A SHORT PROOF OF SCHOENBERG'S CONJECTURE ON POSITIVE DEFINITE FUNCTIONS

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Preprint No. 18
1996/97

Abstract. In 1938 I.J. Schoenberg asked for which positive numbers p is the function $\exp(-\|x\|^p)$ positive definite, where the norm is taken from one of the spaces ℓ_q^n, $q > 2$. The solution of the problem was completed in 1991, by showing that for every $p \in (0, 2]$, the function $\exp(-\|x\|^p)$ is not positive definite for the ℓ_q^n norms with $q > 2$ and $n \geq 3$. We prove a similar result for a more general class of norms, which contains some Orlicz spaces and q–sums, and, in particular, present a simple proof of the answer to the original Schoenberg's question. Some consequences concerning isometric embeddings in L_p spaces for $0 < p \leq 2$ are discussed as well.

1991 *Mathematics Subject Classification.* 42A82, 46B04.

1. Supported in part by the NSF Grant DMS-9531594.

2. Supported in part by the US-Israel BSF, and by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas sponsored by the Minerva Foundation (Germany).

This work was completed when both authors participated in the NSF supported Workshop on Linear Analysis and Probability at Texas A&M University, 1997.

1. Introduction

A complex valued function f defined on $\mathbb{R}^n$ is called **positive definite** on $\mathbb{R}^n$, if for every finite sequence $\{x_i\}_{i=1}^m$ in $\mathbb{R}^n$ and every choice of complex numbers $\{c_i\}_{i=1}^m$, the following inequality holds

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c_i\bar{c}_j f(x_i - x_j) \geq 0.$$

A fundamental result concerning positive definite functions is due to Bochner. It states that continuous, positive definite functions on $\mathbb{R}^n$, are precisely the Fourier transforms of positive, finite, Borel measures on $\mathbb{R}^n$. (For a proof, see, e.g., [2], p. 184).

The 1938 Schoenberg's question on positive definite functions, [16], is the following: for which $p > 0$ is the function $\exp(-\|x\|_q^p)$ $(x \in \mathbb{R}^n)$ positive definite? Here the norm $\|\cdot\|_q$ is the norm of the space ℓ_q^n, where $n \geq 2$ and $q > 2$. The solution was completed in 1991, [6], and it shows in particular that the function $\exp(-\|x\|_q^p)$ is not positive definite if $n \geq 3$, $0 < p \leq 2$ and $q > 2$. In this article we present a short proof of the following more general result:

Theorem 1. *Let X be a three dimensional normed space with a normalized basis $\{e_1, e_2, e_3\}$ so that*

(1) *For every fixed $(x_2, x_3) \in \mathbb{R}^2\backslash\{0\}$, the function $x_1 \to \|x_1e_1 + x_2e_2 + x_3e_3\|$ has a continuous second derivative everywhere on $\mathbb{R}$, and*

$$\|x\|'_{x_1}(0, x_2, x_3) = \|x\|''_{x_1^2}(0, x_2, x_3) = 0,$$

where $\|x\|'_{x_1}$ and $\|x\|''_{x_1^2}$ stand for the first and second derivatives by x_1, respectively.

(2) *There exists a constant K so that for every $x_1 \in \mathbb{R}$ and every $(x_2, x_3) \in \mathbb{R}^2$ with $\|x_2e_2 + x_3e_3\| = 1$, one has $\|x\|''_{x_1^2}(x_1, x_2, x_3) \leq K$.*

Then, for every $0 < p \leq 2$, the function $\exp(-\|x\|^p)$ is not positive definite.

Examples of spaces satisfying conditions (1) and (2), which will be discussed in §3, include q-sums of normed spaces with $q > 2$, and Orlicz spaces ℓ_M^n whose Orlicz function M has a continuous second derivative and $M'(0) = M''(0) = 0$. Both of these classes of spaces include the ℓ_q^n spaces for $q > 2$. Thus Theorem 1 answers Schoenberg's question for a more general class of norms.

There is a close connection between positive definite functions and isometric embeddings into L_p. It was known already to P.Levy, [9], that if $X = (\mathbb{R}^n, \|\cdot\|)$ is isometric to a subspace of L_p, $0 < p \leq 2$, then $\exp(-\|x\|^p)$ is a positive definite function, hence a multiple of a characteristic function of a stable measure. The actual equivalence of the two notions was discovered by Bretagnolle, Dacunha-Castelle, and Krivine [1] who proved that, for $0 < p \leq 2$, a Banach space E is isometric to a subspace of L_p if and only if the function $\exp(-\|x\|^p)$ is positive definite. They have used this fact to show that the space L_q embeds isometrically in L_p if $0 < p < q \leq 2$. Thus Schoenberg's question can be reformulated in terms of isometric embeddings: For which numbers $p \in (0,2], q > 2$, is the space ℓ_q^n isometric to a subspace of L_p?

After L.Dor had answered in [3] the question for $p \in [1,2]$, (by proving that ℓ_q^n is not isometric to a subspace of L_1 if $n \geq 3$ and $q > 2$), the complete solution (including $p \in (0,1)$) was given in [13] for $q = \infty$ and in [6] for $2 < q < \infty$: for $0 < p \leq 2$ the function $\exp(-\|x\|^p)$ is not positive definite if the dimension of the space is greater than 2, and for $n = 2$ the function is positive definite if and only if $0 < p \leq 1$. (For $0 < p \leq 1, n = 2$ this was well known; see [4],[5],[10]).

Not very long after the paper [6] appeared, V. Zastavny, [17,18], proved that for a 3-dimensional normed space X, there are no non-constant positive definite functions of the form $f(\|x\|)$ if there exists a basis e_1, e_2, e_3 so that the function

$$(y,z) \to \|xe_1 + ye_2 + ze_3\|_x^{'}(1,y,z)/\|e_1 + ye_2 + ze_3\|, \quad y,z, \in \mathbb{R}$$

belongs to the space $L_1(\mathbb{R}^2)$. This criterion provided, in particular, a new proof of the answer to Schoenberg's question. For the spaces $X = \ell_q^3$, similar results were established by Misiewicz (for $q = \infty$), in [13], and by Lisitsky (for $2 < q < \infty$) in [11].

Theorem 1 imposes a necessary condition on the norm of the space X, in order that the space will be isometric to a subspace of L_p, with $0 < p \leq 2$. Zastavny's result also imposes a necessary condition, that can be checked for certain spaces, but for others can be quite difficult to check. For example, Zastavny's test can be applied to ℓ_q^n and more generally to q-sums where $q > 2$, but cannot be applied in general to the class of Orlicz spaces. On the other hand, the "second derivative test" formulated in Theorem 1, applies to both of these cases. We refer the reader to [7,14,15] for other results related to positive definite norm dependent functions and their applications to probability and isometric embeddings in L_p.

2. Proof of Theorem 1

Throughout this section, we remain under the conditions and notation of Theorem 1. We need a few simple facts.

Remarks. (i) It is easy to see that, for every continuous, homogeneous of degree 1, positive outside of the origin function f on $\mathbb{R}^n$ and every $\alpha > -n$, the function f^α is locally integrable on $\mathbb{R}^n$. In particular, for any $p > 0$, the function $(x_2, x_3) \to \|x_2e_2 + x_3e_3\|^{p-2}$ is locally integrable on $\mathbb{R}^2$.

(ii) A simple consequence of the triangle inequality is that $-1 \leq \|x\|_{x_1}^{'} \leq 1$ at every point $x \in \mathbb{R}^3$ with $(x_2, x_3) \neq 0$.

(iii) For every fixed $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$, $\|x\|$ is a convex differentiable function of x_1 whose derivative at zero is equal to zero. Therefore, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have $\|x\| \geq \|x_2e_2 + x_3e_3\|$.

(iv) The function $\|x\|''_{x_1^2}$ is non-negative, homogeneous of degree -1. Let K be the constant from the condition (2) of Theorem 1. Then, for every $x_1 \in \mathbb{R}$ and every $(x_2, x_3) \in \mathbb{R}^2 \setminus \{0\}$, the second derivative $\|x\|''_{x_1^2}(x_1, x_2, x_3)$ is equal to

$$\frac{1}{\|x_2e_2 + x_3e_3\|}\|x\|''_{x_1^2}\Big(\frac{x_1}{\|x_2e_2 + x_3e_3\|}, \frac{x_2}{\|x_2e_2 + x_3e_3\|}, \frac{x_3}{\|x_2e_2 + x_3e_3\|}\Big),$$

which is less or equal to $K/\|x_2e_2 + x_3e_3\|$ by the condition (2) of Theorem 1.

We are ready to prove Theorem 1.

Proof of Theorem 1. Since for $0 < p_1 < p_2 \le 2$ there is an isometric embedding of L_{p_2} into L_{p_1}, we may assume that $0 < p < 1$.

Suppose that the function

$$F(x_1, x_2, x_3) = \exp(-\|x_1e_1 + x_2e_2 + x_3e_3\|^p),$$

is positive definite on $\mathbb{R}^3$. Similarly as in [18], we pass to a positive definite function of one variable as follows. Let

$$h(x_2, x_3) = \begin{cases} (1 - |x_2|)(1 - |x_3|), & \max\{|x_2|, |x_3|\} \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Denote $d\mu(x_2, x_3) = h(x_2, x_3)\, dx_2\, dx_3$. Then the function

$$\psi(t) = \int_{\mathbb{R}^2} \exp(-\|te_1 + x_2e_2 + x_3e_3\|^p)\, d\mu(x_2, x_3) \qquad (1)$$

is positive definite on $\mathbb{R}$. Indeed, the function $h(x_2, x_3)$ is positive definite because

$$\int_{\mathbb{R}^2} e^{i(u_2x_2 + u_3x_3)} d\mu(x_2, x_3) = \left(\frac{\sin u_2/2}{u_2/2}\right)^2 \left(\frac{\sin u_3/2}{u_3/2}\right)^2 \ge 0.$$

Therefore, for each $\varepsilon > 0$, the function $F(x_1, x_2, x_3)h(x_2, x_3)e^{-\varepsilon|x_1|}$ is positive definite on $\mathbb{R}^3$, as a product of positive definite functions. Hence:

$$\forall u \in \mathbb{R}^3 : \qquad \int_{\mathbb{R}^3} e^{iu\cdot x} F(x_1, x_2, x_3) e^{-\varepsilon|x_1|}\, dx_1\, d\mu(x_2, x_3) \ge 0,$$

where $u \cdot x$ denotes the inner product of u and x. By taking $u = (s, 0, 0)$ we get

$$\forall s \in \mathbb{R} : \qquad \int_{-\infty}^{\infty} e^{ix_1s} \left(\int_{\mathbb{R}^2} F(x_1, x_2, x_3)\, d\mu(x_2, x_3)\right) e^{-\varepsilon|x_1|}\, dx_1 \ge 0.$$

Therefore, the function $x_1 \to e^{-\varepsilon|x_1|}\psi(x_1)$ is positive definite. Since $\varepsilon > 0$ is arbitrary, $\psi(t)$ is a positive definite function. This argument appeared in [18]. The idea is now to differentiate twice the function under the integral in (1) and deduce that $\psi''(0) = 0$.

Assume from now on that $(x_2, x_3) \in \mathbb{R}^2 \backslash \{0\}$. We have

$$F'_{x_1}(x_1, x_2, x_3) = -p\|x_1e_1 + x_2e_2 + x_3e_3\|^{p-1}\|x\|'_{x_1}F(x_1, x_2, x_3).$$

By remarks (ii),(iii) we get

$$\left|F'_{x_1}(x_1, x_2, x_3)\right| \leq p\|x_1e_1 + x_2e_2 + x_3e_3\|^{p-1} \leq p\|x_2e_2 + x_3e_3\|^{p-1}, \tag{2}$$

for every $x_1 \in \mathbb{R}$ and $(x_2, x_3) \in \mathbb{R}^2 \backslash \{0\}$. For the second derivative, we have

$$\begin{aligned} F''_{x_1^2}(x_1, x_2, x_3) = F(x_1, x_2, x_3)\big[& -p(p-1)\|x\|^{p-2}(\|x\|'_{x_1})^2 - p\|x\|^{p-1}\|x\|''_{x_1^2} \\ & + p^2\|x\|^{2(p-1)}(\|x\|'_{x_1})^2\big]. \end{aligned}$$

By remarks (ii),(iii),(iv), and since $|F| \leq 1$, we have for every $x_1 \in \mathbb{R}$ and $(x_2, x_3) \in \mathbb{R}^2 \backslash \{0\}$,

$$\begin{aligned} |F''_{x_1^2}(x_1, x_2, x_3)| &\leq (1-p)p\|x_1e_1 + x_2e_2 + x_3e_3\|^{p-2} + pK\|x_1e_1 + x_2e_2 + x_3e_3\|^{p-2} \\ &\quad + p^2\|x_1e_1 + x_2e_2 + x_3e_3\|^{2p-2} \\ &\leq p(K + 1 - p)\|x_2e_2 + x_3e_3\|^{p-2} + p^2\|x_2e_2 + x_3e_3\|^{2p-2}. \end{aligned}$$

Thus both the first and second partial derivatives of F by the variable x_1 are bounded above by locally integrable functions in $\mathbb{R}^2$, which do not depend on x_1. (Here we use remark (i)). We have for every $t > 0, \varepsilon \neq 0$,

$$\begin{aligned} \frac{\psi(t+\varepsilon) - \psi(t)}{\varepsilon} &= \int_{\mathbb{R}^2 \backslash \{0\}} \frac{F(t+\varepsilon, x_2, x_3) - F(t, x_2, x_3)}{\varepsilon} d\mu(x_2, x_3), \\ &= \int_{\mathbb{R}^2 \backslash \{0\}} F'_{x_1}(\theta(t, \varepsilon, x_2, x_3), x_2, x_3) d\mu(x_2, x_3), \end{aligned}$$

where $\theta(t, \varepsilon, x_2, x_3) \in (t, t+\varepsilon)$ (or $(t-\varepsilon, t)$). By (2), the function $|F'_{x_1}(x_1, x_2, x_3)|$ is bounded by an integrable function of the variables x_2, x_3, so we can apply Lebesgue's bounded convergence theorem and pass to the limit as $\varepsilon \to 0$, under the integral sign. By continuity of the first derivative we get

$$\psi'(t) = \int_{\mathbb{R}^2 \backslash \{0\}} F'_t(t, x_2, x_3) d\mu(x_2, x_3). \tag{3}$$

Repeating this argument, we obtain

$$\psi''(t) = \int_{\mathbb{R}^2 \backslash \{0\}} F''_{t^2}(t, x_2, x_3) d\mu(x_2, x_3). \tag{4}$$

By condition (1) of the Theorem, the integrands in (4) and in (3) tend pointwise to zero as $t \to 0$. The integrands are also bounded above by integrable functions independent of t. Therefore, applying Lebesgue's bounded convergence theorem once more, we finally get $\psi'(0) = \psi''(0) = 0$. Therefore, $\psi(t) = \psi(0) + o(t^2)$ as $t \to 0$. Since ψ is positive definite, we must have $\psi(t) \equiv \psi(0)$, ([12], Th. 4.1.1). In other words

$$(\forall t) \qquad \int_{\mathbb{R}^2} F(t, x_2, x_3) d\mu(x_2, x_3) \equiv \int_{\mathbb{R}^2} F(0, x_2, x_3) d\mu(x_2, x_3).$$

Since $F(t, x_2, x_3) \leq F(0, x_2, x_3)$ (remark (iii)), and μ is a positive measure with continuous density, we must have $F(t, x_2, x_3) \equiv F(0, x_2, x_3)$ for every t, and this is clearly a contradiction. □

Remark 1. Note that the statement of Theorem 1 is not true for two-dimensional spaces all of which embed isometrically in L_p for every $p \in (0, 1]$. The reason that the proof of Theorem 1 does not work for two-dimensional spaces is that, unlike the function $\|x_2 e_2 + x_3 e_3\|^{p-2}$ on $\mathbb{R}^2$, the function $|x_2|^{p-2}$ is not locally integrable on $\mathbb{R}$ if $0 < p \leq 1$.

3. Applications to certain spaces.

Let us first present a class of Orlicz spaces satisfying the conditions of Theorem 1. An Orlicz function M is a non-decreasing convex function on $[0, \infty)$ such that $M(0) = 0$ and $M(t) > 0$ for every $t > 0$.

For an Orlicz function M, the norm $\|\cdot\|_M$ of the n-dimensional Orlicz space ℓ_M^n is defined by the equality $\sum_{k=1}^n M(|x_k|/\|x\|_M) = 1, \ x \in \mathbb{R}^n \setminus \{0\}$.

Theorem 2. *Let M be an Orlicz function so that $M \in C^2([0, \infty))$, $M'(0) = M''(0) = 0$. Then, for every $0 < p \leq 2$ and $n \geq 3$, the function $\exp(-\|x\|_M^p)$ is not positive definite and the space ℓ_M^n does not embed isometrically into L_p.*

Proof. Clearly, we can assume $n = 3$. We are going to show that the norm of the space ℓ_M^3 satisfies the conditions of Theorem 1. Since the Orlicz norm is an even function with respect to each variable, it suffices to consider the points $x = (x_1, x_2, x_3)$ with non-negative coordinates. We denote by e_1, e_2, e_3 the standard normalized basis in ℓ_M^3. In order to avoid unwieldy notation, we shall omit below the subscript M from the norm $\|\cdot\|_M$.

The function M' is non-decreasing, continuous on $[0, \infty)$ and $M'(0) = 0$. Since $M(0) = 0$ and $M(t) > 0$ for every $t > 0$, the function M' can not be equal to zero on an interval, so $M'(t) > 0$ for every $t > 0$.

Let $x = (x_1, x_2, x_3)$ with $(x_2, x_3) \neq 0$. Then one of the numbers $x_2 M'(x_2/\|x\|)$ or $x_3 M'(x_3/\|x\|)$ is positive. By implicit differentiation,

$$\|x\|'_{x_1} = \frac{\|x\| M'(\frac{x_1}{\|x\|})}{x_1 M'(\frac{x_1}{\|x\|}) + x_2 M'(\frac{x_2}{\|x\|}) + x_3 M'(\frac{x_3}{\|x\|})}.$$

Also,

(5)
$$\|x\|''_{x_1^2} = \frac{(\|x\| - x_1\|x\|'_{x_1})^2 M''(\frac{x_1}{\|x\|}) + x_2^2 (\|x\|'_{x_1})^2 M''(\frac{x_2}{\|x\|}) + x_3^2 (\|x\|'_{x_1})^2 M''(\frac{x_3}{\|x\|})}{\|x\|^2 \big(x_1 M'(\frac{x_1}{\|x\|}) + x_2 M'(\frac{x_2}{\|x\|}) + x_3 M'(\frac{x_3}{\|x\|})\big)}.$$

The condition (1) of Theorem 1 follows from the fact that $M'(0) = M''(0) = 0$.

Let us show that the norm satisfies the condition (2) of Theorem 1. Put

$$c = \min\{x_2 M'(x_2/2) + x_3 M'(x_3/2) : \|x_2 e_2 + x_3 e_3\| = 1,\ x_2, x_3 \geq 0\}.$$

Since M' is a continuous function and $M'(t) > 0$ for $t > 0$, we have $c > 0$. Let $d = \max_{t\in[0,1]} M''(t)$.

Clearly, $x_i \leq \|x\|$, $i = 1, 2, 3$. Therefore, using both Remark (ii) and the positiveness of $\|x\|'_{x_1}$, we have $0 \leq \|x\| - x_1\|x\|'_{x_1} \leq \|x\|$, and $(\|x\| - x_1\|x\|'_{x_1})^2/\|x\|^2 \leq 1$.

Consider any $x_2, x_3 \geq 0$ with $\|x_2 e_2 + x_3 e_3\| = 1$. Then $x_2, x_3 \leq 1$. If $x_1 \in [0,1]$ then $1 \leq \|x\| \leq 2$, hence, $x_i/\|x\| \geq (x_i/2)$, $i = 1, 2, 3$. We get from (5) that $\|x\|''_{x_1^2} \leq 3d/c$.

If $x_1 > 1$ then $x_1/\|x\| > 1/2$, and (5) implies $\|x\|''_{x_1^2} \leq 3d/M'(1/2)$, because M' is an increasing function.

We see that in either case $x_1 \in [0,1]$, or $x_1 > 1$, the second derivative is bounded by constants which do not depend on the choice of x_2, x_3 with $\|x_2 e_2 + x_3 e_3\| = 1$. Thus condition (2) of Theorem 1, is valid. $\square$

The class of Orlicz functions satisfying the conditions of Theorem 2 includes all the functions $M(t) = |t|^q$, $q > 2$. So, Theorem 2 generalizes the solution to Schoenberg's problem.

Given a sequence of Banach spaces $\{Y_k\}_{k=1}^n$, their q-sum, denoted by $\sum_{k=1}^n \oplus_q Y_k$ is the space of all sequences $(y_1, \ldots, y_n)$ equipped with the norm

$$\|(y_1, \ldots, y_n)\| = \left(\sum_{k=1}^n \|y_k\|^q\right)^{1/q}.$$

Concerning q-sums, the following result holds.

Theorem 3. *If $q > 2$ and $0 < p \leq 2$, then for the norm of the q-sum $X \oplus_q Y$, where $\dim X \geq 2$, the function $\exp(-\|x\|^p)$ is not positive definite , and there is no isometric embedding of $X \oplus_q Y$ into L_p.*

Proof. Any q-sum whose dimension is at least three contains (isometrically) a q-sum of the form $X \oplus_q \mathbb{R}$ where $X = (\mathbb{R}^2, \|\cdot\|)$. We shall apply Theorem 1 to show that such q-sums are not isometric to a subspace of L_p if $q > 2$ and $0 < p \leq 2$. Indeed, the space $X \oplus_q \mathbb{R}$ can be identified with $\mathbb{R}^3$ equipped with the norm

$$\|(x_1, x_2, x_3)\| = (\|(x_1, x_2)\|^q + |x_3|^q)^{1/q}.$$

The first and second partial derivatives with respect to x_3, vanish at zero. Moreover,

$$\|x\|''_{x_3}(x_1, x_2, x_3) = (q-1)|x_3|^{q-2}\|(x_1, x_2)\|^q(\|(x_1, x_2)\|^q + |x_3|^q)^{\frac{1}{q}-2},$$

which shows that on the set $\{(x_1, x_2, x_3) : \|(x_1, x_2)\| = 1\}$, the second derivative is uniformly bounded. Thus Theorem 1 can be applied. □

Remark 2. A special case of Theorem 3 was proved by Dor in [3]. Namely, Dor proved that if dim $X \geq 2$ and X is not a Hilbert space then no q-sum of the form $X \oplus_q Y$ where $q > 2$, embeds isometrically into L_1. Theorem 3 can also be proved by using Zastavny's result which was mentioned in the introduction.

Remark 3. A related result to Theorem 1 has been recently established by the first named author. Namely, it can be shown that the unit ball of a normed space is not an intersection body (An origin symmetric convex body K in $\mathbb{R}^n$ is called an intersection body if its radial function is the spherical Radon transform of a non-negative measure on the unit sphere S^{n-1}) if the dimension of the space is greater than or equal to 5, and the norm satisfies the conditions of Theorem 1, and the condition that $\lim_{x_1 \to 0} \|(x_1, x_2, x_3)\|''_{x_1^2} = 0$ uniformly on the set $\{(x_1, x_2, x_3) : \|x_2 e_2 + x_3 e_3\| = 1\}$. This result provides an alternative way to conclude that the Busemann-Petty problem on sections of convex bodies has a negative answer when the dimension is at least 5. For more details see [8].

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