THE MINIMAL CARDINALITY WHERE THE REZNICHENKO PROPERTY FAILS

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ABSTRACT. A topological space $X$ has the Fréchet-Urysohn property if for each subset $A$ of $X$ and each element $x$ in $\overline{A}$, there exists a countable sequence of elements of $A$ which converges to $x$. Reznichenko introduced a natural generalization of this property, where the converging sequence of elements is replaced by a sequence of disjoint finite sets which eventually intersect each neighborhood of $x$. In [5], Kočinac and Scheepers conjecture:

The minimal cardinality of a set $X$ of real numbers such that $C_p(X)$ does not have the weak Fréchet-Urysohn property is equal to $b$.

($b$ is the minimal cardinality of an unbounded family in the Baire space $\mathbb{N}^\mathbb{N}$). We prove the Kočinac-Scheepers conjecture by showing that if $C_p(X)$ has the Reznichenko property, then a continuous image of $X$ cannot be a subbase for a non-feeble filter on $\mathbb{N}$.

1. INTRODUCTION

A topological space $X$ has the Fréchet-Urysohn property if for each subset $A$ of $X$ and each $x \in \overline{A}$, there exists a sequence $\{a_n\}_{n \in \mathbb{N}}$ of elements of $A$ which converges to $x$. If $x \not\in A$ then we may assume that the elements $a_n$, $n \in \mathbb{N}$, are distinct. The following natural generalization of this property was introduced by Reznichenko [7]:

For each subset $A$ of $X$ and each element $x$ in $\overline{A}\setminus A$, there exists a countably infinite pairwise disjoint collection $\mathcal{F}$ of finite subsets of $A$ such that for each neighborhood $U$ of $x$, $U \cap F \neq \emptyset$ for all but finitely many $F \in \mathcal{F}$.

In [7] this is called the weak Fréchet-Urysohn property. In other works [5, 6, 10] this also appears as the Reznichenko property.
For a topological space $X$ denote by $C_p(X)$ the space of continuous real-valued functions with the topology of pointwise convergence. A comprehensive duality theory was developed by Arkhangel'skiĭ and others (see, e.g., [2, 9, 5, 6] and references therein) which characterizes topological properties of $C_p(X)$ for a Tychonoff space $X$ in terms of covering properties of $X$. In [5, 6] this is done for a conjunction of the Reznichenko property and some other classical property (countable strong fan tightness in [5] and countable fan tightness in [6]). According to Sakai [9], a space $X$ has countable fan tightness if for each $x \in X$ and each sequence $\{A_n\}_{n \in \mathbb{N}}$ of subsets of $X$ with $x \in \overline{A_n} \setminus A_n$ for each $n$, there exist finite sets $F_n \subseteq A_n$, $n \in \mathbb{N}$, such that $x \in \bigcup F_n$. In Theorem 19 of [6], Kočinac and Scheepers prove that for a Tychonoff space $X$, $C_p(X)$ has countable fan tightness as well as Reznichenko’s property if, and only if, each finite power of $X$ has the Hurewicz covering property.

The Baire space $^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers is equipped with the product topology (where the topology of $\mathbb{N}$ is discrete). A quasi-ordering $\preceq^*$ is defined on the Baire space $^{\mathbb{N}}\mathbb{N}$ by eventual dominance:

$$f \preceq^* g \text{ if } f(n) \leq g(n) \text{ for all but finitely many } n.$$ 

We say that a subset $Y$ of $^{\mathbb{N}}\mathbb{N}$ is bounded if there exists $g$ in $^{\mathbb{N}}\mathbb{N}$ such that for each $f \in Y$, $f \preceq^* g$. Otherwise, we say that $Y$ is unbounded. $\mathfrak{b}$ denotes the minimal cardinality of an unbounded family in $^{\mathbb{N}}\mathbb{N}$. According to a theorem of Hurewicz [3], a set of reals $X$ has the Hurewicz property if, and only if, each continuous image of $X$ in $^{\mathbb{N}}\mathbb{N}$ is bounded. This and the preceding discussion imply that for each set of reals $X$ of cardinality smaller than $\mathfrak{b}$, $C_p(X)$ has the Reznichenko property. Kočinac and Scheepers conclude their paper [5] with the following.

**Conjecture 1.** $\mathfrak{b}$ is the minimal cardinality of a set $X$ of real numbers such that $C_p(X)$ does not have the Reznichenko property.

We prove that this conjecture is true.

### 2. A proof of the Kočinac-Scheepers conjecture

Throughout the paper, when we say that $\mathcal{U}$ is a cover of $X$ we mean that $X \subseteq \bigcup \mathcal{U}$ but $X$ is not contained in any member of $\mathcal{U}$. A cover $\mathcal{U}$ of a space $X$ is an $\omega$-cover of $X$ if each finite subset $F$ of $X$ is contained in some member of $\mathcal{U}$. This notion is due to Gerlits and Nagy [2], and is starring in [5, 6]. According to [5, 6], a cover $\mathcal{U}$ of $X$ is $\omega$-groupable if there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each finite $F \subseteq X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, there exists $U \in \mathcal{F}$ such that...
$F \subseteq U$. Thus, each $\omega$-$\text{groupable}$ cover is an $\omega$-cover and contains a countable $\omega$-$\text{groupable}$ cover.

In [6] it is proved that if each open $\omega$-cover of a set of reals $X$ is $\omega$-$\text{groupable}$ and $C_p(X)$ has countable fan tightness, then $C_p(X)$ has the Reznichenko property. Recently, Sakai [10] proved that the assumption of countable fan tightness is not needed here. More precisely, say that an open $\omega$-cover $U$ of $X$ is $\omega$-$\text{shrinkable}$ if for each $U \in U$ there exists a closed subset $C_U \subseteq U$ such that $\{C_U : U \in U\}$ is an $\omega$-cover of $X$. Then the following duality result holds.

**Theorem 2** (Sakai [10]). For a Tychonoff space $X$, the following are equivalent:

1. $C_p(X)$ has the Reznichenko property;
2. Each $\omega$-$\text{shrinkable}$ open $\omega$-cover of $X$ is $\omega$-$\text{groupable}$.

It is the other direction of this result that we are interested in here. Observe that any clopen $\omega$-cover is trivially $\omega$-$\text{shrinkable}$.

**Corollary 3.** Assume that $X$ is a Tychonoff space such that $C_p(X)$ has the Reznichenko property. Then each clopen $\omega$-cover of $X$ is $\omega$-$\text{groupable}$.

From now on $X$ will always denote a set of reals. As all powers of sets of reals are Lindelöf, we may assume that all covers we consider are countable [2]. For conciseness, we introduce some notation. For collections of covers of $X$ $\mathcal{U}$ and $\mathcal{V}$, we say that $X$ satisfies $(\mathcal{U}, \mathcal{V})$ (read: $\mathcal{U}$ choose $\mathcal{V}$) if each element of $\mathcal{U}$ contains an element of $\mathcal{V}$ [14]. Let $C_{\mathcal{G}}$ and $C_{\text{Gpr}}$ denote the collections of clopen $\omega$-covers and clopen $\omega$-$\text{groupable}$ covers of $X$, respectively. Corollary 3 says that the Reznichenko property for $C_p(X)$ implies $(C_{\mathcal{G}}, C_{\text{Gpr}})$.

As a warm up towards the real solution, we make the following observation. According to [11], a space $X$ satisfies $\text{Split}(\mathcal{U}, \mathcal{V})$ if every cover $U \in \mathcal{U}$ can be split into two disjoint subcovers $V$ and $W$ which contain elements of $\mathcal{V}$. Observe that $(C_{\mathcal{G}}, C_{\text{Gpr}})$ implies $\text{Split}(C_{\mathcal{G}}, C_{\mathcal{G}})$. The critical cardinality of a property $P$ (or collection) of sets of reals, $\text{non}(P)$, is the minimal cardinality of a set of reals which does not satisfy this property. Write

$$\text{cf}_3 = \text{non}(\{X : C_p(X) \text{ has the Reznichenko property}\}).$$

Then we know that $\mathfrak{b} \leq \text{cf}_3$, and the Kočinac-Scheepers conjecture asserts that $\text{cf}_3 = \mathfrak{b}$. By Corollary 3, $\text{cf}_3 \leq \text{non}(\text{Split}(C_{\mathcal{G}}, C_{\mathcal{G}}))$. In [4] it is proved that $\text{non}(\text{Split}(C_{\mathcal{G}}, C_{\mathcal{G}})) = \mathfrak{u}$, where $\mathfrak{u}$ is the ultrafilter number denoting the minimal size of a base for a nonprincipal ultrafilter on $\mathbb{N}$. Consequently, $\text{cf}_3 \leq \mathfrak{u}$. It is well known that $\mathfrak{b} \leq \mathfrak{u}$, but it is consistent
that $b < u$. Thus this does not prove the conjecture. However, this is the approach that we will use: We will use the language of filters to prove that $\non\left(\mathcal{C}_{\text{pp}}\right) = b$. By Corollary 3, $b \leq \aleph_3 \leq \non\left(\mathcal{C}_{\text{pp}}\right)$, so this will suffice.

A nonprincipal filter on $\mathbb{N}$ is a family $\mathcal{F} \subseteq P(\mathbb{N})$ that contains all cofinite sets but not the empty set, is closed under supersets, and is closed under finite intersections (in particular, all elements of a nonprincipal filter are infinite). A base $\mathcal{B}$ for a nonprincipal filter $\mathcal{F}$ is a subfamily of $\mathcal{F}$ such that for each $A \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $B \subseteq A$. If the closure of $\mathcal{B}$ under finite intersections is a base for a nonprincipal filter $\mathcal{F}$, then we say that $\mathcal{B}$ is a subbase for $\mathcal{F}$. A family $\mathcal{Y} \subseteq P(\mathbb{N})$ is centered if for each finite subset $\mathcal{A}$ of $\mathcal{Y}$, $\cap \mathcal{A}$ is infinite. Thus a subbase $\mathcal{B}$ for a nonprincipal filter is a centered family such that for each $n$ there exists $B \in \mathcal{B}$ with $n \notin B$. For a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$ and a finite-to-one function $f : \mathbb{N} \to \mathbb{N}$, $f(\mathcal{F}) := \{A \subseteq \mathbb{N} : f^{-1}[A] \in \mathcal{F}\}$ is again a nonprincipal filter on $\mathbb{N}$.

A filter $\mathcal{F}$ is feeble if there exists a finite-to-one function $f$ such that $f(\mathcal{F})$ consists of only the cofinite sets, $\mathcal{F}$ is feeble if, and only if, there exists a partition $\{F_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $A \in \mathcal{F}$, $A \cap F_n \neq \emptyset$ for all but finitely many $n$ (take $F_n = f^{-1}[\{n\}]$). Thus $\mathcal{B}$ is a subbase for a feeble filter if, and only if:

1. $\mathcal{B}$ is centered,
2. For each $n$ there exists $B \in \mathcal{B}$ such that $n \notin B$; and
3. There exists a partition $\{F_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $k$ and each $A_1, \ldots, A_k \in \mathcal{B}$, $A_1 \cap \cdots \cap A_k \cap F_n \neq \emptyset$ for all but finitely many $n$.

Define a topology on $P(\mathbb{N})$ by identifying it with Cantor’s space $\mathbb{N} \{0, 1\}$ (which is equipped with the product topology).

**Theorem 4.** For a set of reals $X$, the following are equivalent:

1. $X$ satisfies $\left(\mathcal{C}_{\text{pp}}\right)$;
2. For each continuous function $\Psi : X \to P(\mathbb{N})$, $\Psi[X]$ is not a subbase for a non-feeble filter on $\mathbb{N}$.

**Proof.** ($1 \Rightarrow 2$) Assume that $\Psi : X \to P(\mathbb{N})$ is continuous and $\mathcal{B} = \Psi[X]$ is a subbase for a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$. Consider the (clopen) subsets $O_n = \{A \subseteq \mathbb{N} : n \in A\}$, $n \in \mathbb{N}$, of $P(\mathbb{N})$. For each $n$, there exists $B \in \mathcal{B}$ such that $B \notin O_n$ ($n \notin B$), thus $X \not\subseteq \Psi^{-1}[O_n]$.

As $\mathcal{B}$ is centered, $\{O_n\}_{n \in \mathbb{N}}$ is an $\omega$-cover of $\mathcal{B}$, and therefore $\{\Psi^{-1}[O_n]\}_{n \in \mathbb{N}}$ is a clopen $\omega$-cover of $X$. Let $A \subseteq \mathbb{N}$ be such that the enumeration $\{\Psi^{-1}[O_n]\}_{n \in A}$ is bijective. Apply $\left(\mathcal{C}_{\text{pp}}\right)$ to obtain a partition $\{F_n\}_{n \in \mathbb{N}}$ of $A$ into finite sets such that for each finite $F \subseteq X$, and all but
finally many $n$, there exists $m \in F_n$ such that $F \subseteq \Psi^{-1}[O_m]$ (that is, $\Psi[F] \subseteq O_m$, or $\bigcap_{F_n} \Psi(x) \cap F_n \neq \emptyset$). Add to each $F_n$ an element from $\mathbb{N} \setminus A$ so that $\{F_n\}_{n \in \mathbb{N}}$ becomes a partition of $\mathbb{N}$. Then the sequence $\{F_n\}_{n \in \mathbb{N}}$ witnesses that $\mathcal{B}$ is a subbase for a feeble filter.

(2 $\Rightarrow$ 1) Assume that $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is a clopen $\omega$-cover of $X$. Define $\Psi : X \to P(\mathbb{N})$ by

$$
\Psi(x) = \{n : x \in U_n\}.
$$

As $\mathcal{U}$ is clopen, $\Psi$ is continuous. As $\mathcal{U}$ is an $\omega$-cover of $X$, $\mathcal{B} = \Psi[X]$ is centered (see Lemma 2.2 in [13]). For each $n$ there exists $x \in X \setminus U_n$, thus $n \notin \Psi(x)$. Therefore $\mathcal{B}$ is a subbase for a feeble filter. Fix a partition $\{F_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $\Psi(x_1), \ldots, \Psi(x_k) \in \mathcal{B}$, $\Psi(x_1) \cap \cdots \cap \Psi(x_k) \cap F_n \neq \emptyset$ (that is, there exists $m \in F_n$ such that $x_1, \ldots, x_k \in U_m$) for all but finitely many $n$. This shows that $\mathcal{U}$ is groupable.

\begin{cor}
\textbf{Corollary 5.} $\non\left(\left(\frac{C_0}{C_{app}}\right)\right) = b$.
\end{cor}

\begin{proof}
Every nonprincipal filter on $\mathbb{N}$ with a (sub)base of cardinality smaller than $b$ is feeble (essentially, [12]), and by an unpublished result of Petr Simon, there exists a non-feeble filter with a (sub)base of cardinality $b$ — see [1] for the proofs. Use Theorem 4.
\end{proof}

This completes the proof of the Kočinac-Scheepers conjecture.

3. CONSEQUENCES AND OPEN PROBLEMS

Let $\mathcal{B}_\alpha$ and $\mathcal{B}_{\alpha_{pp}}$ denote the collections of countable Borel $\omega$-covers and $\omega$-groupable covers of $X$, respectively. The same proof as in Theorem 4 shows that the analogue theorem where “continuous” is replaced by “Borel” holds.

$\mathcal{U}$ is a large cover of a space $X$ if each member of $X$ is contained in infinitely many members of $\mathcal{U}$. Let $\mathcal{B}_\Lambda$, $\Lambda$, and $C_\Lambda$ denote the collections of countable large Borel, open, and clopen covers of $X$, respectively. According to [6], a large cover $\mathcal{U}$ of $X$ is groupable if there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each $x \in X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, $x \in \cup \mathcal{F}$. Let $\mathcal{B}_{\Lambda_{pp}}$, $\Lambda_{pp}$, and $C_{\Lambda_{pp}}$ denote the collections of countable groupable Borel, open, and clopen covers of $X$, respectively.

\begin{cor}
\textbf{Corollary 6.} The critical cardinalities of the classes $(\mathcal{B}_\alpha)$, $(\mathcal{B}_{\alpha_{pp}})$, $(\mathcal{B}_\alpha)$, $(\alpha_{pp})$, $(\alpha_{pp})$, $(\mathcal{B}_\alpha)$, $(\alpha_{pp})$, $(\mathcal{B}_{\alpha_{pp}})$, $(\alpha_{pp})$, and $(\mathcal{B}_{\alpha_{pp}})$ are all equal to $b$.
\end{cor}

\begin{proof}
By the Borel version of Theorem 4, $\non(\mathcal{B}_\alpha) = b$. In [15] it is proved that $\non(\mathcal{B}_{\alpha_{pp}}) = b$. These two properties imply all other
properties in the list. Now, all properties in the list imply either \( C_{\kappa^p} \)

or \( C_{\kappa^{2^p}} \), whose critical cardinality is \( b \) by Theorem 4 and [15]. □

If we forget about the topology and consider arbitrary countable covers, we get the following characterization of \( b \), which extends Theorem 15 of [6] and Corollary 2.7 of [15]. For a cardinal \( \kappa \), denote by \( \Lambda_{\kappa}, \Omega_{\kappa}, \Lambda_{\kappa}^{gp}, \) and \( \Omega_{\kappa}^{gp} \) the collections of countable large covers, \( \omega \)-covers, groupable covers, and \( \omega \)-groupable covers of \( \kappa \), respectively.

**Corollary 7.** For an infinite cardinal \( \kappa \), the following are equivalent:

1. \( \kappa < b \),
2. \( \Lambda_{\kappa}^{gp} \),
3. \( \Omega_{\kappa}^{gp} \); and
4. \( \Omega_{\kappa}^{gp} \).

It is an open problem [10] whether item (2) in Sakai’s Theorem 2 can be replaced with \( \Omega_{\kappa}^{gp} \) (by the theorem, if \( X \) satisfies \( \Omega_{\kappa}^{gp} \), then \( C_{\kappa}(X) \) has the Reznichenko property; the other direction is the unclear one).

For collections \( \mathcal{U} \) and \( \mathcal{V} \) of covers of \( X \), we say that \( X \) satisfies \( S_{fin}(\mathcal{U}, \mathcal{V}) \) if:

For each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of members of \( \mathcal{U} \), there is a sequence \( \{F_n\}_{n \in \mathbb{N}} \) such that each \( F_n \) is a finite subset of \( U_n \), and \( \bigcup_{n \in \mathbb{N}} F_n \in \mathcal{V} \).

In [15] it is proved that \( \Omega_{\kappa}^{gp} = S_{fin}(\Lambda, \Lambda^{gp}) \) (which is the same as the Hurewicz covering property [6]). We do not know whether the analogue result for \( \Omega_{\kappa}^{gp} \) is true.

**Problem 8.** Does \( \Omega_{\kappa}^{gp} = S_{fin}(\Omega, \Omega^{gp}) \) ?

In [6] it is proved that \( X \) satisfies \( S_{fin}(\Omega, \Omega^{gp}) \) if, and only if, all finite powers of \( X \) satisfy the Hurewicz covering property \( S_{fin}(\Lambda, \Lambda^{gp}) \), which we now know is the same as \( \Lambda_{\kappa^{2^p}} \).

*Added after publication.* The answer to Problem 8 is “No”, in the following strong sense: Masami Sakai proves in: Weak Fréchet-Urysohn property in function spaces (preprint), that every analytic set of reals (and, in particular, the Baire space \( \mathbb{N}^\mathbb{N} \)) satisfies \( S_{fin}(\mathbb{N}^\mathbb{N}) \). But we know that \( \mathbb{N}^\mathbb{N} \) does not satisfy the Hurewicz covering property.
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