \text{ is odd,} \\ n-1 & \text{if } n \text{ is even,}\end{cases} \tag{1.6}$$

$$\int\limits_0^\infty s^\beta |w(s)|ds < \infty \ \text{ for some } \ \beta > n. \tag{1.7}$$

Denote

$$c_w = \begin{cases} \dfrac{2\pi^{n/2}\Gamma\left((1-n)/2\right)}{\Gamma(1+n/2)}\displaystyle\int\limits_0^\infty s^n w(s)\,ds & \text{if } n \text{ is even,} \\[2ex] \dfrac{4\pi^{n/2}(-1)^{(n+1)/2}}{\Gamma(1+n/2)\Gamma\left((n+1)/2\right)}\displaystyle\int\limits_0^\infty s^n w(s)\log s\,ds & \text{if } n \text{ is odd.}\end{cases} \tag{1.8}$$

Theorem B. (i) *If* $\varphi = F\mu$, $\mu \in \mathcal{M}_{\text{odd}}$, *then*

$$\Big(\int_0^\infty (W\varphi)(x,t)\frac{dt}{t^{1+n}}, \omega\Big) \equiv \lim_{\varepsilon\to 0}\Big(\int_\varepsilon^\infty (W\varphi)(x,t)\frac{dt}{t^{1+n}}, \omega\Big) = c_w(\mu,\omega), \qquad \forall \omega \in C. \quad (1.9)$$

(ii) *If* $\varphi = Ff$, $f \in L^p_{\text{odd}}$, $1 \le p \le \infty$ *(we keep the convention* $L^\infty_{\text{odd}} = C_{\text{odd}}$*), then*

$$\int_0^\infty \frac{(W\varphi)(x,t)}{t^{1+n}}dt \equiv \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{(W\varphi)(x,t)}{t^{1+n}}dt = c_w f(x) \quad \text{in the } L^p\text{-norm and a.e.} \qquad (1.10)$$

At the first glance the wavelet transform (1.5) looks exotic. It becomes clearer if we introduce the more general wavelet transform

$$(W_g\varphi)(x,t) = \frac{1}{t}\int_{S^n} \varphi(y) g\left(\frac{x\cdot y}{t}\right) dy, \qquad g : \mathbb{R} \to \mathbb{C}. \qquad (1.11)$$

For g odd, (1.11) can be written in the form (1.5). Without going into details we note that for sufficently nice g and φ (cf. Theorem 1.2 from [Ru2]),

$$\int_0^\infty (W_g\varphi)(x,t)\frac{dt}{t} = \alpha_+(R\varphi)(x) + \alpha_-(V^0\varphi)(x), \qquad (1.12)$$

where $(R\varphi)(x)$ is the spherical Radon transform (see (2.1) below) and $(V^0\varphi)(x)$ is the singular integral (1.3). The coefficients $\alpha_\pm$ depend on g and enjoy the following property: if g is even (odd), then $\alpha_- = 0$ ($\alpha_+ = 0$). The equality (1.12) can be regarded as a "spherical integral-geometrical" analogue of the generalized Calderón reproducing formula

$$\int_0^\infty (\mathcal{W}_g\varphi)(x,t)\frac{dt}{t} = \alpha_+\varphi(x) + \alpha_-(H\varphi)(x),$$

in which $(\mathcal{W}_g\varphi)(x,t) = t^{-1}\int_{-\infty}^{\infty}\varphi(y)g((x-y)/t)dy$ is a "usual" wavelet transforms on the real line, and $(H\varphi)(x)$ stands for the Hilbert transform (see Theorem 12.1 and the formula (12.13) in [Ru1]). For more explanations, related to (1.5), the reader is addressed to Section 5.

Our next result concerns the action of F in Sobolev spaces.

Definition 1.1. Given $\gamma\in\mathbb{R}$ and $p\in(1,\infty)$, the Sobolev space $L^\gamma_p=L^\gamma_p(S^n)$ is defined by

$$L^\gamma_p = \{f \in \mathcal{S}'\colon \sum_{j,k}(j+1)^\gamma f_{j,k}Y_{j,k} \sim f^{(\gamma)} \in L^p\}; \qquad \|f\|_{L^\gamma_p} = \|f^{(\gamma)}\|_p.$$

Theorem C. *Let* $\gamma = (n+1)/2 - |1/p - 1/2|(n-1)$, $\gamma' = (n+1)/2 + |1/p-1/2|(n-1)$. *Then*

$$L^{\gamma'}_{p,\mathrm{odd}} \subset F(L^p_{\mathrm{odd}}) \subset L^\gamma_{p,\mathrm{odd}}. \tag{1.14}$$

These embeddings are sharp in the sense that γ' *cannot be reduced and* γ *cannot be increased.*

Denote $H^\gamma_{\mathrm{odd}} = L^\gamma_{2,\mathrm{odd}}$. By (1.14),

$$F(L^2_{\mathrm{odd}}) = H^{(n+1)/2}_{\mathrm{odd}}. \tag{1.15}$$

This equality represents an n-dimensional generalization of the Campi's result [C].

The following statement characterizes the ranges $F(L^p_{\mathrm{odd}})$, $F(\mathcal{M}_{\mathrm{odd}})$.

Theorem D. *Assume that* $1 \le p \le \infty$, *and* μ *satisfies* (1.6), (1.7) *with* $c_w \ne 0$ (*see* (1.8)).
(i) *For* $\varphi \in L^p_{\mathrm{odd}}$ (*the space* L^∞_{odd} *is identified with* C_{odd}), *the following statements are equivalent:* (*a*) $\varphi \in F(L^p_{\mathrm{odd}})$; (*b*) *the integrals* $(J_\varepsilon\varphi)(x) = \int_\varepsilon^\infty (W\varphi)(x,t)dt/t^{1+n}$ *converge in the* L^p*-norm as* $\varepsilon \to 0$. *If* $1 < p < \infty$, *then* (*a*) *and* (*b*) *are equivalent to* (*c*) $\sup\limits_{0<\varepsilon<1} \|J_\varepsilon\varphi\|_p < \infty$.
(ii) *For* $\varphi \in L^1_{\mathrm{odd}}$, *the following statements are equivalent:* (*a*$'$) $\varphi \in F(\mathcal{M}_{\mathrm{odd}})$; (*b*$'$) *the integrals* $\int_{S^n}(J_\varepsilon\varphi)(x)\omega(x)dx$ *converge as* $\varepsilon\to 0$ *for any* $\omega\in C$; (*c*$'$) $\sup\limits_{0<\varepsilon<1}\|J_\varepsilon\varphi\|_1<\infty$.

In view of (1.15), Theorem D gives the following characterization of the space $H^{(n+1)/2}_{\mathrm{odd}}$:

$$H^{(n+1)/2}_{\mathrm{odd}} = \Big\{\varphi \in L^2_{\mathrm{odd}}\colon \sup_{0<\varepsilon<1}\Big\|\int_\varepsilon^\infty (W\varphi)(\cdot,t)\frac{dt}{t^{1+n}}\Big\|_2 < \infty\Big\}. \tag{1.16}$$

The paper is organized as follows. In Section 2 we introduce an analytic family $A^\alpha f$ of spherical fractional integrals (see (2.2) below), associated with the hemispherical transform F. These integrals are borrowed from the Fourier analysis on $\mathbb{R}^n$ ([GS], [Str1], [Es], [P], [S1], [S2]). Their application to studying the hemispherical transform seems to be new. Section 3 contains a proof of Theorem A and a justification of (1.4). Section 4 is devoted to the mapping properties of the spherical fractional integrals in Sobolev spaces. These properties provide the validity of Theorem C. Theorems B and D are established in Section 5. In Section 6 we reproduce the original argument of Funk [F] in the n-dimensional setting and reduce the equation $Ff = \varphi$ for zonal f to the Abel integral equation.

Acknowledgements. I express my gratitude to H. Furstenberg, J. Lindenstrauss, L. Zalcman, and M. Agranovsky for useful discussions. I am also grateful to R.S. Strichartz for numerous pleasant conversations and hospitality during my staying in Cornell University in summer 1997.

2. Preliminaries

2.1. Analytic families $A^\alpha, U^\alpha, V^\alpha$. Apart from the hemispherical transform $(Ff)(x) = \int_{x\cdot y>0} f(y)dy$, the following two operators play an important role in integral geometry [Gr2]:

$$(Rf)(x) = \int\limits_{x\cdot y=0} f(y)d_x y, \qquad (B_\alpha f)(x) = \int\limits_{S^n} f(y)|x\cdot y|^{\alpha-1}dy, \tag{2.1}$$

$Re\alpha > 0$; $\alpha \neq 1,3,5,\ldots$. The first one is the spherical Radon transform, in which integration is performed over the planar section $\{y \in S^n\colon\ x\cdot y = 0\}$. The second integral is known as the Blaschke-Levy representation [K], [Ru3]. For $\alpha = 2$, $B_\alpha f$ is also known as the cosine transform [Ga]. The key observation is that F, R, and B_α are members of the same operator family $\{A^\alpha\}$ and can be treated in the framework of a unified approach. For $0 < Re\alpha < 1$, this family is defined by

$$(A^\alpha f)(x) = \frac{\Gamma(1-\alpha)}{2^{1-\alpha}\pi^{(n+1)/2}} \int\limits_{S^n} (-ix\cdot y)^{\alpha-1} f(y)dy, \tag{2.2}$$

where

$$\begin{aligned}(-ix\cdot y)^{\alpha-1} &= \exp[(\alpha-1)\log|x\cdot y| + \frac{(1-\alpha)\pi i}{2}\mathrm{sgn}(x\cdot y)] = \\ &= |x\cdot y|^{\alpha-1}\big[\sin\frac{\alpha\pi}{2} + i\sin\frac{(1-\alpha)\pi}{2}\mathrm{sgn}(x\cdot y)\big].\end{aligned} \tag{2.3}$$

The operator (2.2) arises in evaluation of the Fourier transform of homogeneous functions (see, e.g., [Es], [GS], [P], [S1], [S2]). Namely, if $\varphi(\xi) = |\xi|^{-\alpha-n} f(\xi/|\xi|) \in C^\infty(\mathbb{R}^n\setminus\{0\})$, and $0 < Re\alpha < 1$, then

$$\int\limits_{\mathbb{R}^{n+1}} \varphi(\xi)e^{i\xi\cdot\eta}d\xi = c_\alpha|\eta|^{\alpha-1}(A^\alpha f)(\eta/|\eta|), \quad c_\alpha = 2^{1-\alpha}\pi^{(n+1)/2}. \tag{2.4}$$

The equality (2.4) and the definition (2.2) can be extended to all $\alpha \in \mathbb{C}$ in a suitable way. We recall that "L^p-L^q"-estimates for the spherical convolutions (2.2) were studied in [Str1] and [MP]. More general estimates in the scale of Sobolev spaces can be found in [Kr] (see also [Str3]).

Lemma 2.1. *If $f \in C^\infty$ and $0 < Re\alpha < 1$, then*

$$(A^\alpha f)(x) = \sum_{j,k} i^j a_{j,\alpha} f_{j,k} Y_{j,k}(x), \quad a_{j,\alpha} = \frac{\Gamma(j/2 + (1-\alpha)/2)}{\Gamma(j/2 + (n+\alpha)/2)}. \tag{2.5}$$

Furthermore,

$$A^\alpha f = U^\alpha f + iV^\alpha f, \tag{2.6}$$

$$(U^\alpha f)(x) = \frac{\Gamma\big((1-\alpha)/2\big)}{2\pi^{n/2}\Gamma(\alpha/2)} \int_{S^n} f(y)|x\cdot y|^{\alpha-1} dy, \tag{2.7}$$

$$(V^\alpha f)(x) = \frac{\Gamma(1-\alpha/2)}{2\pi^{n/2}\Gamma\left((1+\alpha)/2\right)} \int_{S^n} f(y)|x\cdot y|^{\alpha-1} sgn(x\cdot y) dy, \tag{2.8}$$

and the following relations hold:

$$(U^\alpha f)(x) = \sum_{j,k} u_{j,\alpha} f_{j,k} Y_{j,k}(x), \qquad u_{j,\alpha} = \begin{cases} (-1)^{j/2} a_{j,\alpha} & \text{if } j \text{ is even,} \\ 0 & \text{if } j \text{ is odd,} \end{cases} \tag{2.9}$$

$$(V^\alpha f)(x) = \sum_{j,k} v_{j,\alpha} f_{j,k} Y_{j,k}(x), \qquad v_{j,\alpha} = \begin{cases} 0 & \text{if } j \text{ is even,} \\ (-1)^{(j-1)/2} a_{j,\alpha} & \text{if } j \text{ is odd.} \end{cases} \tag{2.10}$$

Proof. By (2.3),

$$(-ix\cdot y)^{\alpha-1} = \frac{\pi^{1/2}|x\cdot y|^{\alpha-1}}{2^\alpha \Gamma(1-\alpha)} \left[\frac{\Gamma\big((1-\alpha)/2\big)}{\Gamma(\alpha/2)} + i \frac{\Gamma(1-\alpha/2)}{\Gamma\big((1+\alpha)/2\big)} sgn(x\cdot y) \right].$$

This implies (2.6). The decompositions (2.5), (2.9) and (2.10) can be obtained with the aid of the Funk-Hecke formula [M], which reads

$$\int_{S^n} a(x\cdot y) Y_{j,k}(y) dy = \lambda Y_{j,k}(x), \quad \lambda = \sigma_{n-1} \int_{-1}^{1} a(\tau)(1-\tau^2)^{n/2-1} H_j(\tau) d\tau, \tag{2.11}$$

$$H_j(\tau) = \frac{\Gamma(j+1)\ \Gamma(n-1)}{\Gamma(j+n-1)} C_j^{(n-1)/2}(\tau), \qquad \sigma_{n-1} = |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{2.12}$$

$C_j^{(n-1)/2}(\tau)$ being the Gegenbauer polynomials. By (2.11), $(U^\alpha Y_{j,k})(x) = u_{j,\alpha} Y_{j,k}(x)$, where $u_{j,\alpha}$ can be evaluated using the formula 2.21.2(5) from [PBM]. As a result we obtain (2.9). The proof of (2.10) is similar, and (2.5) then follows due to (2.6). $\square$

If $Re\alpha > 0$ and f is an integrable function, then the integrals (2.7) and (2.8) are absolutely convergent for $\alpha \neq 1, 3, 5, \dots$ and $\alpha \neq 2, 4, 6, \dots$ respectively. If $f \in C^\infty$ and

$Re\alpha \le 0$, then $U^\alpha f$ and $V^\alpha f$ can be defined as the multiplier operators (2.9) and (2.10), or by analytic continuation of the corresponding integrals (2.7), (2.8).

The operator $U^\alpha (V^\alpha)$ represents an even (odd) part of A^α and annihilates odd (even) functions f. One can readily see that

$$Rf = 2\pi^{(n-1)/2} U^0 f, \qquad B_\alpha f = \frac{2\pi^{n/2}\Gamma(\alpha/2)}{\Gamma\big((1-\alpha)/2\big)} U^\alpha f \tag{2.13}$$

for each $f \in L^1$, and

$$Ff = \pi^{(n-1)/2} V^1 f \ \text{ for each } \ f \in L^1_{\text{odd}}. \tag{2.14}$$

Some remarks are in order. We first note that for the singular values $\alpha = 1, 3, 5, \dots$ (for U^α) and $\alpha = 2, 4, 6, \dots$ (for V^α) the corresponding operators can be defined by continuity. Namely,

$$(U^{2m+1} f)(x) = \frac{(-1)^m}{\pi^{n/2}\ m!\ \Gamma(m+1/2)} \int\limits_{S^n} f(y)|x\cdot y|^{2m} \log \frac{1}{|x\cdot y|} dy, \tag{2.15}$$
$$m = 0, 1, 2, \dots;$$
$$(V^{2m} f)(x) = \frac{(-1)^{m-1}}{\pi^{n/2}\ (m-1)!\ \Gamma(m+1/2)} \int\limits_{S^n} f(y)(x\cdot y)^{2m-1} \log \frac{1}{|x\cdot y|} dy, \tag{2.16}$$
$$m = 1, 2, 3 \dots.$$

If $f_{j,k}$ are the Fourier-Laplace coefficients of $f \in L^1$, then

$$\lim_{\alpha\to 2m+1} U^\alpha f = U^{2m+1} f \quad \text{provided that} \quad f_{j,k} = 0 \ \forall j = 0, 1, \dots 2m, \tag{2.17}$$
$$\lim_{\alpha\to 2m} V^\alpha f = V^{2m} f \quad \text{provided that} \quad f_{j,k} = 0 \ \forall j = 0, 1, \dots 2m-1. \tag{2.17$'$}$$

The relations (2.15), (2.17) play an important role in [Ru4]. In order to check (2.17$'$) we put $\alpha = 2m + \varepsilon$ and write down $V^\alpha f$ in the form

$$(V^\alpha f)(x) = \frac{\Gamma(1-m-\varepsilon/2)}{2\pi^{n/2}\Gamma(m+(1+\varepsilon)/2)} \Big[\int\limits_{S^n} f(y)(x\cdot y)^{2m-1}(|x\cdot y|^\varepsilon - 1) dy + \int\limits_{S^n} f(y)(x\cdot y)^{2m-1} dy \Big].$$

If $f_{j,k} = 0 \ \forall j = 0, 1, \dots, 2m-1$, the second term $= 0$, and (2.17$'$) follows by the Lebesgue theorem on dominated convergence (for almost all $x \in S^n$).

The operator $V^0 = V^\alpha|_{\alpha=0}$ is of special interest, because it does not exist as the absolutely convergent integral of the form (2.8) and should be understood in the "principal value" sense:

$$(V^0 f)(x) = \lim_{\varepsilon \to 0} \frac{1}{2\pi^{(n+1)/2}} \int\limits_{|x\cdot y|>\varepsilon} \frac{f(y)}{x \cdot y} dy. \tag{2.18}$$

Lemma 2.2. *Let $f \in C^\infty$, $\alpha \in \mathbb{C}$; $\alpha \notin \mathfrak{A} = \{1,2,3,\ldots\} \cup \{-n,-n-1,-n-2,\ldots\}$. Then*

$$(A^{1-n-\alpha} A^\alpha f)(x) = f(-x). \tag{2.19}$$

Proof. For $0 < Re\alpha < 1$, the statement holds due to (2.5), because $(i^j a_{j,\alpha})(i^j a_{j,1-n-\alpha}) = (-1)^j$. For other values of α the result follows by analytic continuation. □

We note that if $\alpha \in \mathfrak{A}$, then (2.19) is still valid for all C^∞-functions f having a certain number (depending on α and n) of vanishing Fourier-Laplace coefficients. Furthermore, (2.19) gives rise to the following inversion formulas

$$(U^\alpha)^{-1} f = U^{1-n-\alpha} f, \quad (V^\alpha)^{-1} f = V^{1-n-\alpha} f, \tag{2.20}$$

which hold for $f \in C^\infty_{\text{even}}$ and $f \in C^\infty_{\text{odd}}$ respectively, and play an important heuristic role. For $Re\,(1-n-\alpha) \le 0$, the right-hand sides in (2.20) are understood as analytic continuations of the corresponding integrals. These analytic continuations can be realized in different forms: (1) in the form of the Fourier-Laplace series (see (2.9), (2.10)); (2) as the finite parts of divergent integrals [S1]; (3) in the sense of the Schwartz-Gelfand-Shilov regularization. The last interpretation was employed by V.I. Semyanistyi [Se2, Section 4] for $U^{1-n-\alpha} f$, who arrived at the first formula in (2.20) by using the more complicated way (via the Fourier transform on $\mathbb{R}^n$). We suggest the following realizations of the mentioned analytic continuations: (4) as integro-differential constructions involving a polynomial of the Beltrami-Laplace operator (here we follow the idea of Helgason [H1]); (5) as wavelet type representations. In accordance with (4), explicit expressions for $U^{1-n-\alpha} f$ have been given in [Ru4] for all $\alpha \in \mathbb{C}$ and all $n \ge 2$. Similar expressions for $V^{1-n-\alpha} f$ can be obtained by using the argument from Section 3. The method (5) was applied in [Ru2] and [Ru3] to the operator family $\{U^\alpha\}$. A general scheme of this method is described in [Ru8], and examples of its implementation can be found in [BR, BrR, Ru1-Ru7] . In the present paper we apply (5) for inversion and characterization of the hemispherical transform. In contrast to (1)-(4), the last method is applicable in the nonsmooth situation.

2.2. Passage to odd measures. We recall some definitions. A finite Borel measure $\mu \in \mathcal{M}$ is even if $(\mu, \omega) = (\mu, \omega^-)$ $\forall \omega \in C$, $\omega^-(x) = \omega(-x)$. Similarly, $\mu \in \mathcal{M}$ is odd if $(\mu, \omega) = -(\mu, \omega^-)$. The set of all even (odd) measures $\mu \in \mathcal{M}$ is denoted by $\mathcal{M}_{\text{even}}$ ($\mathcal{M}_{\text{odd}}$). For $\mu \in \mathcal{M}$, we define $\mu_+ \in \mathcal{M}_{\text{even}}$ and $\mu_- \in \mathcal{M}_{\text{odd}}$ by $(\mu_\pm, \omega) = (\mu, \omega_\pm)$, $\omega_\pm(x) = [\omega(x) \pm \omega(-x)]/2$, $\omega \in C$. Clearly, $\mathcal{M}_{\text{even}} = \{\mu \in \mathcal{M} : \mu = \mu_+\}$ and $\mathcal{M}_{\text{odd}} = \{\mu \in \mathcal{M} : \mu = \mu_-\}$.

The following properties of the operator $F\colon \mathcal{M} \to L^\infty$ seem to be known (cf. [F], [C] for $n = 2$). For the sake of completeness we present them in the most general form.

Lemma 2.3. $\text{Ker}F = \overset{\circ}{\mathcal{M}}_{\text{even}} = \{\mu \in \mathcal{M}_{\text{even}}\colon \mu(S^n) = 0\}$.

Proof. Let $\mu \in \overset{\circ}{\mathcal{M}}_{\text{even}}$. Given an arbitrary function $\omega \in C$, by putting $c_\omega = \int_{S^n} \omega(x)dx$, we have $(F\mu, \omega) = (\mu, F\omega) = (\mu, (F\omega)_+) = 2^{-1}[(\mu, F\omega) + (\mu, (F\omega)^-)] = 2^{-1}(\mu, c_\omega) = 0$, i.e., $\mu \in \text{Ker}F$. Conversely, let $\mu \in \text{Ker}F$. Then all Fourier-Laplace coefficients $(F\mu)_{j,k} = m_j \mu_{j,k}$ are zero. Since $m_0 = |S^n|/2$, $m_j = 0$ for $j = 2, 4, 6, \ldots$, and $m_j = \pi^{(n-1)/2} v_{j,1} \neq 0$ for $j = 1, 3, 5, \ldots$ (cf. (2.14), (2.10)), then $\mu_{j,k} = 0$ for $j = 0, 1, 3, 5, \ldots$. It follows that $\mu \in \overset{\circ}{\mathcal{M}}_{\text{even}}$. □

The next statement shows that inversion and characterization of F on the space $\mathcal{M}$ can be reduced to the similar problems on the space $\mathcal{M}_{\text{odd}}$.

Lemma 2.4. *Let $\mu \in \mathcal{M}$, $\varphi \in L^\infty$. Denote*

$$c_\mu = \mu(S^n), \quad c_\varphi = \int_{S^n} \varphi(x)dx, \quad \tilde{\varphi}(x) = \varphi(x) - \sigma_n^{-1} c_\varphi, \quad \sigma_n = |S^n|.$$

If $\varphi = F\mu$, then

$$\mu = \tilde{\mu} + 2\sigma_n^{-2} c_\varphi + \nu \tag{2.21}$$

where $\tilde{\mu} = \mu_- \in \mathcal{M}_{\text{odd}}$, $\nu = \mu_+ - \sigma_n^{-1} c_\mu \in \overset{\circ}{\mathcal{M}}_{\text{even}}$, and $F\tilde{\mu} = \tilde{\varphi}$. Conversely, if $F\tilde{\mu} = \tilde{\varphi}$ for some $\tilde{\mu} \in \mathcal{M}_{\text{odd}}$, then $\varphi = F\mu$ with $\mu = \tilde{\mu} + 2\sigma_n^{-2} c_\varphi \pmod{\overset{\circ}{\mathcal{M}}_{\text{even}}}$.

Proof. Let $\varphi = F\mu$. Then $\sigma_n^{-1} c_\varphi = 2^{-1} c_\mu$, and (2.21) can be checked as follows: $\mu = \mu_- + \mu_+ - \sigma_n^{-1} c_\mu + 2\sigma_n^{-2} c_\varphi = \tilde{\mu} + 2\sigma_n^{-2} c_\varphi + \nu$. Furthermore, for each $\omega \in C$,

$$
\begin{aligned}
(F\mu_-, \omega) &= (\mu_-, F\omega) = (\mu, (F\omega)_-) = 2^{-1}[(\mu, F\omega) - (\mu, (F\omega)^-)] = \\
&= 2^{-1}[(\varphi, \omega) - (\mu, c_\omega - F\omega)] = \qquad\qquad \Big(c_\omega = \textstyle\int_{S^n} \omega(x)dx\Big) \\
&= (\varphi - 2^{-1}c_\mu, \omega) = (\varphi - \sigma_n^{-1}c_\varphi, \omega) = (\tilde\varphi, \omega), \quad \text{i.e., } F\tilde\mu = \tilde\varphi.
\end{aligned}
$$

Conversely, if $F\tilde\mu = \tilde\varphi$ for some $\tilde\mu \in \mathcal{M}_{\text{odd}}$, and $\mu = \tilde\mu + 2\sigma_n^{-2}c_\varphi \pmod{\mathring{\mathcal{M}}_{\text{even}}}$, then, by Lemma 2.3, $F\mu = F\tilde\mu + \sigma_n^{-1}c_\varphi = \varphi$. $\square$

Lemma 2.4 shows that $F(\mathcal{M}) = F(\mathcal{M}_{\text{odd}}) + \mathbb{C}$ (instead of $\mathcal{M}$, one can put any space $X \subset \mathcal{M}$). Moreover, by (2.14) (written for finite measures), it enables us to treat the operator V^1 on $\mathcal{M}_{\text{odd}}$ rather than F on $\mathcal{M}$.

3. Inversion of $V^\alpha f$ for $f \in C^\infty_{\text{odd}}(S^n)$.

The following statement is an "odd copy" of Lemma 2.2 from [Ru4].

Lemma 3.1. *Given $\alpha \in \mathbb{C}$ and a nonnegative integer r, let*

$$
P_r^{(\alpha)}(\Delta) = \begin{cases} \text{the identity operator} & \text{for } r=0, \\ 4^{-r}\prod_{k=1}^{r}[-\Delta + (\alpha - 2r + 2k + n - 2)(2r - 2k + 1 - \alpha)] & \text{for } r \geq 1. \end{cases} \tag{3.1}
$$

Then for a spherical harmonic Y_j of odd degree j,

$$
P_r^{(\alpha)}(\Delta)V^{2r+1-\alpha-n}V^\alpha Y_j = Y_j \tag{3.2}
$$

provided $j \notin \{\alpha-1,\ \alpha-3,\ \alpha-5, \dots\} \cup \{2r-\alpha-n,\ 2r-\alpha-n-2,\ 2r-\alpha-n-4, \dots\}$.

The validity of (3.2) can be checked by direct computation, using (2.10), (2.5) and the equality

$$
-\Delta Y_j = j(j+n-1)Y_j. \tag{3.3}
$$

As in [Ru4], the equality (3.2) can be extended to all $f \in C^\infty_{\text{odd}}$. We omit these technicalities and confine ourselves to the case $\alpha = 1$, corresponding to the hemispherical transform.

Proof of Theorem A. Let $f \in C^\infty_{\text{odd}}$. If n is odd, we may write (3.2) with $\alpha = 1$, $r = (n+1)/2$, and replace Y_j by f. This gives $P^{(1)}_{(n+1)/2}(\Delta)V^1V^1 f = f$. By (2.14) and (3.1), the last equality coincides with (1.1). If n is even, we proceed as before, with $r = n/2$. $\square$

Since the integer r in (3.2) can be picked up in an infinite number of ways, a variety of inversion formulas is possible. For example, if n is even and $r = 1 + n/2$, then (3.2) yields $P^{(1)}_{1+n/2}(\Delta)V^2V^1Y_j = Y_j \quad \forall j = 3, 5, \dots$, where

$$P^{(1)}_{1+n/2}(\Delta) = 2^{-2-n} \prod_{l=1}^{1+n/2} [-\Delta + (2l-3)(n+2-2l)].$$

By (3.3), $P^{(1)}_{1+n/2}(\Delta)Y_{1,k} = 2^{-2-n}Y_{1,k}\prod_{l=1}^{1+n/2}[n + (2l-3)(n+2-2l)] = 0$, and therefore

$$P^{(1)}_{1+n/2}(\Delta)V^2V^1f = f - \sum_{k=1}^{d_n(1)} f_{1,k}Y_{1,k}, \quad \forall f \in C^\infty_{\text{odd}}, \qquad d_n(1) = n+1 \tag{3.4}$$

(cf. $d_n(j)$ in Notation). If $\psi = V^1f$, then, by (2.10), $\psi_{1,k} = a_{1,1}f_{1,k}$, where $a_{1,1} = \pi^{1/2}/\Gamma(1+n/2)$. Hence (3.4) yields

$$f = P^{(1)}_{1+n/2}(\Delta)V^2\psi + a_{1,1}^{-1}\sum_{k=1}^{n+1}\psi_{1,k}Y_{1,k}, \quad \psi_{1,k} = \int_{S^n}\psi(y)Y_{1,k}(y)dy.$$

Using the addition theorem for spherical harmonics

$$\sum_{k=1}^{n+1} Y_{1,k}(x)Y_{1,k}(y) = \frac{n+1}{\sigma_n}H_1(x\cdot y), \quad \sigma_n = |S^n| = 2\pi^{(n+1)/2}/\Gamma((n+1)/2),$$

(see, e.g., [E], [M]), and taking into account that $H_1(\tau) = \tau$ (cf. (2.12)), we get

$$f = P^{(1)}_{1+n/2}(\Delta)V^2\psi + \frac{\Gamma(1+n/2)\ \Gamma((n+3)/2)}{\pi^{1+n/2}}\int_{S^n}\psi(y)(x\cdot y)dy, \quad \psi = V^1f.$$

By (2.14) and (2.16), the last formula coincides with (1.4).

4. Mapping properties of V^α in Sobolev spaces

Theorem 4.1. *Let* $1 < p < \infty$, $\alpha \in \mathbb{C}$. (i) *If*

$$Re\alpha \geq \gamma - \beta - \frac{n-1}{2} + \left|\frac{1}{p} - \frac{1}{2}\right|(n-1), \tag{4.1}$$

then the operator V^α, *initially defined on* $f \in C^\infty$ *by the integral* (2.8) *or by the series* (2.10), *can be extended as a linear bounded operator acting from* L^β_p *into* L^γ_p.

(ii) *If* (4.1) *fails, then there exists* $\tilde f \in L^\beta_p$ *such that* $V^\alpha \tilde f \notin L^\gamma_p$.

Proof. Let $f = f_+ + f_-$, $f_\pm(x) = (f(x) \pm f(-x))/2$. Then $V^\alpha f = V^\alpha f_- = A^\alpha f_-$, and $\|f_-\|_{L^\beta_p} \le \|f\|_{L^\beta_p}$ (see Definition 1.1). By the Strichartz' multiplier theorem [Str2, Corollary from Theorem 1 on p. 115], the estimate $\|A^\alpha f_-\|_{L^\gamma_p} \lesssim \|f_-\|_{L^\beta_p}$ is equivalent to $\|A^1 f_-\|_{L^\delta_p} \lesssim \|f_-\|_p$, $\delta = \gamma - \beta - \mathrm{Re}\ \alpha + 1$. Since (4.1) implies the mentioned estimate of $A^1 f_-$ [Kr, G1-G3], then $\|V^\alpha f\|_{L^\gamma_p} = \|A^\alpha f_-\|_{L^\gamma_p} \lesssim \|f_-\|_{L^\beta_p} \le \|f\|_{L^\beta_p}$.

In order to prove (ii) we proceed as follows. Let $\lambda = (n-1)/2$,

$$\overset{\circ}{f}_\mu(x) \sim \sum_{j=1}^{\infty} (j+\lambda) j^{-\mu} C_j^\lambda(x_{n+1}), \qquad 0 < \mu < n,$$

$$f_\mu(x) \sim \sum_{j=1}^{\infty} s(j)(j+\lambda) j^{-\mu} C_j^\lambda(x_{n+1}), \quad s(j) = \frac{j^{-\lambda-\alpha'}}{a_{j,\alpha}} \Big(\frac{1+j(j+n-1)}{j^2}\Big)^{(\beta-\gamma)/2} \frac{j}{j+\lambda},$$

where $a_{j,\alpha}$ is defined in (2.5), $\alpha' = Re\ \alpha$. By Theorem 1 from [AW-I, p. 202], $\overset{\circ}{f}_\mu \in L^p$ provided $\mu > n - n/p$. By the Strichartz' multiplier theorem, $s(j)$ is an L^p-multiplier (the relevant computations can be performed using the properties of gamma functions exhibited in [Ru1, Appendix 2]; see also [Kr]). Hence, for

$$n - n/p < \mu < n, \tag{4.2}$$

we have $f_\mu \in L^p$, and $f_{\mu,\beta} \equiv (I-\Delta)^{-\beta/2} f_\mu \in L^\beta_p$. Let $\mathcal{F} = (I-\Delta)^{\gamma/2} A^\alpha f_{\mu,\beta}$. The above definitions, together with (2.5) and (3.3), yield

$$\mathcal{F}(x) \sim \sum_{j=1}^{\infty} i^j j^{\delta-\lambda-\mu} C_j^\lambda(x_{n+1}), \quad \delta = \gamma - \beta - \alpha' + 1.$$

Denote $x_{n+1} = \cos\sigma$. An examination of the proof of Theorem 1 from [AW-II] shows that

$$\mathcal{F}(x) \sim \Gamma(\delta-\mu)|\sigma-\pi/2|^{\mu-\delta}\Big[\cos\frac{\pi(\mu-\delta)}{2} + i\,\mathrm{sgn}(\sigma-\pi/2)\sin\frac{\pi(\mu-\delta)}{2}\Big], \quad \sigma \to \pi/2. \tag{4.3}$$

This relation is more precise than the similar one in [AW-II] (cf. (1.1) on p. 223). In order to derive (4.3), one should proceed as in [AW-II], but borrow more details from the Zygmund's book when stating Lemma 1 (see p. 224) .

Owing to (2.6) and (4.3), we have

$$(I-\Delta)^{\gamma/2} V^\alpha f_{\mu,\beta} \sim \Gamma(\delta-\mu)|\sigma-\pi/2|^{\mu-\delta}\mathrm{sgn}(\sigma-\pi/2)\sin\frac{\pi(\mu-\delta)}{2}, \quad \sigma \to \pi/2. \tag{4.4}$$

Assume that (4.1) fails. Let $1 < p \le 2$ (the required result for $2 < p < \infty$ can be obtained by duality), and put $\mu = \delta - 1/p, \quad \gamma_0 = \beta + \alpha' + n - 1 + 1/p$. If $\gamma < \gamma_0$, then $\delta < n + 1/p$, (4.2) is satisfied, $\tilde{f} \equiv f_{\mu,\beta} \in L_p^\beta$, and (4.4) yields $V^\alpha \tilde{f} \notin L_p^\gamma$. If $\gamma \ge \gamma_0$, we fix $\varepsilon \in (0, n/p)$ and put $\delta_\varepsilon = \gamma_0 - \varepsilon - \beta - \alpha' + 1 = n - \varepsilon + 1/p, \quad \mu = \delta_\varepsilon - 1/p = n - \varepsilon$. Then (4.2) is satisfied, and the above argument (with γ replaced by $\gamma_0 - \varepsilon$) yields $V^\alpha \tilde{f} \notin L_p^{\gamma_0 - \varepsilon}$ with $\tilde{f} \equiv f_{\mu,\beta} \in L_p^\beta$. In our case $L_p^\gamma \subset L_p^{\gamma_0 - \varepsilon}$, and therefore $V^\alpha \tilde{f} \notin L_p^\gamma$. □

Theorem 4.1 implies Theorem C. Indeed, the right embedding in (1.14) is clear due to (2.14). The left embedding follows from the observation that if $f \in L^{\gamma'}_{p,\mathrm{odd}}$, then $f = const\, V^1 V^{-n} f$ where $V^{-n} f \in L^p_{\mathrm{odd}}$ (use Theorem 4.1 with $\beta = \gamma'$ and $\gamma = 0$). The sharpness of (1.14) is a consequence of Theorem 4.1(ii) (since V^α annihilates even harmonics, one might handle the odd function $\tilde{f}_-(x) = (\tilde{f}(x) - \tilde{f}(-x))/2$ rather than $\tilde{f}$).

5. Proof of Theorems B and D

We first explain why the wavelet transform in these theorems has been chosen in the form (1.5). For sufficiently good $f : S^n \to \mathbb{C}$ and $w : \mathbb{R}_+ \to \mathbb{C}$,

$$(V^\alpha f)(x) = const \int_0^\infty (Wf)(x,t) \frac{dt}{t^{1-\alpha}}, \qquad Re\alpha > 0. \tag{5.1}$$

In order to check this equality one should plug (1.5) in the right-hand side and change the order of integration. Due to (2.20) and (5.1), the following inversion formula is expected for $\varphi = V^1 f$:

$$f(x) = (V^{-n}\varphi)(x) = const \int_0^\infty (W\varphi)(x,t) \frac{dt}{t^{1+n}}. \tag{5.2}$$

Theorem B gives this formula precise meaning. In order to prove Theorem B some preparations are needed. For $\tau \in (-1,1)$, let

$$(M_\tau f)(x) = \frac{(1-\tau^2)^{(1-n)/2}}{\sigma_{n-1}} \int_{x\cdot y = \tau} f(y) d\sigma(y), \qquad \sigma_{n-1} = |S^{n-1}|, \tag{5.3}$$

be the mean value of f on the planar section $\{y \in S^n\colon\ x \cdot y = \tau\}$. The following equality holds:

$$\int_{S^n} a(x\cdot y) f(y) dy = \sigma_{n-1} \int_{-1}^{1} a(\tau)(M_\tau f)(x)(1-\tau^2)^{n/2-1} d\tau \tag{5.4}$$

(see, e.g., [S1], p. 183). Furthermore [BBP],

$$\|M_\tau f\|_p \le \|f\|_p \ \text{ for } \ f \in L^p, \qquad 1 \le p \le \infty, \tag{5.5}$$

$$(M_\tau Y_j)(x) = H_j(\tau) Y_j(x) \quad \text{for each spherical harmonic } Y_j(x) \text{ of degree } j. \tag{5.6}$$

Here $H_j(\tau)$ is defined by (2.12). If j is odd, then

$$H_j(\tau) = \frac{\tau\Gamma\left((j+1)/2\right)\Gamma(n/2)}{\Gamma\left((j+n-1)/2\right)} P_{(j-1)/2}^{(n/2-1,1/2)}(2\tau^2 - 1) \tag{5.7}$$

with the Jacobi polynomial in the right-hand side. The equality (5.7) is a consequence of the relevant formulas from [E] (see 3.15.1(6) and the first formula from 10.8(16)).

We will deal with the Riemann-Liouville fractional integrals

$$(I_+^\alpha \psi)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\psi(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \qquad (I_-^\alpha \psi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 \frac{\psi(\tau)d\tau}{(\tau-t)^{1-\alpha}}, \qquad Re\alpha > 0. \tag{5.8}$$

Lemma 5.1. *If $\mu \in \mathcal{M}_{\text{odd}}$, $0 < \tau < 1$, then for almost all $x \in S^n$,*

$$(1-\tau^2)^{n/2-1}(M_\tau V^1 \mu)(x) = \tau(I_-^{(n-1)/2}\psi(\cdot, x))(\tau^2), \tag{5.9}$$

$$\psi(s,x) = \frac{2}{\sigma_{n-1}} s^{-n/2}(1-s)^{-1/2} \int\limits_{xy > \sqrt{1-s}} d\mu(y). \tag{5.10}$$

Proof. Since both sides of (5.9) represent spherical convolutions, belonging to L^1, it suffices to check (5.9) for $d\mu(x) = Y_j(x)dx$ with a spherical harmonic $Y_j(x)$ of odd degree. Due to (2.10) and (5.7), the Fourier-Laplace multiplier of the left-hand side of (5.9) reads

$$\tau(1-\tau^2)^{n/2-1}(-1)^{(j-1)/2} \frac{\Gamma\left((j+1)/2\right)\ \Gamma(n/2)\ \Gamma(j/2)}{\Gamma\left((j+n-1)/2\right)\ \Gamma\left((j+n+1)/2\right)} P_{(j-1)/2}^{(n/2-1,1/2)}(2\tau^2-1).$$

The same multiplier can be obtained for the right-hand side by using 2.22.2(2) from [PBM] and noting that the multiplier $m_j(s)$, corresponding to $\psi(s,x)$, has the form

$$m_j(s) = (1-s)^{-1/2} \frac{\Gamma\left((j+1)/2\right)\ \Gamma(n/2)}{\Gamma\left((j+n+1)/2\right)} P_{(j-1)/2}^{(n/2,-1/2)}(1-2s). \tag{5.11}$$

The latter can be checked with the aid of the Funk-Hecke formula (2.11), the equality (5.7), and the formula 2.22.2(7) [PBM]. $\square$

Proof of Theorem B. Put $w_1(s)=w(\sqrt{s}),\ h=I_+^{(n-1)/2}w_1$. We remark that $h\in L^1(\mathbb{R}_+)$ (see Lemma 4.12 from [Ru1]). If $\varphi(x)=(F\mu)(x)=\mu\{y\in S^n : x\cdot y\ge 0\}$, and $\psi(s,x)$ is the function (5.10), then

$$(W\varphi)(x,t)=\sigma_{n-1}\pi^{(n-1)/2}t^{n-2}\int_0^1\psi(s,x)h(s/t^2)ds. \tag{5.12}$$

Indeed, owing to (5.4) and (5.9), we have

$$(W\varphi)(x,t)=\frac{2\sigma_{n-1}}{t}\int_0^1(1-\tau^2)^{n/2-1}(M_\tau\varphi)(x)w\left(\frac{\tau}{t}\right)d\tau=$$

$$=\frac{2\sigma_{n-1}\pi^{(n-1)/2}}{t\Gamma\left((n-1)/2\right)}\int_0^1 w\left(\frac{\tau}{t}\right)\tau d\tau\int_{\tau^2}^1(s-\tau^2)^{(n-3)/2}\psi(s,x)ds=$$

$$=\frac{t^n\sigma_{n-1}\pi^{(n-1)/2}}{\Gamma\left((n-1)/2\right)}\int_0^{1/t^2}w_1(\eta)d\eta\int_\eta^{1/t^2}(\xi-\eta)^{(n-3)/2}\psi(t^2\xi,x)d\xi=$$

$$=t^n\sigma_{n-1}\pi^{(n-1)/2}\int_0^{1/t^2}\psi(t^2\xi,x)h(\xi)d\xi.$$

This gives (5.12). Denote $\lambda(s)=s^{-1}(I_+^{(n+1)/2}w_1)(s)$, $\quad(J_\varepsilon\varphi)(x)=\int_\varepsilon^\infty(W\varphi)(x,t)dt/t^{1+n}$,

$$(A_t\mu)(x)=\frac{1}{(1-t^2)^{n/2}}\int_{xy>t}d\mu(y)\quad\big(=2^{-1}\sigma_{n-1}t\psi(1-t^2,x)\big). \tag{5.13}$$

Up to abuse of notation, for $d\mu(x)=f(x)dx$ we shall write A_tf instead of $A_t\mu$. According to (5.12),

$$(J_\varepsilon\varphi)(x)=\pi^{(n-1)/2}\int_0^{1/\varepsilon^2}\frac{\lambda(s)}{\sqrt{1-\varepsilon^2 s}}(A_{\sqrt{1-\varepsilon^2 s}}\mu)(x)ds. \tag{5.14}$$

Indeed,

$$(J_\varepsilon\varphi)(x)=\frac{\sigma_{n-1}\pi^{(n-1)/2}}{2}\int_0^{1/\varepsilon^2}\psi(\varepsilon^2 s,x)\frac{ds}{s}\int_0^s h(\tau)d\tau\overset{(5.10)}{=}$$

$$=\pi^{(n-1)/2}\int_0^{1/\varepsilon^2}\frac{\lambda(s)ds}{(\varepsilon^2 s)^{n/2}\sqrt{1-\varepsilon^2 s}}\int_{xy>\sqrt{1-\varepsilon^2 s}}d\mu(y).$$

Our next goal is to pass to the limit in (5.14) as $\varepsilon \to 0$. In order to justify this passage, the following observations will be needed. By Lemma 2.4 from [Ru2],

$$\lambda(s) = \begin{cases} O(s^{(n-3)/2}) & \text{if } 0 < s \le 1, \\ O(s^{\gamma-1}) \text{ for some } \gamma < 0, & \text{if } s > 1, \end{cases} \tag{5.15}$$

$$\int\limits_0^\infty \lambda(s)ds = \begin{cases} 2\Gamma\left((1-n)/2\right) \int\limits_0^\infty s^n w(s)\,ds & \text{if } n \text{ is even}, \\ \\ \dfrac{4(-1)^{(n+1)/2}}{\Gamma\left((n+1)/2\right)} \int\limits_0^\infty s^n w(s) \log s\,ds & \text{if } n \text{ is odd} \end{cases} \tag{5.16}$$

(more general relations are established in [Ru1, Chapter 3]). For any $f \in L^p$ and $t \ge 0$, by (5.4) and (5.5) we have a uniform estimate

$$\|A_t f\|_p \le \frac{\sigma_{n-1}\|f\|_p}{(1-t^2)^{n/2}} \int\limits_t^1 (1-\tau^2)^{n/2-1} d\tau \le c\,\|f\|_p, \quad 1 \le p \le \infty, \quad c = c(n). \tag{5.17}$$

Moreover,

$$\sup_{t\in[0,1)} |(A_t f)(x)| < \sup_{t\in[0,1)} \frac{1}{(1-t)^{n/2}} \int\limits_{x\cdot y>t} |f(y)|dy \le c\,f^*(x) \tag{5.18}$$

where $f^*(x)$ is the Hardy-Littlewood maximal function on S^n, defined by

$$f^*(x) = \sup_{t\in(-1,1)} \frac{1}{|\sigma_t(x)|} \int\limits_{\sigma_t(x)} |f(y)|dy, \qquad \sigma_t(x) = \{y \in S^n : x \cdot y > t\}.$$

Similarly, for any $\mu \in \mathcal{M}$,

$$\sup_{t\in[0,1)} \|A_t \mu\|_1 \le c\,\|\mu\| \tag{5.19}$$

where $\|\mu\|$ designates the total variation of $|\mu|$ (this estimate will be needed later). For each spherical harmonic Y_j of odd degree j, by (5.13) and (5.11) we have

$$A_t Y_j = \mu_j(t) Y_j, \quad \mu_j(t) = \frac{\sigma_{n-1} t}{2} m_j(1-t^2) = \frac{\pi^{n/2}\Gamma\left((j+1)/2\right)}{\Gamma\left((j+n+1)/2\right)} P^{(n/2,-1/2)}_{(j-1)/2}(2t^2-1), \tag{5.20}$$

and therefore (use 10.8(3) from [E])

$$\mu_j(1) = \pi^{n/2}/\Gamma(1+n/2). \tag{5.21}$$

Now we can resume our argument. Let $f \in L^p_{\text{odd}}$, $1 \le p \le \infty$ (we recall that L^∞_{odd} is identified with C_{odd}). Owing to (5.17), (5.18), (5.20), and (5.21), a standard machinery of approximation to the identity yields

$$\lim_{t\to 1} A_t f = \frac{\pi^{n/2}}{\Gamma(1+n/2)}\, f \ \text{ in the } \ L^p\text{-norm and a.e.} \tag{5.22}$$

By (5.14), (5.15), and (5.17),

$$\|J_\varepsilon F f\|_p \lesssim \|f\|_p\Big(\int\limits_0^{1/2\varepsilon^2} |\lambda(s)|ds + \int\limits_{1/2\varepsilon^2}^{1/\varepsilon^2} \frac{s^{\gamma-1}}{\sqrt{1-\varepsilon^2 s}}ds\Big) \le \|f\|_p\Big(\|\lambda\|_1 + \varepsilon^{2|\gamma|}\int\limits_{1/2}^{1}\frac{\eta^{\gamma-1}}{\sqrt{1-\eta}}d\eta\Big).$$

Hence

$$\sup_{0<\varepsilon<1} \|J_\varepsilon F f\|_p \lesssim \|f\|_p, \qquad 1\le p\le\infty. \tag{5.23}$$

Similarly, by (5.18), $\sup\limits_{0<\varepsilon<1} |(J_\varepsilon F f)(x)| \lesssim f^*(x)$. This relation, together with (5.22), (5.23) and (5.16), enables us to pass to the limit in (5.14) (with μ replaced by f) and to obtain the statement (ii) of Theorem B. Let us prove (i). Owing to (ii), for any $\omega\in C$ we have $(J_\varepsilon F\mu,\omega) = (\mu, J_\varepsilon F\omega) \to c_w(\mu,\omega)$ as $\varepsilon\to 0$. This imples (1.9). $\square$

Proof of Theorem D. (i) The implication $(a) \Rightarrow (b)$ follows from Theorem B. The validity of "$(a) \Rightarrow (c)$" follows from (5.23). In order to prove "$(b) \Rightarrow (a)$" we denote $f = c_w^{-1} \overset{(L^p)}{\lim\limits_{\varepsilon\to 0}} J_\varepsilon\varphi$. Clearly, f is odd. Then $Ff = c_w^{-1}\overset{(L^p)}{\lim\limits_{\varepsilon\to 0}} FJ_\varepsilon\varphi = c_w^{-1}\overset{(L^p)}{\lim\limits_{\varepsilon\to 0}} J_\varepsilon F\varphi = \varphi$ (here the L^p-boundedness of F and Theorem B have been used). Let us prove "$(c) \Rightarrow (a)$". Since the ball in L^p is compact in the weak* topology, there exist a sequence $\varepsilon_k\to 0$ and a function $f_0\in L^p$ such that $\lim_{\varepsilon_k\to 0}(J_{\varepsilon_k}\varphi,\psi) = (f_0,\psi)$ for each $\psi\in L^{p'}$. Clearly, f_0 is odd, because the functions $J_{\varepsilon_k}\varphi$ are odd. Put $f = c_w^{-1} f_0$. Then $(Ff,\psi) = (f, F\psi) = \lim\limits_{\varepsilon_k\to 0} c_w^{-1}(J_{\varepsilon_k}\varphi, F\psi) = \lim\limits_{\varepsilon_k\to 0} c_w^{-1}(J_{\varepsilon_k}F\varphi,\psi) = (\varphi,\psi)$, i.e. $\varphi = Ff$.

(ii) The implication $(a') \Rightarrow (b')$ follows from Theorem B. The validity of "$(a') \Rightarrow (c')$" follows from the estimate $\sup\limits_{0<\varepsilon<1}\|J_\varepsilon F\mu\|_1 \lesssim \|\mu\|$, the proof of which is similar to (5.23) and based on (5.19). Let us prove "$(b') = (a')$". Since the space of finite Borel measures on S^n is weakly complete, then there is a finite Borel measure μ such that $\lim\limits_{\varepsilon\to 0}(J_\varepsilon\varphi,\omega) = (\mu,\omega) \quad \forall\omega\in C$. Obviously, μ is odd. Furthermore, for an arbitrary infinitely differentiable function ψ, by Theorem B we have $(F\mu,\psi) = (\mu, F\psi) = \lim\limits_{\varepsilon\to 0}(J_\varepsilon\varphi, F\psi) = \lim\limits_{\varepsilon\to 0}(J_\varepsilon F\varphi,\psi) = c_w(\varphi,\psi)$. This implies $\varphi \overset{a.e.}{=} c_w^{-1}F\mu$. The proof of the implication "$(c') \Rightarrow (a')$" is similar to that of "$(c) \Rightarrow (a)$" in (i). $\square$

6. The Funk equation for zonal functions in higher dimensions

Assume that f is a zonal function, belonging to $L^1_{\rm odd}$, i.e. $f(y) \equiv f_0(y_{n+1})$ for a certain odd function f_0 on $[-1,1]$. Then $(Ff)(x) \equiv \int_{x\cdot y>0} f(y)dy$ is also zonal and odd. Let us transform $(Ff)(x)$ by passing to "polar coordinates" on S^n. Put

$$x = (\sin r)e_n + (\cos r)e_{n+1}, \quad y = (\sin\rho)\omega + (\cos\rho)e_{n+1}.$$

Here r and ρ lie in $(0,\pi)$, $\omega \in S^{n-1}$, e_n and e_{n+1} designate the coordinate unit vectors. We define $\chi(s)$ by setting $\chi(s) \equiv 0$ for $s<0$ and $\chi(s) \equiv 1$ for $s \ge 0$. Then

$$(Ff)(x) = \int_0^\pi f_0(\cos\rho)(\sin\rho)^{n-1}d\rho \int_{S^{n-1}} \chi\left((\omega e_n)\sin r\sin\rho + \cos r\cos\rho\right)d\omega =$$

$$= \sigma_{n-2}\int_{-1}^1 f_0(\tau)(1-\tau^2)^{n/2-1}d\tau \int_{-1}^1 (1-t^2)^{(n-3)/2}\chi(t\sqrt{1-\tau^2}\sin r + \tau\cos r)dt =$$

$$= \frac{\sigma_{n-2}}{(\sin r)^{n-2}}\int_{-1}^1 f_0(\tau)d\tau \int_{-\sqrt{1-\tau^2}\sin r}^{\sqrt{1-\tau^2}\sin r} \left((1-\tau^2)\sin^2 r - \eta^2\right)^{(n-3)/2}\chi(\eta+\tau\cos r)d\eta.$$

Put

$$(Ff)(x) = \begin{cases} \varphi_0(\sin r) & \text{if } 0<r<\pi/2, \\ -\varphi_0(\sin r) & \text{if } \pi/2<r<\pi. \end{cases}$$

Assuming $0<r<\pi/2$, i.e. $\cos r>0$, and taking into account that f_0 is odd, we obtain

$$\varphi_0(s) \overset{\rm def}{=}$$

$$= \frac{\sigma_{n-2}}{s^{n-2}}\int_0^1 f_0(\tau)d\tau \int_{-s\sqrt{1-\tau^2}}^{s\sqrt{1-\tau^2}} \left((1-\tau^2)s^2-\eta^2\right)^{(n-3)/2}[\chi(\eta+\tau\sqrt{1-s^2})-\chi(\eta-\tau\sqrt{1-s^2})]d\eta =$$

$$= \frac{\sigma_{n-2}}{s^{n-2}}\Bigg(\int_0^s f_0(\tau)d\tau \int_{-\tau\sqrt{1-s^2}}^{\tau\sqrt{1-s^2}} + \int_s^1 f_0(\tau)d\tau \int_{-s\sqrt{1-\tau^2}}^{s\sqrt{1-\tau^2}}\Bigg)\left((1-\tau^2)s^2-\eta^2\right)^{(n-3)/2}d\eta =$$

$$= 2\sigma_{n-2}\Bigg(\int_0^s \int_0^{\tau\sqrt{1-s^2}/s\sqrt{1-\tau^2}} + \int_s^1\int_0^1\Bigg) f_0(\tau)(1-\tau^2)^{n/2-1}(1-t^2)^{(n-3)/2}d\tau dt.$$

Differentiation of the last expression yields

$$\frac{d\varphi_0(s)}{ds} = -\frac{2\sigma_{n-2}s^{1-n}}{\sqrt{1-s^2}}\int_0^s f_0(\tau)(s^2-\tau^2)^{(n-3)/2}\tau d\tau.$$

The following theorem resumes our argument.

Theorem 6.1. *Let $n \geq 2$, $\varphi = Ff$, where $f(\in L^1_{\text{odd}})$ is a zonal function. If $f(x) = f_0(x_{n+1})$ and $\varphi(x) = \varphi_0\left(\sqrt{1-x_{n+1}^2}\right)$ for $x_{n+1} > 0$, then f_0 satisfies the integral equation*

$$\int_0^s f_0(\tau)(s^2-\tau^2)^{(n-3)/2}\tau d\tau = -\frac{s^{n-1}\sqrt{1-s^2}}{2\sigma_{n-2}}\frac{d\varphi_0(s)}{ds} \tag{6.1}$$

of the Abel type.

The equation (6.1) can be solved in a standard way ([Ru1], [SKM]).

References

[AW] Askey, R., and Wainger, S. On the behaviour of special classes of ultraspherical expansions, I, II. *J. Anal. Math.* **15** (1965), 193–220, 221–244.

[BS] Bagchi, S.C., and Sitaram, A. The Pompeiu problem revisited. *L'Enseignement Math.* **36**, 67-91, (1990).

[BBP] Berens, H., Butzer, P.L., and Pawelke, S. Limitierung verfahren von Reihen mehrdimensionaler kugelfunktionen und deren Saturationsverhalten. *Publs. Res. Inst. Math. Sci. Ser. A,* **4**, No. 2, 201–268 (1968).

[BR] Berenstein, C.A. and Rubin, B. Radon transform of L^p-functions on the Lobachevsky space and hyperbolic wavelet transforms, The Hebrew University of Jerusalem, Prepirnt, No. 1, 1996/97. Submitted to Forum Math.

[BZ] Berenstein, C.A. and Zalcman, L. Pompeiu's problem on spaces of constant curvature, *J. Anal. Math.* **30**, 113–130 (1976).

[BrR] Bray, W.O. and Rubin, B. Inversion of horocycle transforms in the hyperbolic space by using wavelet transforms, in *Proc. of the International Workshop in Analysis and its Applications (Seventh Meeting), Maine, June 1-6, 1997.*, (to appear) .

[C] Campi, S. On the reconstruction of a star-shaped body from its "half-volumes". *J. Austral. Math. Soc. (Ser. A)* **37**, 243–257 (1984).

[E] Erdélyi, A. (Editor). "Higher transcendental functions", Vol. I, II, McGraw-Hill, New York, 1953.

[Es] Eskin, G. I. "Boundary value problems for elliptic pseudodifferential equations", Amer. Math. Soc., Providence, R.I., 1981.

[F] Funk, P. Über eine geometrische Anwendung der Abelschen Integralgleichung. *Math. Ann.* **77**, 129–135 (1916).

[G1] Gadzhiev, A.D. On differential properties of the symbol of a multidimensional singular integral operator, *Mat. Sbornik*, **114**, 483–510 (1981).

[G2] Gadzhiev, A.D. Differential properties of the symbol of a singular operator in spaces of Bessel potentials on the sphere, *Izvestia Akademii Nauk Azerbaidzhanskoi SSR. Ser. fiziko-tekhnicheskikh i matem. nauk*, Baku, Izd-vo AN Azerb. SSR, no. 1, 134–140 (1982), in Russian.

[G3] Gadzhiev, A.D. Exact theorems on multipliers of spherical expansions and some applications, in "Special problems of function theory", vyp. 4, Baku, Izd-vo Elm, 73–100 (1989), in Russian.

[Ga] Gardner, R.J. "Geometric tomography", Cambridge University Press, New York, 1995.

[GS] Gelfand, I.M. and Shapiro, Z. Ja. Homogeneous functions and their applications. *Uspekhi Mat. Nauk,* **10**, no. 3, 3–70 (1955), in Russian.

[Gr1] Groemer, H. Fourier series and spherical harmonics in convexity, in *Handbook of convex Geometry*, Edited by P.M. Gruber and J.M. Wils, Elsevier Science Publishers B.V., 1993, 1259–1295.

[Gr2] Groemer, H. Stability results for convex bodies and related spherical integral transformations. *Adv. Math.* **109**, 45–74 (1994).

[Gr3] Groemer, H. "Geometric applications of Fourier series and spherical harmonics", Cambridge University Press, New York, 1996.

[H1] Helgason, S. "The Radon transform", Birkhäuser, Boston, 1980.

[H2] Helgason, S. "Geometric analysis on symmetric spaces", Amer. Math. Soc., Providence, R.I., 1994.

[K] Koldobsky, A. Inversion formula for the Blaschke-Levy representation with applications to zonoids and sections of star bodies. Preprint.

[Kr] Kryuchkov, V. S. Differential properties of the symbol of a Calderón-Zygmund singular integral operator. *Proc. of the Steklov Inst. of Math.* **170**, Issue 1, 169–183 (1987).

[MP] Meda, S. and Pini, R. Spherical convolutions with kernels having singularities on an equator, *Bollettino U.M.I.* (7) 5-B , 275–290 (1991).

[M] Müller, C. "Spherical harmonics", Springer, Berlin-Heidelberg- New York, 1966.

[N] Nakajima, S. On some characteristic properties of curves and surfaces, *Tohôku Math. J.* **18**, 272–287 (1920).

[P] Plamenevskii, B. A. "Algebras of pseudodifferential operators", Kluwer Acad. Publ., Dordrecht, 1989.

[PBM] Prudnikov, A. P., Brychkov, Y. A., and Marichev, O. I. "Integrals and series: special functions", Gordon and Breach Sci. Publ., New York-London, 1986.

[Ru1] Rubin, B. "Fractional integrals and potentials", Addison Wesley Longman, Essex, U.K., 1996.

[Ru2] Rubin, B. Spherical Radon transforms and related wavelet transforms. *Applied and Computational Harmonic Analysis* (to appear).

[Ru3] Rubin, B. Fractional integrals and wavelet transforms associated with Blaschke-Levy representations on the sphere, *Israel J. of Math.* (to appear).

[Ru4] Rubin, B. Inversion of fractional integrals related to the spherical Radon transform, The Hebrew University of Jerusalem, Preprint No. 4, 1996/97 (submitted to *Journal of Func. Analysis*).

[Ru5] Rubin, B. Inversion of k-plane transforms via continuous wavelet transforms. *Journal of Math. Analysis and Applications* (to appear).

[Ru6] Rubin, B. Inversion of Radon transforms using wavelet transforms generated by wavelet measures. *Math. Scand.* (to appear).

[Ru7] Rubin, B. The Calderón reproducing formula, windowed X-ray transforms and Radon transforms in L^p-spaces. *The Journal of Fourier Analysis and Applications* (to appear).

[Ru8] Rubin, B. Fractional calculus and wavelet transforms in integral geometry, *Fractional Calculus and Applied Analysis* (to appear).

[S1] Samko, S. G. Generalized Riesz potentials and hypersingular integrals with homogeneous characteristics, their symbols and inversion, *Proceeding of the Steklov Inst. of Math.* **2**, 173–243 (1983).

[S2] Samko, S. G. Singular integrals over a sphere and the construction of the characteristic from the symbol, *Soviet Math. (Iz. VUZ).* **27**, No. 4, 35–52 (1983) .

[SKM] Samko, S. G, Kilbas, A. A., and Marichev, O. I. "Fractional integrals and derivatives. Theory and applications", Gordon and Breach, London, 1993.

[Sch1] Schneider, R. Functions on a sphere with vanishing integrals over certain subspheres, *J. math. Anal. Appl.* **26**, 381–384 (1969).

[Sch2] Schneider, R. Über eine Integralgleichung in der Theorie der konvexen Körper, *Math. Nachr.* **44**, 55–75 (1970).

[Se1] Semyanistyi, V.I. On some integral transformations in Euclidean space. *Dokl. Akad. Nauk SSSR* **134**, No. 3, 536–539 (1960), in Russian.

[Se2] Semyanistyi, V.I. Some integral transformations and integral geometry in an elliptic space. *Trudy Sem. Vektor. Tenzor. Anal.* **12**, 397–441 (1963), in Russian.

[Str1] Strichartz, R. S. Convolutions with kernels having singularities on a sphere, *Trans. Amer. Math. Soc.* **148**, 461–471 (1970).

[Str2] Strichartz, R. S. Multipliers for spherical harmonic expansions, *Trans. Amer. Math. Soc.* **167**, 115–124 (1972).

[Str3] Strichartz, R. S. L^p-estimates for Radon transforms in euclidean and non-euclidean spaces. *Duke Math. J.* **48**, 699–727 (1981).

[U] Ungar, P. Freak theorem about functions on a sphere, *J. London Math. Soc.* **29**, 100–103 (1959).

[Z] Zalcman, L. A bibliographic survey of the Pompeiu problem, in *Approximation by solutions of partial differential equations*, B. Fuglede et al. (eds.), Kluwer Acad. Publ., 1992, 185-194.

Department of Mathematics, The Hebrew University, 91904, Jerusalem, Israel

Email: boris@math.huji.ac.il