# The inversion of fractional integrals on a sphere

Boris Rubin[1]

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Abstract

The purpose of the paper is to invert Riesz potentials and some other fractional integrals on a spherical surface in $\mathbb{R}^{n+1}$ in the closed form. New descriptions of spaces of the fractional smoothness on a sphere are obtained in terms of spherical hypersingular integrals. It is shown that Riesz potentials of the orders $n, n+2, n+4, \dots$ on a sphere may be Noether operators with a $d$-characteristic which depends on the radius of the sphere.

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## Introduction

There are many types of fractional integrals defined on the surface of the $n$-dimensional unit sphere $\Sigma_n \subset \mathbb{R}^{n+1}$. One of them is a Riesz potential

$$(1) \qquad (I^\alpha \varphi)(x) = c_{n,\alpha} \int_{\Sigma_n} |x-y|^{\alpha-n} \varphi(y) dy,$$

where $\alpha > 0$; $\alpha \neq n, n+2, n+4, \ldots$;

$$(2) \qquad {}_{n,\alpha} = 2^{-\alpha} \pi^{-n/2} \Gamma\left(\frac{n-2}{2}\right) / \Gamma(\frac{\alpha}{2}).$$

Due to the outward simplicity and to the plurality of applications the Riesz potential is a typical object in fractional calculus. Nevertheless, the inversion method for $I^\alpha$, covering all admissible $\alpha$ seems unknown. There is a simple idea to change variables in (1), using the stereographic projection, and to turn the potential (1) in such a way into the Riesz potential over $\mathbb{R}^n$ (up to some multipliers). The latter may be inverted by diverse known methods (see [14], [13]). This approach, suggested by the author, enables one to obtain a number of estimates of $I^\alpha \varphi$ using the corresponding estimates of the space potentials (see [10], [19]). However, this way leads to the unnatural awkward construction of $(I^\alpha)^{-1}$ which depends on the pole of the projection. Furthermore, the proof of such an inversion formula is connected with large technical difficulties. It is more preferable to construct the operator $(I^\alpha)^{-1}$ directly in spherical terms. In [10] Pavlov P.M. and Samko S.G. proved that if $f = I^\alpha \varphi$, $\varphi \in L_p(\Sigma_n)$, $0 < \alpha < 2, 1 \leq p < \infty$, then

$$(3) \qquad \varphi(x) = c_1 f(x) + c_2 \int_{\Sigma_n} \frac{f(x) - f(y)}{|x-y|^{n+\alpha}} dy,$$

where

$$c_1 = \Gamma\left(\frac{n+\alpha}{2}\right) / \Gamma\left(\frac{n-\alpha}{2}\right)$$

,

$$c_2 = \frac{2^{\alpha-1} \alpha \Gamma\left(\frac{n+\alpha}{2}\right)}{\pi^{n/2} \Gamma\left(1 - \frac{\alpha}{2}\right)}$$

,

$$\int_{\Sigma_n} (\ldots) = \overset{(L_p)}{\lim_{\varepsilon \to \infty}} \int_{|x-y| > \varepsilon} (\ldots)$$

.

The method of [10] gives no answer how to invert $I^\alpha$ for all $\alpha \geq 2$. In the present paper we suggest two different inversion methods for Riesz potentials of finite Borel measures in spherical terms. These methods are suitable for all $\alpha > 0$ ( the definition of $I^\alpha\varphi$ for $\alpha = n, n+2, n+4, \ldots$, see below) and may be generalized for all complex $\alpha$ with $\mathrm{Re}\,\alpha > 0$ as in [13]. Our formulas contain hypersingular integrals, the convergence of which is associated with a type of the measure to be restored. For arbitrary finite Borel measure these integrals converge in a weak sense. If the measure is absolutely continuous with a density belonging to $L_p(\Sigma_n)$, $1 \leq p < \infty$, then the convergence of hypersingular integrals is treated in the "almost everywhere" sense and in $L_p$-norm. If the density is continuous, then a uniform convergence is used.

In section 1, we construct the operator $(I^\alpha)^{-1}$ using a direct regularization of the potential $I^\alpha\varphi$. This method was developed in [13]. The case $\alpha = n$ when $I^\alpha\varphi$ turns into the logarithmic potential, is considered in section 2. Another inversion method for $I^\alpha\varphi$, based on properties of a Poisson integral, is given in section 3.

The inversion problem for potentials (1) is closely connected with the characterization of functions of a fractional smoothness on a sphere. In section 4 we give a number of diverse descriptions of the spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$ generated by $L_p$-functions, by continuous functions and by finite Borel measures respectively. By the way we obtain inversion formulas for some fractional integral operators introduced by du Plessis N.[11], Greenwald H.C. [6], [7], Muckenhoupt B. and Stein E.M.[9]. All these operators have the same range as $I^\alpha$ (with the exception of some values of $\alpha$) and are built by means of a Poisson integral.

The investigation of Riesz potentials of the orders $\alpha = n + 2k$, $k = 0, 1, \ldots$, leads to the following integral equation on a sphere $\Sigma_n(a) = \{x \in \mathbb{R}^{n+1} : |x| = a\}$:

$$\int_{\Sigma_n(a)} \varphi(y)|x-y|^{2k} \log|x-y| dy = f(x) \tag{4}$$

In section 5 we show that in contrast to the case $\alpha \neq n + 2k$ the operator in the left-hand side of (4) may be the Noether one for some radii $a$. We define its two-sided regularizer and the $d$-characteristic explicity. It is interesting that the $d$-characteristic depends on the value of a radius $a$.

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**Notation**

$$\Sigma_n\{x\in\mathbb{R}^{n+1}:|x|=1\},\ \sigma_n=|\Sigma_n|=2\pi^{\frac{n+1}{2}}/\Gamma\left(\frac{n+1}{2}\right);$$

$dx$ denotes the Lebesgue measure on $\Sigma_n$; $\mathcal{Y}(\Sigma_n)=\{Y_{m,\mu}(x)\}$ denotes a complete orthonormal system of spherical harmonics on $\Sigma_n$; $m=0,1,\ldots;\quad \mu=1,2,\ldots,d_n(m),\ d_n(m)$ being a dimension of the subspace of harmonics of the order $m$, $d_n(m)=(n+2m-1)\frac{(n+m-2)!}{m!(n-1)!}$ (see [18]). $\mathcal{B}(\Sigma_n)$ is the Borel $\sigma$-algebra of $\Sigma_n$. $M(\Sigma_n)$ denotes a Banach space of all regular complex valued finite Borel measures on $\mathcal{B}(\Sigma_n)$ with the norm $\|\nu\|_M$ equaled to a total variation of the measure $\nu$ on $\Sigma_n$ ([3]); $C(\Sigma_n)$ denotes the space of all continuous functions on $\Sigma_n$; $S(\Sigma_n)$ denotes the space of all infinitely differentiable functions on $\Sigma_n$ with the standard Shwartz topology; $S'(\Sigma_n)$ is a dual to $S(\Sigma_n)$; $(f,\omega)$ denotes a value of a functional $f\in S'(\Sigma_n)$ on a function $\omega\in S(\Sigma_n)$. If $f\in M(\Sigma_n)(f\in L_1(\Sigma_n))$, then

$$(f,\omega)=\int_{\Sigma_n}\omega(x)df\ \left((f,\omega)=\int_{\Sigma_n}\omega(x)f(x)dx\right);$$

$f_{m,\mu}=(f,Y_{m,\mu})$ denote Fourier-Laplace coefficients of a functional $f\in S'(\Sigma_n)$;

$e_{n+1}(0,\ldots,0,1)$; $a_+^\lambda=(\sup\{a,0\})^\lambda$; $P^{(\rho,\sigma)}(t)$ denotes a Jacobi polynomial; $\mathbb{Z}_+$ denotes the set of all nonnegative integers;

$$\|\varphi\|_p=\|\varphi\|_{L_p(\Sigma_n)};$$

$$P_z(x,y)=\frac{1-r^2}{\sigma_n|y-rx|^{n+1}}\quad\text{is a Poisson kernel, } 0<r<1;$$

$f(x,r)=(f,P_r(x,\cdot))$ denotes a Poisson integral of a function (measure) $f$.

$$(I_+^\lambda\psi)(\tau)=\frac{1}{\Gamma(\lambda)}\int_{-\infty}^{\tau}\psi(t)(\tau-t)^{\lambda-1}dt \tag{5}$$

is a Riemann-Liouville fractional integral of the order $\lambda>0$. We define a truncated Marchund derivative by the equality

$$(D_{+,\varepsilon}^\lambda\psi)(\tau)=\frac{1}{\kappa_\ell(\lambda)}\int_\varepsilon^\infty\left(\sum_{j=0}^{\ell}\binom{\ell}{j}(-1)^jf(\tau-jt)\right)\frac{dt}{t^{1+\lambda}}$$

, where $\varepsilon > 0$, $\ell > \lambda$,

$$\kappa_\ell(\lambda) = \int_0^\infty \frac{(1-e^{-t})^\ell}{t^{1+\lambda}} dt \text{ (see [14])}$$

.

Let $E \subset \mathbb{R}$ be some set with a limit point $\varepsilon_0$, and let $\{A_\varepsilon\}_{\varepsilon \in E}$ be a family of linear operators defined on $\mathcal{Y}(\Sigma_n)$. If $\lim_{\varepsilon \to \varepsilon_0} A_\varepsilon Y_{m,\mu} = Y_{m,\mu}$ $\forall Y_{m,\mu} \in \mathcal{Y}(\Sigma_n)$, then the famlily $\{A_\varepsilon\}$ will be called an approximative identity as $\varepsilon \to \varepsilon_0$.

Let us introduce functional spaces to be used later. Given $\alpha \in \mathbb{R}$; $1 \leq p \leq \infty$, we denote by $L_p^\alpha(\Sigma_n)$ $(C^\alpha(\Sigma_n), M^\alpha(\Sigma_n))$ the space of functionals $f \in S'(\Sigma_n)$ with the following property: for each $f \in S'(\Sigma_n)$ there exists a function $f^{(\alpha)} \in L_p(\Sigma_n)(f^{(\alpha)} \in C(\Sigma_n)$, a measure $f^{(\alpha)} \in M(\Sigma_n))$ such that $f_{m,\mu}^{(\alpha)} = (m+1)^\alpha f_{m,\mu}$ for any $m, \mu$. The space $L_p^\alpha(\Sigma_n)$ $(C^\alpha(\Sigma_n), M^\alpha(\Sigma_n))$ is a Banach one with respect to the norm

$$\|f\| = \|f^{(\alpha)}\|_p \;\; (\|f\| = \|f^{(\alpha)}\|_{C(\Sigma_n)}, \; \|f\| = \|f^{(\alpha)}\|_{M(\Sigma_n)}) \tag{6}$$

If $\alpha > 0$, the elements of the spaces $L_p^\alpha(\Sigma_n)$, $C^\alpha(\Sigma_n)$, $M^\alpha(\Sigma_n)$ are usual functions represented by spherical fractional integrals (see section 4). Besides the Riesz potential that has the expansion

$$I^\alpha \varphi \sim \sum_{m,\mu} \frac{\Gamma\left(m + \frac{n-\alpha}{2}\right)}{\Gamma\left(m + \frac{n+\alpha}{2}\right)} \varphi_{m,\mu} Y_{m,\mu} \tag{7}$$

(see[15]) we shall use the following fractional integrals

$$I_1^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-\rho)^{\alpha-1} \varphi(x,\rho) d\rho \left( \sim \sum_{m,\mu} \frac{\Gamma(m+1)}{\Gamma(m+1+\alpha)} \varphi_{m,\mu} Y_{m,\mu} \right), \tag{8}$$

$$I_2^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 (\log \frac{1}{\rho})^{\alpha-1} \varphi(x,\rho) d\rho \left( \sim \sum_{m,\mu} (m+1)^{-\alpha} \varphi_{m,\mu} Y_{m,\mu} \right), \tag{9}$$

$$I_3^\alpha \varphi = \frac{1}{\Gamma(\alpha)} \int_0^1 (\log \frac{1}{\rho})^{\alpha-1} \varphi(x,\rho) \frac{d\rho}{\rho} \left( \sim \sum_{m,\mu} m^{-\alpha} \varphi_{m,\mu} Y_{m,\mu} \right), \tag{10}$$

$$I_4^\alpha \varphi \frac{\pi^{\frac{1}{2}} (n-1)^{\frac{1-\alpha}{2}}}{\Gamma(\alpha/2)} \int_0^1 \rho^{\frac{n-3}{2}} (\log \frac{1}{\rho})^{\alpha-1} I_{\frac{\alpha-1}{2}} (\frac{n-1}{2} \log \frac{1}{\rho}) \varphi(x,\rho) d\rho$$