ALMOST LOCALLY MINIMAL PROJECTIONS IN FINITE DIMENSIONAL BANACH SPACES

M. Zippin*

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Abstract. A projection P on a Banach space X is called "almost locally minimal" if, for every $\alpha > 0$ small enough, the ball $B(P, \alpha)$ in the space $L(X)$ of all operators on X contains no projection Q with $\|Q\| \leq \|P\|(1 - D\alpha^2)$ where D is a constant. A necessary and sufficient condition for P to be almost locally minimal is proved in the case of finite dimensional spaces. This criterion is used to describe almost locally minimal projections on ℓ_1^n.

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1. Introduction

This work deals with projections on a finite dimensional Banach space X which have "almost" locally minimal norms (see the definition in Section 2). It will make little sense to discuss these projections without mentioning the following fundamental open problem which we hope to better understand by studying those "almost locally minimal" objects.

Problem 1.1. Does there exist a function $\psi(\lambda)$ defined for $\lambda \geq 1$ such that, for every $n \geq 1$, $1 \leq k \leq n$ and every projection P on ℓ_1^n with rank $P = k$ and $\|P\| = \lambda$, $d(P(\ell_1^n), \ell_1^k) \leq \psi(\lambda)$?

($d(Y, Z)$ is the Banach-Mazur distance between the isomorphic spaces Y and Z).

The problem has a positive answer in the special case of small λ's:

Theorem 1.2 ([Z-1] [Z-2]). *There exists a function $\psi(\lambda)$ defined for $1 \leq \lambda \leq 1.01$ with $\lim_{\lambda \to 1} \psi(\lambda) = 1$ such that, for every $n \geq 1$, $1 \leq k \leq n$ and every projection P on ℓ_1^n with rank $P = k$ and $\|P\| = \lambda \leq 1.01$, $d(P(\ell_1^n), \ell_1^k) \leq \psi(\lambda)$.*

Theorem 1.2 is proved by constructing a projection Q of norm 1 which is close to P. Whenever P and Q are projections on a space X and $\|P - Q\| = \alpha < 1$, the operator $J = Q|_{P(X)}$ is an isomorphism of $P(X)$ onto $Q(X)$ with $\|Je - e\| = \|Qe - Pe\| \leq \|Q - P\|\|e\| \leq \alpha\|e\|$ for all $e \in P(X)$; hence $d(P(X), Q(X)) \leq (1-\alpha)^{-1}(1+\alpha)$. This raises the following question: Given a projection P with $\|P\| = \lambda > 1$ on ℓ_1^n, can we find a sequence of projections $\{P_i\}_{i=0}^N$ on ℓ_1^n with $\max\{\|P_{i-1} - P_i\| : 1 \leq i \leq N\} \leq \alpha$ such that $P_0 = P$, $\|P_N\| = 1$ and $N\alpha \leq \varphi(\lambda)$, where $\varphi(\lambda)$ is independent on n and $k = \text{rank}(P)$? The existence of such a sequence will settle Problem 1.1 because

$$d(P(X), P_N(X)) \leq [(1-\alpha)^{-1}(1+\alpha)]^N \leq e^{4\varphi(\lambda)}$$

if $\alpha < \frac{1}{2}$. Let us pause for the following

Remark 1.3. It follows from Theorem 1 of [D-Z] that whenever P is a projection on ℓ_1^n with $\|P\| = \lambda$, rank$P = k$ and $d(P(X), \ell_1^k) \leq \mu$ one can embed ℓ_1^n isometrically in ℓ_1^{nk} and extend P to a projection $\tilde{P}$ of ℓ_1^{nk} onto $P(X)$ in a natural way so that $\tilde{P}$ admits a sequence $\{P_i\}_{i=0}^N$ of projections with $P_0 = \tilde{P}$, $\|P_N\| = 1$ and $\max\{\|P_{i-1} - P_i\| \ 1 \leq i \leq N\} = \alpha$ where $N\alpha \leq 2\lambda\mu$.

Unfortunately we are far from obtaining such a sequence without knowing in advance that $d(P(X), \ell_1^k)$ is under control. The main difficulty is the lack of information about the structure of the set $\pi(k)$, of projections of a fixed rank k, on a space X.

We know that $\pi(k)$ is a closed connected set which is not convex. it turns out that $\pi(k)$ is "almost" locally convex in the following sense:

Proposition 1.4. *Let $0 < \alpha \le 1/8$ and let P and Q be projections on a Banach space X with $\|P\|$, $\|Q\| < \lambda$, $\|P - Q\| = \alpha$ and* $\operatorname{rank} P = k$. *Then there is a continuous function $\theta : [0,1] \to \pi(k)$ such that $\theta(0) = P$, $\theta(1) = Q$ and, for each $0 < t < 1$ $\|\theta(t) - ((1-t)P + tQ)\| \le t(1-t)C\alpha^2$ where $C = C(\lambda)$ does not depend on k, X or the particular projections.*

Proposition 1.4 is a consequence of the following lemma which was proved in [Z-2]:

Lemma 1.5. *Let X be a Banach space and let $\lambda_0 > 1$. There exist a constant $C = C(\lambda_0)$ and a continuous function $\beta(T)$ defined for all operators T on X which satisfy the conditions $\|T\| \le \lambda_0$ and $\|T^2 - T\| = \alpha \le \frac{1}{8}$, such that $\beta(T)$ is a proejction and $\|\beta(T) - T\| \le C\alpha$.*

Proof of Proposition 1.4. For every $0 \le t \le 1$ put $T_t = (1-t)P + tQ$ then $\|T_t\| < \lambda_0$ and $\|T_t^2 - T_t\| = \|t(1-t)(P-Q)^2\| \le t(1-t)\alpha^2$. By Lemma 1.5, for each $0 < t < 1$ there is a projection $\theta(t) = \beta(T_t)$ such that

$$\|(1-t)P + tQ - \theta(t)\| = \|T_t - \beta(T_t)\| \le t(1-t)C\alpha^2.$$

Since β is continuous, so is $\theta(t)$. □

In the sequel $C = C(\lambda_0)$ will denote the constant appearing in Lemma 1.5. An explicit expression for C can be $C = 4(2\lambda_0 + 1)e$.

2. Almost minimal projections

The construction of the desired sequence of projections $\{P_i\}_{i=0}^N$ for which $\|P_0\| = \|P\| = \lambda > 1$ and $\|P_N\| = 1$ must be essentially a norm reduction process. In each step we like to find in a ball $B(P_i, \alpha)$ of radius α around P_i another projection P_{i+1} of smaller norm. What happens if this cannot be done at a reasonable pace? The following discussion of this question deals with one type of a norm reduction pace which seems to be useful.

Definition 2.1. Let $\lambda_0 > 1$, $D = 10 + 4\lambda_0$ and $0 < \alpha < (8\lambda_0)^{-1}$. A projection P with $\|P\| = \lambda < \lambda_0$ on X is called **almost α-minimal** if the ball $B(P, \alpha)$ does not contain a projection Q with $\|Q\| \leq \lambda(1 - D\alpha^2)$. The projection P is said to be **almost locally minimal** if it is almost α-minimal for all $0 < \alpha < (8\lambda)^{-1}$.

Remark 2.2. The size of the constant D appearing in Definition 2.1 becomes significant only in the necessity part of Theorem 2.4 bellow. Any positive constant D will do for Theorem 2.3 and the sufficiency part of Theorem 2.4.

The main difficulty in the proofs of Theorems 2.3 and 2.4 below is the following: How do we construct a projection with good control over its norm and its position? Lemma 1.5 provides us with a tool: If T is an operator which is δ-close to its square T^2, then there is a projection which is $C\delta$-close to T, where C is a universal constant. Next, we need operators which are close to their squares. The following surprising fact will be proved below: If P is a projection and U and V are any operators with $\|U\|, \|V\| < \delta$ then the operator $T = P + (I - P)VP + PV(I - P)$ is $C_1\delta^2$-close to its square, where C_1 is a universal constant. With these ideas in mind, let us proceed to precise statements and their proofs.

Theorem 2.3. *Let $\alpha < 4\lambda C^{-1}$, where $C = C(\lambda_0)$ is the constant of Lemma 1.5, and let P be a projection on a finite dimensional space X with $\|P\| = \lambda < \lambda_0$. Assume that P is almost α-minimal. Then there is an operator S on X with nuclear norm $\|S\|_\wedge = 1$ such that $SP = PS$ and* $\mathrm{tr}PS \geq (1+8\lambda^2(\lambda+1)D_1\alpha)^{-1}\lambda(1-D_1\alpha^2)$ *where $D_1 = D + \lambda^{-2}C$.*

Proof. Put

$$G(\alpha) = \{T = P+(I-P)UP+PV(I-P) : \ U, V \in L(X) \text{ and } \|U\|, \|V\| \leq [4\lambda(1+\lambda)]^{-1}\alpha\}$$

then, clearly, $G(\alpha)$ is a convex set, symmetric around P. Suppose that $G(\alpha)$ contains an element $T = P + (I - P)UP + PV(I - P)$ where

$$\|U\|, \|V\| \leq [4\lambda(1 + \lambda)]^{-1}\alpha \text{ and } \|T\| \leq \lambda(1 - (D + \lambda^{-2}C)\alpha^2).$$

Since

$$\begin{aligned}\|T^2 - T\| &= \|P + PV(I-P) + (I-P)UP + (I-P)UPV(I-P) \\ &\quad + PV(I-P)UP - P - (I-P)UP - PV(I-P)\| \\ &= \|(I-P)UPV(I-P) + PV(I-P)UP\| \le [\lambda(1+\lambda)^2 + \lambda^2(1+\lambda)]\|U\|\|V\| \\ &\le 2\lambda(1+\lambda)^2 \cdot [4\lambda(1+\lambda)]^{-2}\alpha^2 \le (8\lambda)^{-1}\alpha^2\end{aligned}$$

we get by Lemma 1.5 that there is a projection Q on X such that $\|Q-T\| \le C\alpha^2(8\lambda)^{-1}$. Hence

$$\begin{aligned}\|Q\| \le \|T\| + \|Q-T\| &\le \lambda(1-(D+\lambda^{-2}C)\alpha^2) + \lambda C\alpha^2(8\lambda^2)^{-1} \\ &< \lambda(1-D\alpha^2)\end{aligned}$$

and, since $\alpha < 4\lambda C^{-1}$,

$$\|P-Q\| \le \|P-T\| + \|T-Q\| \le \frac{1}{2}\alpha + C\alpha^2(8\lambda)^{-1} < \alpha.$$

The last two inequalities contradict the assumption that P is almost α-minimal. Put $D_1 = D + \lambda^{-2}C$. It follows that the ball $B(0, \lambda(1-D_1\alpha^2))$ and the convex set $G(\alpha)$ are disjoint and therefore there exists a functional W^* on $L(X)$ which separates these two sets, i.e. $W^*(F) \le \lambda(1-D_1\alpha^2) < W^*(T)$ for all $F \in \lambda(1-D_1\alpha^2)B(0,1)$ and $T \in G(\alpha)$. Without loss of generality we may assume that $\|W^*\| = 1$ and $W^*(T) > \lambda(1-D_1\alpha^2)$ for all $T \in G(\alpha)$. Because of

$$\lambda(1-D_1\alpha^2) \le W^*(P) \le \|W^*\|\|P\| \le \lambda \tag{2.1}$$

and because $P + \gamma(I-P)UP + \delta PV(I-P) \in G(\alpha)$ for every choice of signs $\gamma = \pm 1$, $\delta = \pm 1$ whenever $\|U\|, \|V\| \le [4\lambda(\lambda+1)]^{-1}\alpha$, we get that

$$|W^*((I-P)UP)| + |W^*(PV(I-P))| \le \lambda D_1\alpha^2 \tag{2.2}$$

for all $U, V \in L(X)$ with $\|U\|, \|V\| \le [4\lambda(\lambda+1)]^{-1}\alpha$. As is well known, the functional W^* on $L(X)$ is represented by an operator W on X via the identity $W^*(T) = \mathrm{tr}(WT)$, where the nuclear norm $\|W\|_\wedge = 1$. It follows from (2.2) that, for every operator $U \in B(0,1)$, we have that

$$\begin{aligned}4\lambda^2(\lambda+1)D_1\alpha &\ge |W^*((I-P)UP)| = |\mathrm{tr}(W(I-P)UP)| \\ &= |\mathrm{tr}(PW(I-P)U)|\end{aligned} \tag{2.3}$$

and

$$4\lambda^2(\lambda+1)D_1\alpha \geq |W^*(PU(I-P))| = |\mathrm{tr}(WPU(I-P))| = |\mathrm{tr}((I-P)WPU)|. \tag{2.4}$$

Hence $\|PW(I-P)\|_\wedge \leq 4\lambda^2(\lambda+1)D_1\alpha$ and

$$\|(I-P)WP\|_\wedge \leq 4\lambda^2(\lambda+1)D_1\alpha.$$

Define $S_1 = PWP + (I-P)W(I-P)$ then $S_1P = PS_1$ and

$$\|S_1 - W\|_\wedge = \|PW(I-P) + (I-P)WP\|_\wedge \leq 8\lambda^2(\lambda+1)D_1\alpha$$

hence $\|S_1\|_\wedge \leq \|W\|_\wedge + 8\lambda^2(\lambda+1)D_1\alpha$ and the operator $S = \|S_1\|_\wedge^{-1}S_1$ will satisfy the desired conditions: $\|S\|_\wedge = 1$, $SP = PS$ and, by (2.1),

$$\begin{aligned}\mathrm{tr}SP &\geq (1+8\lambda^2(\lambda+1)D_1\alpha)^{-1}\mathrm{tr}(S_1P) = \\ &= (1+8\lambda^2(\lambda+1)D_1\alpha)^{-1}\mathrm{tr}(WP) \geq \\ &\geq (1+8\lambda^2(\lambda+1)D_1\alpha)^{-1}\cdot\lambda(1-D_1\alpha^2) \quad \square\end{aligned}$$

We will conclude the section with the following characterization of almost locally minimal projections.

Theorem 2.4. *A projection P on a finite dimensional space X with $\|P\| = \lambda$ is almost locally minimal if and only if there is an operator S on X satisfying the following three conditions: $\|S\|_\wedge = 1$, $\mathrm{tr}SP = \lambda$ and $SP = PS$.*

Proof. Suppose that P is almost locally minimal then it satisfies the assumptions of Theorem 2.3 for every $\alpha > 0$ small enough. Hence, for each such α there is an operator S_α on X such that $\|S_\alpha\|_\wedge = 1$, $S_\alpha P = PS_\alpha$ and

$$\mathrm{tr}(S_\alpha P) \geq (1+8\lambda^2(\lambda+1)D_1\alpha)^{-1}\lambda(1-D_1\alpha^2).$$

Since the constant $D_1 = D_1(\lambda_0)$ does not depend on α, by passing to a convergent subsequence as α tends to 0, we obtain a limit operator S on X with $\|S\|_\wedge = 1$, $SP = PS$ and $\mathrm{tr}PS = \lambda$. Conversely, suppose that there is an operator S satisfying the above three conditions and suppose that P is not almost locally minimal. Then there is an $0 < \alpha < (8\lambda)^{-1}$ and a projection $Q \in B(P,\alpha)$ with $\|Q\| \le \lambda(1 - D\alpha^2)$. We will show that this yields a contradiction. This is the place where the size of the constant D of Definition 2.1 plays a role.

Let $T = PQ + I - Q$ then T maps the subspace $(I-Q)(X)$ identically onto itself and the subspace $Q(X)$ into $P(X)$. Moreover, $\|I-T\| = \|Q-PQ\| = \|(Q-P)Q\| \le \lambda\alpha$ hence T is invertible and, if $V = T^{-1}$, then $\|V\| \le (1-\lambda\alpha)^{-1}$ and we claim that the operator $R = P + (I-P)VP$ is a projection of X onto $Q(X)$ along $(I-P)(X)$. Indeed, $R^2 = R$, $I - R = (I-P)(I-VP)$ and, by the definition of T, for every $y \in Q(X)$, $Ty = Py$. Therefore

$$Ry = Py + (I-P)T^{-1}Py = Py + (I-P)T^{-1}Ty = y.$$

Moreover, if $\|x\| \le 1$ then $y = T^{-1}Px \in Q(X)$ and $\|y\| \le (1-\lambda\alpha)^{-1}\lambda\|x\| \le (1-\lambda\alpha)^{-1}\lambda$ and so

(2.5)
$$\begin{aligned}\|Rx - Px\| &= \|(I-P)T^{-1}Px\| = \|(I-P)Qy\| \\ &= \|(Q-P)Qy\| \le \alpha\lambda\|y\| \le \alpha\lambda^2(1-\lambda\alpha)^{-1} \le 2\lambda^2\alpha.\end{aligned}$$

Replacing P and Q by $I-Q$ and $I-P$, respectively, in the above argument and putting $W = (I-Q)(I-P) + P$ we get that $W|_{P(X)}$ is the identity on $P(X)$ and

$$\|I - W\| = \|Q(I-P)\| = \|Q(Q-P)\| \le \lambda\alpha.$$

Hence W is invertible and it maps $(I-P)(X)$ isomorphically onto $(I-Q)(X)$. Put $U = W^{-1}$ then $\|U\| \le (1-\lambda\alpha)^{-1}$ and consider $\tilde{R} = I - Q + QU(I-Q)$. Then $\tilde{R}$ is a projection with kernel $Q(X)$ and if $y \in (I-P)(X)$ then $Wy = (I-Q)y$ therefore

$$\tilde{R}y = (I-Q)y + QW^{-1}(I-Q)y = y.$$

It follows that $\tilde{R}$ is a projection of X onto $(I-P)(X)$ with $\ker\tilde{R} = Q(X)$ and hence $\tilde{R} = I - R$. Moreover, for every $x \in \mathrm{Ball}(X)$ let $y = W^{-1}(I-Q)x$, then $y \in (I-P)(X)$

(2.6)
$$\begin{aligned}\|(R-Q)(x)\| &= \|(I-R)(x) - (I-Q)(x)\| = \\ &= \|QU(I-Q)x\| = \|Q(I-P)y\| \\ &\le \lambda\alpha\|y\| \le \lambda\alpha(1+\lambda)(1-\lambda\alpha)^{-1}\|x\| \le 4\alpha\lambda(1+\lambda)\|x\|.\end{aligned}$$

Consequently we get that

$$P + (I - P)VP = R = I - \tilde{R} = Q - QU(I - Q)$$

and hence

$$Q = P + (I - P)VP + QU(I - Q). \tag{2.7}$$

Note that

$$(Q - P)U(P - Q) = QU(I - Q) - QU(I - P) - PU(I - Q) + PU(I - P) = \tag{2.8}$$
$$= QU(I - Q) - QU(I - P) + PU(I - P)$$

because $U = W^{-1}$ maps $(I - Q)(X)$ onto $(I - P)(X)$. Also,

$$QU(I - P) = Q^2U(I - P) = (Q - P)QU(I - P) + PQU(I - P) = \tag{2.9}$$
$$= (Q - P)QU(I - Q) + (Q - P)QU(Q - P) + PQU(I - P).$$

Combining identities (2.7), (2.8) and (2.9) we get that

$$Q = P + (I - P)VP + (Q - P)U(P - Q) - PU(I - P) \tag{2.10}$$
$$+ (Q - P)QU(I - Q) + (Q - P)QU(Q - P) + PQU(I - P).$$

Since $SP = PS$, we have that

$$\mathrm{tr}(S(I - P)VP) = \mathrm{tr}(SPU(I - P)) = \mathrm{tr}(SPQU(I - P)) = 0.$$

Now,

$$\|(Q - P)U(P - Q)\| \le \alpha^2\|U\| \le (1 - \lambda\alpha)^{-1}\alpha^2 \le 2\alpha^2,$$
$$\|(Q - P)QU(I - Q)\| \le \|Q - P\|\|QU(I - Q)\| \le 4\alpha^2\lambda(1 + \lambda)$$

by (2.6) and, finally,

$$\|(Q - P)QU(Q - P)\| \le \lambda\alpha^2(1 - \lambda\alpha)^{-1} \le 2\lambda\alpha^2.$$

It follows from (2.10) that

$$\begin{aligned}\lambda(1 - D\alpha^2) \ge \|S\|_\wedge \cdot \|Q\| \ge \mathrm{tr}SQ \ge \\ &\mathrm{tr}SP - \|S\|_\wedge[\|(Q - P)U(P - Q)\| + \|(Q - P)QU(I - Q)\| + \\ &+ \|(Q - P)QU(Q - P)\|] \\ &\ge \lambda - \alpha^2(2 + 4\lambda(1 + \lambda) + 2\lambda) \\ &\ge \lambda[1 - (8 + 4\lambda)\alpha^2]\end{aligned}$$

- a contradiction, in view of definition 2.1. □

3. Almost locally minimal projections on ℓ_1^n

We start with a natural example of an almost locally minimal projection on $X = \ell_1^4$.

Example 3.1. Let P be the orthogonal projection of X onto the 3-dimensional subspace E, spanned by the vectors $x_1 = (1,1,1,1)$, $x_2 = (1,-1,1,-1)$ and $x_3 = (1,-1,-1,1)$. We claim that P is almost locally minimal. With respect to the unit vector basis $\{u_i\}_{i=1}^4$ of X, P is represented by the matrix

$$P = 4^{-1}\begin{bmatrix} 3 & -1 & 1 & 1 \\ -1 & 3 & 1 & 1 \\ 1 & 1 & 3 & -1 \\ 1 & 1 & -1 & 3 \end{bmatrix}.$$

Clearly, $\|P\| = 3/2$. Consider the following vectors in $\ell_\infty^4 = X^*$: $g_1 = (1,-1,1,1)$, $g_2 = (-1,1,1,1)$, $g_3 = (1,1,1,-1)$ and $g_4 = (1,1,-1,1)$. Then $\|g_i\| = 1$ for all $1 \le i \le 4$ and hence the operator $S = 4^{-1}\sum_{i=1}^{4} g_i \otimes u_i$ has nuclear norm $\|S\|_\wedge \le 1$. Let us compute the trace of PS.

$$\operatorname{tr}(PS) = 4^{-1}\left[\sum_{i=1}^{4} g_i(Pu_i)\right] = 3/2 = \|P\|.$$

It follows that $\|S\|_\wedge = 1$. We claim that $PS = SP$. Indeed

$$S = 4^{-1}\begin{bmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} = P - \frac{1}{2}I$$

and hence $PS = P(P - \frac{1}{2}I) = (P - \frac{1}{2}I)P = SP$. It follows from Theorem 2.4. that P is almost locally minimal.

Note that in each row and each column of P the sum of the absolute values of the entries is $3/2 = \|P\|$. Is this typical of almost locally minimal projections on ℓ_1^n? The answer is negative, as is shown in the next example.

Example 3.2. Let $X = \ell_1^5$ and let

$$\tilde{P} = 4^{-1}\begin{bmatrix} 3 & -1 & 1 & 1 & 0 \\ -1 & 3 & 1 & 1 & 0 \\ 1 & 1 & 3 & -1 & 0 \\ 1 & 1 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

Then $\tilde{P}$ is a projection of X onto a 4-dimensional subspace of X, $\|\tilde{P}\| = \frac{3}{2}$ and, if $\tilde{g}_i = (g_i, 0) \in \ell_\infty^5$ and $\{u_i\}_{i=1}^5$ is the unit vector basis of X, then the operator $\tilde{S} = 4^{-1} \sum_{i=1}^{4} \tilde{g}_i \otimes u_i$ has unclear norm $\|\tilde{S}\|_\wedge = 1$, $\text{tr}\tilde{P}\tilde{S} = 3/2$ and

$$\tilde{S}\tilde{P} = \tilde{P}\tilde{S}.$$

Hence $\tilde{P}$ is almost locally minimal by Theorem 2.4. Here the space ℓ_1^5 splits into an ℓ_1-direct sum of ℓ_1^4 and a (one dimensional) subspace $[u_5]$, and the range of $\tilde{P}$ is isometric to the space range$P \oplus [u_5]$. Moreover, the ℓ_∞ norm $\|P\|_\infty = \|P\|_1 = 3/2$. It turns out that a similar property is shared by every almost locally minimal projection on ℓ_1^n. Before we state the main result of this section let us discuss the following special projections on $L_1[0,1]$.

Definition 3.3. A projection $\tilde{P}$ on $L_1[0,1]$ is called **λ-doubly stochastic** if it is represented by a kernel $p(x,y)$ (i.e. $(\tilde{P}f)(x) = \int_0^1 p(x,y)f(y)dy$) satisfying the following equalities:

$$\int_0^1 |p(x,y)|dy = \lambda \text{ a.e}$$

and

$$\int_0^1 |p(x,y)|dx = \lambda \text{ a.e.}$$

A projection P on ℓ_1^n is said to be **equivalent to a λ-doubly stochastic projection $\tilde{P}$** on $L_1[0,1]$ if there is an isometric embedding $J : \ell_1^n \to L_1[0,1]$ and n pairwise disjoint measurable sets $A_1, \ldots, A_n$ with $\mu(A_i) > 0$ ($\mu =$ Lebesgue's measure) such that, for each unit vector basis element u_i, $1 \le i \le n$, of ℓ_1^n, $Ju_i = \mu(A_i)^{-1}\chi_{A_i}$ and $\tilde{P} = JPJ^{-1}R$, where R denotes the natural projection of $L_1[0,1]$ onto $J(\ell_1^n)$ defined by

$$Rf = \sum_{i=1}^{n} \mu(A_i)^{-1} \left(\int_{A_i} f(y)dy \right) \chi_{A_i}.$$

Clearly, $P(\ell_1^n)$ is isometric to $\tilde{P}(L_1)$.

Lemma 3.4. *Let Q be a projection on ℓ_1^m and assume that Q is represented by the matrix $(q_{i,j})_{i,j=1}^m$ with respect to the unit vector basis $\{u_i\}_{i=1}^m$. Assume that*

$\sum_{i=1}^{m} |q_{i,j}| = \lambda$ *for* $1 \leq j \leq m$ *and that there exist positive number* $\{\lambda_i\}_{i=1}^m$ *with* $\sum_{i=1}^{m} \lambda_i = 1$ *such that*

$$\lambda_i^{-1} \sum_{j=1}^{m} |q_{i,j}| \lambda_j = \lambda \quad \textit{for all } 1 \leq i \leq m. \tag{3.1}$$

Then Q *is equivalent to a* λ*-doubly stochastic projection* $\tilde{Q}$ *on* $L_1[0,1]$.

Proof. Let A_i denote the interval $\left(\sum_{j=1}^{i-1} \lambda_j, \sum_{j=1}^{i} \lambda_j\right)$ of $[0,1]$ and embed ℓ_1^m into $L_1[0,1]$ by the map $J : \ell_1^m \to L_1[0,1]$, defined by $Ju_i = \lambda_i^{-1} \chi_{A_i}$ where χ_{A_i} is the indicator function of A_i. Clearly, J is an isometric embedding and there is a natural projection R of norm 1 of $L_1[0,1]$ onto $J(\ell_1^m)$ defined by

$$Rf = \sum_{i=1}^{m} \left(\int_{A_i} f \right) J(u_i).$$

Put $\tilde{Q} = JQJ^{-1}R$, then $\tilde{Q}$ is a projection of $L_1[0,1]$ onto $JQ(\ell_1^m)$ and $\|\tilde{Q}\|_1 = \|Q\|_1 = \lambda$. Note that

$$\tilde{Q}f(x) = \int_0^1 q(x,y) f(y) dy \quad \text{where}$$

$$q(x,y) = \sum_{i=1}^{m} \left(\sum_{j=1}^{m} q_{i,j} \chi_{A_j}(y) \right) (Ju_i)(x).$$

To show that Q is λ-doubly stochastic, let us compute the relevant integrals: If $x \in A_k$ then, because the A_i are pairwise disjoint,

$$\begin{aligned}
\int_0^1 |q(x,y)| dy &= \mu(A_k)^{-1} \int_0^1 \left| \sum_{j=1}^{m} q_{k,j} \chi_{A_j}(y) \right| dy \\
&= \mu(A_k)^{-1} \int_0^1 \sum_{j=1}^{m} |q_{k,j}| \chi_{A_j}(y) dy \\
&= \lambda_k^{-1} \sum_{j=1}^{n} |q_{k,j}| \lambda_j = \lambda,
\end{aligned}$$

by (3.1). Also, if $y \in A_k$ then

$$\int_0^1 \left| \sum_{i=1}^{m} q_{i,k} J(u_i)(x) \right| dx = \sum_{i=1}^{m} |q_{i,k}| = \lambda.$$

This proves that $\tilde{Q}$ is λ-doubly stochastic. $\square$

One important property of a λ-doubly stochastic projection $\tilde{P}$ on L_1 is the following fact:

(3.2) Regarding $\tilde{P}$ as a projection on $L_\infty[0,1]$, $\|\tilde{P}\|_\infty = \lambda$.

Proof. Let $\|f\|_\infty = 1$, then, for every $0 \le x \le 1$,

$$|(\tilde{P}f)(x)| = \left| \int_0^1 p(x,y)f(y)dy \right| \le \int_0^1 |p(x,y)|dy = \lambda.$$

The equality $\|\tilde{P}\|_\infty = \lambda$ holds because, for almost every x, if $f(y) = \operatorname{sign}(p(x,y))$ then $|(\tilde{P}f)(x)| = \lambda$. $\square$

Remark 3.5. We are interested in the isomorphic nature of the range E of a projection P on ℓ_1^n. We have just seen that if (3.1) holds then E is isometric to the range of a λ-doubly stochastic projection $\tilde{P}$ on L_1.

In general it may not true that the range E of a projection Q satisfying the assumptions of Lemma 3.4 is isometric to the range of a projection P on some ℓ_1^m which is represented (with respect to the unit vector basis $\{v_i\}_{i=1}^m$) by a λ-doubly stochastic matrix $(q_{i,j})$ satisfying $\sum_{i=1}^n |q_{i,j}| = \sum_{i=1}^n |q_{j,i}| = \lambda$ for all $1 \le j \le n$. However, if λ_i are all rational numbers then the last statement holds. Indeed, let $\lambda_i = \frac{k_i}{m}$ and let $T : \ell_1^n \to \ell_1^m$ be the isometry defined by

$$T(u_i) = k_i^{-1} \sum_{j=t_{i-1}+1}^{t_i} v_j \overset{\text{def}}{=} w_i$$

where $t_0 = 0$ and, for $i \ge 1$, $t_i = \sum_{j=1}^{i} k_j$. Let V denote the natural projection of ℓ_1^m onto $[w_i]_{i=1}^n$ defined by $V(v_k) = w_i$ if $t_{i-1} < k \le t_i$. Then $\|V\| = 1$ and $P = TQT^{-1}V$ is the desired projection of ℓ_1^m onto $T(E)$. Indeed, if $t_{h-1} < k \le t_h$ then

$$\begin{aligned} Pv_k &= TQT^{-1}w_h = TQu_h = \\ &= T\left(\sum_{j=1}^n q_{j,h}u_j\right) = \sum_{j=1}^n q_{j,j}k_j^{-1} \sum_{i=t_{j-1}+1}^{t_j} v_i. \end{aligned}$$

Therefore, if $P = (p_{i,j})_{i,j=1}^m$ then $p_{i,k} = q_{j,h}k_j^{-1} = m^{-1}q_{j,h}\lambda_j^{-1}$ if $t_{h-1} < k \le t_h$ and $t_{j-1} < i \le t_j$. It follows that

$$\sum_{i=1}^m |p_{i,k}| = m^{-1} \cdot \sum_{j=1}^n |q_{j,h}|k_j\lambda_j^{-1} = \lambda$$

for every $1 \le k \le m$ and

$$\sum_{k=1}^m |p_{i,k}| = m^{-1}\sum_{h=1}^n |q_{j,h}|k_h\lambda_j^{-1} = \lambda_j^{-1}\sum_{h=1}^m |q_{j,h}|\lambda_h = \lambda,$$

by (3.1).

We are now ready to prove the following representation theorem for almost locally minimal projections on ℓ_1^n.

Theorem 3.6. *Let P be an almost locally minimal projection on $X = \ell_1^n$ with $\|P\| = \lambda > 1$. Then there is an integer $2 \le m \le n$, there are positive numbers $\{\lambda_i\}_{i=1}^m$ with $\sum_{i=1}^m \lambda_i = 1$ and a permutation $\{u_i\}_{i=1}^n$ of the unit vector basis of ℓ_1^n with respect to which*

$$P = \begin{bmatrix} Q & T \\ 0 & P_0 \end{bmatrix}$$

where $Q^2 = Q$, $P_0^2 = P_0$, Q is an $m \times m$ matrix satisfying (3.1) and, therefore, equivalent to a λ-doubly stochastic projection, P_0 is an $(n-m) \times (n-m)$ matrix, $\|Q\| = \lambda$, $\|P_0\| \le \lambda$ and

$$d(Q([u_i]_{i=1}^m) \oplus_{\ell_1} P_0([u_i]_{i=m+1}^n), P(\ell_1^n)) \le \lambda^2.$$

Proof. By Theorem 2.4, there is an operator S on X such that $\|S\|_\wedge = 1$, $\text{tr}SP = \|P\| = \lambda$ and $SP = PS$. Because $X = \ell_1^n$ there is a permutation $\{u_i\}_{i=1}^n$ of the unit vector basis of X with respect to which $S = \sum_{i=1}^m \lambda_i g_i \otimes u_i$ where $1 \le m \le n$, $\lambda_i > 0$, $\sum_{i=1}^m \lambda_i = 1$, $g_i = \sum_{j=1}^n g_{i,j}u_j^* \in \ell_\infty^n$ and $\|g_i\| = \max_j |g_{i,j}| = 1$ for all $1 \le i \le m$. ($\{u_j^*\}_{j=1}^n$ denotes the unit vector basis of ℓ_∞^n). Let $P = (p_{i,j})_{i,j=1}^n$ be the matrix of

P with respect to $\{u_i\}_{i=1}^n$, then

$$PS = \sum_{i=1}^m \lambda_i g_i \otimes Pu_i = \sum_{i=1}^m \lambda_i g_i \otimes \sum_{j=1}^n p_{j,i} u_j = \sum_{j=1}^n \left(\sum_{i=1}^m \lambda_i p_{j,i} g_i \right) \otimes u_j \tag{3.2}$$

It follows that

$$\lambda = \mathrm{tr}(PS) = \sum_{j=1}^n \sum_{i=1}^m \lambda_i p_{j,i} g_{i,j} = = \sum_{i=1}^m \lambda_i \left(\sum_{j=1}^n p_{j,i} g_{i,j} \right) \leq \sum_{i=1}^m \lambda_i \left(\sum_{j=1}^n |p_{j,i}| \right) \leq \lambda \tag{3.3}$$

because $\lambda = \|P\| = \max_i \sum_{j=1}^n |p_{j,i}|$ and $\|g_i\| = \max_j |g_{i,j}| = 1$.

Hence, for every $1 \leq i \leq m$, if $p_{i,j} \neq 0$ then

$$g_{i,j} = \mathrm{sign}(p_{j,i}) \text{ and } \sum_{j=1}^n |p_{j,i}| = \lambda. \tag{3.4}$$

On the other hand,

$$PS = SP = \sum_{j=1}^m \lambda_j P^* g_j \otimes u_j. \tag{3.5}$$

Comparing (3.2) and (3.5) we get that, for $1 \leq j \leq m$

$$\sum_{i=1}^m \lambda_i p_{j,i} g_i = \lambda_j P^* g_j. \tag{3.6}$$

Applying both sides of (3.6) to u_j we obtain by (3.4) that

$$\begin{aligned} \sum_{i=1}^m \lambda_i |p_{j,i}| &= \sum_{i=1}^m \lambda_i p_{j,i} g_{i,j} = \sum_{i=1}^m \lambda_i p_{j,i} g_i(u_j) \\ &= \lambda_j P^* g_j(u_j) = \lambda_j g_j(Pu_j) = \lambda_j g_j \left(\sum_{k=1}^n p_{k,j} u_k \right) = \\ &= \lambda_j \sum_{k=1}^n p_{k,j} g_{j,k} = \lambda_j \sum_{k=1}^n |p_{k,j}|. \end{aligned} \tag{3.7}$$

Summing both sides of (3.7) over $1 \leq j \leq m$ we get, by (3.4), that

$$\sum_{i=1}^m \lambda_i \sum_{j=1}^m |p_{j,i}| = \sum_{j=1}^m \lambda_j \left(\sum_{k=1}^n |p_{k,j}| \right) = \sum_{j=1}^m \lambda_j \lambda = \lambda. \tag{3.8}$$

The equality (3.8) forces $\sum_{j=1}^{m} |p_{j,i}| = \lambda$ for all $1 \leq i \leq m$ hence

$$p_{j,i} = 0 \quad \text{for all} \quad 1 \leq i \leq m \quad \text{and} \quad m < j \leq n. \tag{3.9}$$

The relation $P^2 = P$ and the last equality imply that, for $m < h, k \leq n$,

$$p_{h,k} = \sum_{j=1}^{n} p_{h,j} p_{j,k} = \sum_{j=m+1}^{n} p_{h,j} p_{j,k}$$

hence, the $(n-m) \times (n-m)$ matrix $P_0 = (p_{h,k})_{h,k=m+1}^{n}$ is a projection with $\|P_0\| \leq \|P\| = \lambda$. Similarly, if we put $q_{i,j} = p_{i,j}$ for $1 \leq i, j \leq m$ then $Q = (q_{i,j})_{i,k=1}^{m}$ is easily checked to be a projection satisfying the equality $\sum_{i=1}^{m} |q_{i,j}| = \lambda$. Also, by (3.6), $\sum_{j=1}^{n} \lambda_j |q_{i,j}| = \lambda_i g_j(Pu_j) = \lambda_i \lambda$. It follows from Lemma 3.4 that Q is equivalent to a λ-doubly stochastic projection on $L_1[0,1]$. Let us now discuss the isomorphic type of $Q([u_i]_{i=1}^{m}) \oplus_{\ell_1} P_0([u_i]_{i=m+1}^{n})$.

Let $\hat{P}$ denote the projection $Q \oplus P_0$ on ℓ_1^n (i.e. $\hat{P}u_i = Qu_i$ if $1 \leq i \leq m$ and $\hat{P}u_i = P_0 u_i$ if $m < i \leq n$). Since $P - \tilde{P}$ is an upper right $m \times (n-m)$ matrix, we have that $(P - \tilde{P})^2 = 0$. The following lemma is needed for the completion of the proof of Theorem 3.6.

Lemma 3.7. *Let P and $\hat{P}$ be projections on a Banach space X and assume that $\|P\|, \|\hat{P}\| \leq \lambda$ and $(P - \hat{P})^2 = 0$. Then $d(P(X), \hat{P}(X)) \leq \lambda^2$.*

Proof. Since $P + \hat{P} = P\hat{P} + \hat{P}P$, multiplying both sides by P on the left we get that $P + P\hat{P} = P\hat{P} + P\hat{P}P$ hence $P = P\hat{P}P$. It follows that if $x = Px$ then $x = P\hat{P}x$ and, therefore,

$$\|x\| \leq \|P\| \|\hat{P}x\| \leq \lambda \|\hat{P}x\| \leq \lambda^2 \|x\|.$$

By symmetry we get that for every $y = \hat{P}y$, $\|y\| \leq \lambda \|Py\| \leq \lambda^2 \|y\|$. It follows that $d(P(X), \hat{P}(X)) \leq \lambda^2$. $\square$

It remains to discuss the magnitude of m. Suppose that $m = 1$ then, w.l.g., $S = \lambda_1 g_1 \otimes u_1$ and so, $\lambda_1 P^* g_1 \otimes u_1 = SP = PS = \lambda_1 g_1 \otimes Pu_1$. It follows that $u_1 = Pu_1$, contradicting the fact that $\|Pu_1\| = \lambda > 1$. This completes the proof of Theorem 3.6. $\square$

4. Concluding remarks

How far are we from a positive solution of Problem 1.1 raised in the Introduction and what is needed for a complete solution?

It seems that we still have a long way ahead, however, Theorem 3.6 suggests that, under certain circumstances, and an inductive argument might work well. We need positive answers to the following

Problem 4.1. Let P be a λ-doubly stochastic projection on $L_1[0,1]$ with $\operatorname{rank}(P) = k$. Is $d(P(L_1), \ell_1^k) \leq \psi(\lambda)$ where ψ is independent on k?

Problem 4.2. Do there exist constants $0 < \beta = \beta(\lambda) < 1$ and $0 < \alpha = \alpha(\lambda)$ such that if $\tilde{P}$ is a projection on ℓ_1^n with $\|\tilde{P}\| \leq \lambda$ which is almost α-minimal then there is an almost locally minimal projection P on ℓ_1^n with $\|P - \tilde{P}\| \leq \beta$ and $\|\tilde{P}\| \leq \|P\|$?

Note that Theorem 1.2 settles Problem 4.2 in the special case of projections of small norm, since any projection of norm 1 is locally minimal.

Suppose that Problems 4.1 and 4.2 have positive solutions. Starting with a projection Q of norm λ, either we can find a sequence $\{Q_i\}_{i=1}^N$ with $Q_0 = Q$ and $\|Q_N\| = 1$ such that $\|Q_i - Q_{i+1}\| \leq \alpha$ and $\|Q_{i+1}\| \leq \lambda(1 - D\alpha^2)^{i+1}$ for every $0 \leq i \leq N-1$ or there is a $Q_i = P$ such that P is almost α-minimal. In the first case, Problem 1.1 is solved because the discussion in the Introduction shows that $d(Q(\ell_1^n), \ell_1^k) \leq (1-\alpha)^{-N}(1+\alpha)^N$. Since $\lambda(1 - D\alpha^2)^N \cong 1$ we get that

$$d(Q(\ell_1^n), \ell_1^k) \cong [(1+\alpha)/(1-\alpha)]^{[\log(1-D\alpha^2)]^{-1} \log \lambda^{-1}}.$$

In the second case, if Problem 4.2 has a positive solution, there is an almost locally minimal projection P on ℓ_1^n with $\|P - Q\| \leq \beta$. If problem 4.1 has a positive solution then Theorem 3.6 ensures that $P\ell_1^n$ splits into a precise ℓ_1-direct sum of a k_1 dimensional subspace F of ℓ_1^m with $d(F, \ell_1^{k_1}) \leq \psi(\lambda)$ and a range E of a projection Q_0 of rank $< k$ with $\|Q_0\| \leq \lambda$. With some luck an induction procedure may then settle Problem 1.1 but, of course, additional work is required.

The case of orthogonal projections. Suppose that P is an almost locally minimal orthogonal projection on ℓ_1^n, i.e., the representing matrix $(p_{i,j})_{i,j=1}^n$ is symmetric. In this case Theorem 3.6 states that the space ℓ_1^n splits into a precise ℓ_1-direct sum $\ell_1^m \oplus \ell_1^{n-m}$ and $P = Q \oplus P_0$, where Q and P_0 are orthogonal projections on ℓ_1^m

and ℓ_1^{n-m} resp. with $\|Q\|, \|P_0\| \leq \|P\|$ and where Q is an almost locally minimal projection on ℓ_1^m which is equivalent to a λ-doubly stochastic projection. It will be interesting to settle the following.

Problem 4.3. What is the isomorphic type of the range $P(\ell_1^n)$ of an orthogonal λ-doubly stochastic projection on ℓ_1^n?

Problem 4.4. Let P be a λ-doubly stochastic projection on ℓ_1^n. Is P almost locally minimal?

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The Hebrew University, Jerusalem Israel
and
The University of Connecticut, Storrs, Ct. 06269
E-mail address: zippin@math.huji.ac.il