A POSITIVE SOLUTION TO THE GENERALIZED BUSEMANN-PETTY PROBLEM FOR TWO AND THREE-DIMENSIONAL SECTIONS

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ABSTRACT. The generalized Busemann-Petty problem asks: If $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and the volume of $K \cap \xi$ is smaller than the volume of $L \cap \xi$ for every $i$-dimensional subspace $\xi$, $1 \leq i \leq n - 1$, does it follow that the volume of $K$ is also smaller than the volume of $L$? For $i = 1$, the affirmative answer is obvious. Bourgain and Zhang gave a negative answer for all $3 < i \leq n - 1$. We show that in the cases (a) $i = 2$, $n \geq 3$, and (b) $i = 3$, $n \geq 4$, the answer is positive. This completes the problem.

1. INTRODUCTION

Let $G_{n,i}$ be the Grassmann manifold of $i$-dimensional subspaces $\xi$ of $\mathbb{R}^n$, and $vol_i(\cdot)$ denote the $i$-dimensional volume function, $1 \leq i \leq n$. Is it true that for origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$, the inequality

(1.1) \[ vol_i(K \cap \xi) \leq vol_i(L \cap \xi) \quad \forall \xi \in G_{n,i} \]

implies

(1.2) \[ vol_n(K) \leq vol_n(L) \quad ? \]

This question is known as the generalized Busemann-Petty problem (GBP). For $i = n - 1$, the problem was posed by Busemann and Petty [BP] in 1956 and has a long history (see [G2]). In the past decade a good deal of studying various aspects of this problem was made by Bourgain, Gardner, Koldobsky, Zhang and others. It was proven [GKS], [BFM], [P], [Z2] that for $i = n - 1$, the answer is YES if and only if $n \leq 4$, and in the general case the answer is NO for $3 < i \leq n - 1$ [BZ]. In [Z1]

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a negative answer was also claimed for \( i = 3 \). The case \( i = 2, n \geq 5 \) remained open. In the present article we show that GBP has an affirmative answer if \( i = 2 \) and \( i = 3 \). This disproves the aforementioned statement from [Z1]. One should note that there is a series of accompanying results which are of independent interest but do not solve GBP. We shall not discuss these results here.

The present article is almost self-contained, and the argument is simpler than that in [GKS], [Z2] and [BFM]. The Busemann-Petty case \( i = n - 1 \) is also included. The key idea is a remarkable interrelation between euclidean \( i \)-plane transforms on \( \mathbb{R}^n \), the dual geodesic Radon transforms on the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \), and the classical fractional calculus [R1], [SKM]. This interrelation was used implicitly in [BFM] for \( i = n - 1 \). We extend it to all \( i \leq n - 1 \) and simplify technicalities.

**Remark 1.1.** In contrast to [GKS] (and some other papers by A. Koldobsky), our approach does not employ the Fourier analysis on \( \mathbb{R}^n \). In this connection one should mention earlier papers by V.I. Semyanistyi [Se1], [Se2] containing a systematic study of interrelation between homogeneous distributions on \( \mathbb{R}^n \) and analytic families of convolution operators on the sphere (including the Minkowski-Funk and cosine transforms). According to the Gel’fand-Šapiro theory [GS], the Fourier transform serves as a bridge between these two classes of objects. Similar results were obtained by Semyanistyi and his collaborators for hyperbolic spaces and more general settings.

Section 2 contains auxiliary statements about Radon transforms of different kinds. In Section 3 we prove the following

**Theorem A.** If origin-symmetric convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) satisfy \( vol_i(K \cap \xi) \leq vol_i(L \cap \xi) \forall \xi \in G_{n,i} \), where \( i = 2, n \geq 3 \) (or \( i = 3, n \geq 4 \)), then \( vol_n(K) \leq vol_n(L) \).

Since for \( 3 < i \leq n - 1 \), GBP has a negative answer [BZ], Theorem A completes the problem.

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2. PASSAGE TO RADON TRANSFORMS AND PRELIMINARIES

We shall use the following notation: \( \sigma_{n-1} = |S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2) \) is the area of the unit sphere in \( \mathbb{R}^n \); \( e_1, e_2, \ldots, e_n \) denote the coordinate unit vectors, \( p_0 = \mathbb{R}_e n_{n+1} + \ldots + \mathbb{R}_e n \) is the coordinate \( i \)-plane; \( du \) and \( d\xi \) are invariant measures so that \( \int_{S^n_{n-1}} du = \sigma_{n-1} \) and \( \int_{G_{n,i}} d\xi = 1 \). The totally geodesic Radon transform of an integrable function \( f \) on
\( S^{n-1} \) is defined by
\[
(R_i f)(\xi) = \int_{S^{n-1} \cap \xi} f(u) d_\xi u, \quad \xi \in G_{n,i},
\]
where \( d_\xi u \) is the induced Lebesgue measure on the section \( S^{n-1} \cap \xi \). We identify the \( i \)-plane \( \xi \) with the \((i-1)\)-dimensional circle \( S^{n-1} \cap \xi \). The case \( i = n-1 \) in \( (2.1) \) corresponds to the Minkowski-Funk transform \( (Rf)(u) = (R_{n-1} f)(u^\perp) \). The dual Radon transform of a function \( \varphi \in L^1(G_{n,i}) \) is defined by
\[
(R^*_i \varphi)(u) = \int_{SO(n-1)} \varphi(r_u \gamma p_0) d\gamma, \quad u \in S^{n-1},
\]
where \( r_u \in SO(n) \) is a rotation so that \( r_u e_n = u \), and \( SO(n-1) \) is the isotropy subgroup of \( e_n \) [He], [R2]. The relevant duality reads
\[
\frac{1}{\sigma_{n-1}} \int_{G_{n,i}} (R^*_i f)(\xi) d\xi = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(u)(R^*_{i} \varphi) (u) du.
\]

Each origin-symmetric star body \( K \) can be identified with its radial function
\[
\rho_K(u) = \sup \{ \lambda \geq 0 : \lambda u \in K, \ u \in S^{n-1} \} \quad \in C_{even}(S^{n-1}).
\]
By passing to polar coordinates, we have
\[
vol_n(K) = \frac{1}{n} \int_{S^{n-1}} \rho_K^\alpha(u) du = V(K),
\]
\[
vol_i(K \cap \xi) = \frac{1}{i} \int_{S^{n-1} \cap \xi} \rho_K^\alpha(u) d_\xi u = \frac{1}{i} (R_i \rho_K^\alpha)(\xi).
\]
It is worth mentioning the paper by Hadwiger [H] who studied more general “radialpotenzintegrale” like \( (2.4) \) and \( (2.5) \) for zonal functions on \( S^2 \) with \( \rho^\alpha_K \) and \( \rho^\beta_K \) replaced by \( \rho^\alpha_K \) and \( \rho^\beta_K \) \( \forall \alpha, \beta \geq 0 \).

We shall deal with the following classes of star bodies:
\[
\mathcal{K}_e = \text{the set of all origin-symmetric star bodies;}
\]
\[
\mathcal{K}_c = \{ K : K \in \mathcal{K}_e, \ K \text{ is convex} \};
\]
\[
\mathcal{I}_{n,i} = \{ K : K \in \mathcal{K}_e, \ \rho^\alpha_K \in R^*_i(L^1(G_{n,i})) \}
\]
where \( R^*_i(L^1(G_{n,i})) \) is the range of the dual transform \( (2.2) \) on non-negative functions \( \varphi \in L^1(G_{n,i}) \).

The definition \( (2.6) \) is motivated by the following

**Lemma 2.1.** If \( K \in \mathcal{I}_{n,i} \) then \( (1.1) \) implies \( (1.2) \) \( \forall L \in \mathcal{K}_e \).
Proof. Let $\rho_{K}^{n-i} = R_{i}^{*}\varphi$, $\varphi \in L^{1}_{+}(G_{n,i})$. By (2.3)-(2.5),
\[
V(K) = \frac{1}{n} \int_{S^{n-1}} \rho_{K}^{i}(u)\rho_{K}^{n-i}(u)du = \frac{1}{n} \int_{S^{n-1}} \rho_{K}^{i}(u)(R_{i}^{*}\varphi)(u)du
\]
\[
= \frac{\sigma_{n-1}}{n\sigma_{i-1}} \int_{G_{n,i}} (R_{i}\rho_{K}^{i})(\xi)\varphi(\xi)d\xi \leq \frac{\sigma_{n-1}}{n\sigma_{i-1}} \int_{G_{n,i}} (R_{i}\rho_{L}^{i})(\xi)\varphi(\xi)d\xi
\]
\[
= \frac{1}{n} \int_{S^{n-1}} \rho_{L}^{i}(u)\rho_{K}^{n-i}(u)du.
\]
By Hölder’s inequality, $V(K) \leq V(L)^{i/n}V(K)^{1-i/n}$, and (1.2) follows.  
\]

The above lemma is well known. Such a simple argument was used, e.g., by Hadwiger [H] and Lutwak [L].

Lemma 2.2. If $\varphi \in L^{1}(G_{n,i})$ then $R_{i}^{*}\varphi = \sigma_{n-2}^{-1}R\psi$ where $R\psi$ is the Minkowski-Funk transform of a function
\[
(2.7) \quad \psi(\lambda e_{n-1}) = \int_{SO(n-2)} \varphi(\lambda\gamma p_{0})d\gamma, \quad \lambda \in SO(n),
\]
satisfying $\int_{S^{n-1}} \psi(u)du = \sigma_{n-1} \int_{G_{n,i}} \varphi(\xi)d\xi$.

Proof. By (2.1) and (2.2),
\[
\frac{1}{\sigma_{n-2}}(R\psi)(u) = \int_{SO(n-1)} \psi(r_{u}\rho e_{n-1})d\rho
\]
\[
= \int_{SO(n-1)} d\rho \int_{SO(n-2)} \varphi(r_{u}\rho\gamma p_{0})d\gamma = \int_{SO(n-1)} d\gamma \int_{SO(n-2)} \varphi(r_{u}\rho\gamma p_{0})d\rho.
\]
Replacing $\rho\gamma$ by $\rho$, we get $(R_{i}^{*}\varphi)(u)$. Furthermore,
\[
\int_{S^{n-1}} \psi(u)du = \sigma_{n-1} \int_{SO(n)} \psi(\lambda e_{n-1})d\lambda = \sigma_{n-1} \int_{SO(n)} d\lambda \int_{SO(n-2)} \varphi(\lambda\gamma p_{0})d\gamma
\]
\[
= \sigma_{n-1} \int_{SO(n)} \varphi(\lambda p_{0})d\lambda = \sigma_{n-1} \int_{G_{n,i}} \varphi(\xi)d\xi.
\]
\]

\]
We shall need some facts from [Mu], [Herz] about functions of matrix argument. Let \( \mathcal{M}_{n,k} \), \( n \geq k \), be the space of real matrices \( x = (x_{i,j}) \) having \( n \) rows and \( k \) columns with the volume element \( dx = \prod_{i=1}^{n} \prod_{j=1}^{k} dx_{i,j} \). In the following \( x' \) denotes the transpose of \( x \), \( |x| = \det(x) \) (for square matrices), and \( I_k \) is the identity \( k \times k \) matrix. Let \( V_{n,k} = \{ v \in \mathcal{M}_{n,k} : v'v = I_k \} \) be the Stiefel manifold of orthonormal \( k \)-frames in \( \mathbb{R}^n \). For \( n = k \), \( V_{n,n} = O(n) \) represents the orthogonal group in \( \mathbb{R}^n \). We fix a measure \( dv \) on \( V_{n,k} \) which is \( O(n) \) left-invariant, \( O(k) \) right-invariant, and normalized by \( \int_{V_{n,k}} dv = \sigma_{n,k} \), where \( \sigma_{n,k} = 2^k \pi^{n^2} / \Gamma_k(n/2) \), and

\[
\Gamma_k(\alpha) = \pi^{k(k-1)/4} \Gamma(\alpha - \frac{1}{2}) \Gamma(\alpha - \frac{k-1}{2})
\]

is the Siegel Gamma function. We denote by \( P_k \) the cone of positive definite symmetric \( k \times k \) matrices \( r = (r_{i,j}) \) with the volume element \( dr = \prod_{i \leq j} dr_{i,j} \). The following polar decomposition can be found in ([Herz], p. 482]; [Mu], p. 66]; [GR]).

**Lemma 2.3.** Almost all \( x \in \mathcal{M}_{n,k} \), \( n \geq k \), can be decomposed as

\[
x = vr^{1/2}, \quad v \in V_{n,k}, \quad r = x'x \in P_k \quad \text{with} \quad dx = 2^{-k} |r|^{(n-k-1)/2} dr dv.
\]

The next statement mimics Lemma 6.1 from [GZ]. We present it in a more complete form, and give an alternative proof.

**Lemma 2.4.** Let \( f = Rg \), \( g \in C_+(S^{n-1}) \), and let \( K \) be a star body so that \( \rho_{K}^{n-1}(u) = (n-1)g(u) \). Then for \( 1 < i < n \),

\[(2.8) \quad f^{n-i}(u) = (R_i \varphi)(u), \quad \varphi(\eta) = \int_{\eta \cap K} dx_1 \cdots \int_{\eta \cap K} |x'x|^{(i-1)/2} dx_{n-i},
\]

\( x = [x_1, \ldots, x_{n-i}] \in \mathcal{M}_{n,n-i} \), and

\[(2.9) \quad \int_{\mathcal{G}_{n,i}} \varphi(\eta) d\eta = c(vol_n(K))^{n-i}, \quad c = \frac{\sigma_{n-i,n-i}}{\sigma_{n,n-i}}.
\]

**Proof.** By (2.5), \( f(u) = vol_{n-1}(K \cap u^+) \), and therefore

\[
f^{n-i}(u) = \prod_{j=1}^{n-i} \int_{K \cap u^+} dx_j = \int_{\mathcal{M}_{n-1,n-i}} F(y) dy
\]
where \( F(y) \) is the product of characteristic functions \( \chi_{r_u^{-1}K}(y_j), j = 1, \ldots, n-i \), of the rotated body \( r_u^{-1}K \). Now Lemma 2.3 yields

\[
\begin{align*}
\int_{G_{n-i}} f^{n-i}(u) &= 2^{i-n} \int_{V_{n-1,n-i}} \int_{O(n-i)} d\gamma \int_{\mathbb{P}_{n-i}} F(v\gamma r^{1/2}) |r|^{(i-2)/2} \, dr \\
&= \frac{2^{i-n}}{\sigma_{n-i,n-i}} \int_{V_{n-1,n-i}} \int_{O(n-i)} d\gamma \int_{\mathbb{P}_{n-i}} F(v\gamma r^{1/2}) |r|^{(i-2)/2} \, dr \\
&= \frac{1}{\sigma_{n-i,n-i}} \int_{V_{n-1,n-i}} \int_{\mathbb{M}_{n-1,n-i}} F(vz) |z'|^{(i-1)/2} \, dz \\
&= \int_{SO(n-1)} \int_{\mathbb{M}_{n-1,n-i}} F(\alpha \begin{bmatrix} I_{n-i} \\ 0 \end{bmatrix}) z) |z'|^{(i-1)/2} \, dz \\
&= \int_{G_{n-1,n-i}} \int_{\mathbb{S}_{n-1}^{n-i}} F(y) |y'|^{(i-1)/2} \, dy_1 \cdots dy_{n-i}.
\end{align*}
\]

After changing variables this reads

\[
\int_{\eta \in u} \left( \int_{\eta \cap K} \int_{\eta \cap K} d\xi_1 \cdots d\xi_{n-i} \right) |x' x|^{(i-1)/2} \, dx_{n-i} \, d\eta, \quad \eta \in G_{n,i},
\]

and (2.8) follows. In order to prove (2.9) we proceed in the opposite direction with slight changes:

\[
\begin{align*}
\int_{G_{n,i}} \varphi(\eta) \, d\eta &= \int_{SO(n)} \int_{\mathbb{M}_{n-1,n-i}} \int_{\mathbb{S}_{n-i}} F(\beta \begin{bmatrix} I_{n-i} \\ 0 \end{bmatrix}) z) |z'|^{(i-1)/2} \, dz \\
&= \frac{1}{\sigma_{n,n-i}} \int_{V_{n,n-i}} \int_{\mathbb{M}_{n-1,n-i}} \int_{\mathbb{S}_{n-i}} F(wz) |z'|^{(i-1)/2} \, dz \\
&= \frac{2^{i-n}}{\sigma_{n,n-i}} \int_{V_{n,n-i}} \int_{O(n-i)} \int_{\mathbb{P}_{n-i}} F(wr^{1/2}) |r|^{(i-2)/2} \, dr \\
&= \frac{2^{i-n} \sigma_{n,n-i}}{\sigma_{n,n-i}} \int_{V_{n,n-i}} \int_{\mathbb{P}_{n-i}} F(wr^{1/2}) |r|^{(i-2)/2} \, dr \\
&= \frac{c}{\sigma_{n,n-i}} \int_{\mathbb{M}_{n-1,n-i}} F(y) dy = c \int_{r_u^{-1}K} dy_1 \cdots \int_{r_u^{-1}K} dy_{n-i} = c(\text{vol}_n(K))^{n-i}.
\end{align*}
\]

\[\square\]
The next statement is especially important. It concerns interrelation between spherical and euclidean Radon transforms. We consider the euclidean $i$-plane transform $[He]$  

\[(2.10) \quad (P_i f)(\xi, x'') = \int f(x' + x'') dx', \quad \xi \in G_{n,i}, \quad x'' \in \xi^{\perp}.\]

Since $\int_{\xi^{\perp}} (P_i f)(\xi, x'') dx'' = \int_{\mathbb{R}^n} f(x) dx \forall \xi$, $P_i f$ is well defined for all $f \in L^1(\mathbb{R}^n)$. The case $i = 1$ in (2.10) corresponds to the X-ray transform of $f$.

**Lemma 2.5.** Let $f \in L^1(\mathbb{R}^n)$, $\xi \in G_{n,i}$. Then for all $\omega \in S^{n-1} \cap \xi^{\perp}$ and $t \in \mathbb{R}$,

\[(2.11) \quad R^*_t : (P_i f)(\xi, \omega t) \to \sigma_{i-2} \int_{|t|}^{\infty} \psi_u(r)(r^2 - t^2)^{(i-3)/2} rdr\]

where

\[(2.12) \quad \psi_u(r) = \frac{1}{\sigma_{i-2}} \int_{S^{n-1} \cap u^{\perp}} (P_1 f)(\theta, r\theta) d\theta\]

is the average of the X-ray transform of $f$ over all lines parallel to $u$ at distance $r$ from the origin.

**Proof.** Let $p_0 = \mathbb{R}e_{n-i+1} + \ldots + \mathbb{R}e_n$, $y = y' + y''$, $y' \in p_0$, $y'' \in p_0^{\perp}$. Given $u \in S^{n-1}$, we fix a rotation $r_u \in SO(n)$ so that $r_u : e_n \to u$, and denote $f_u(x) = f(r_u x)$. By (2.2),

\[(R^*_t[(P_i f)(\xi, x'')])(u) = \int_{SO(n-1)} (P_i f_u)(\rho p_0, \rho y) d\rho = \int_{SO(n-1)} d\rho \int_{p_0} f_u(\rho(y' + y'')) dy'.\]

For $y = \sum_{k=1}^n y_k e_k$, we write

\[y = z + y_n e_n, \quad z = z' + z'', \quad z' = \sum_{k=n-i+1}^{n-1} y_k e_k, \quad z'' = \sum_{k=1}^{n-i} y_k e_k = y''.\]
Then the last integral reads
\[
\int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \int_{SO(n-1)} f_u(\rho z + y_n e_n) d\rho d\sigma d\alpha
\]
\[
= \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\infty} \int_{SO(n-1)} f_u(\rho \sqrt{|z'|^2 + |z''|^2} e_1 + y_n e_n) d\rho d\sigma d\alpha
\]
\[
= \sigma_{i-2} \int_0^\infty s^{i-2} \psi_u(\sqrt{s^2 + |y''|^2}) ds = \sigma_{i-2} \int_0^\infty \psi_u(r)(r^2 - r^2)^{(i-3)/2} dr,
\]
where \( t = |z''| = |y''| = |x''| \), and
\[
\psi_u(r) = \int_{SO(n-1)} dp \int_{-\infty}^{\infty} f_u(r p e_1 + y_n e_n) d\sigma d\alpha = \frac{1}{\sigma_{n-2}} \int_{S^{n-1} \cap u^\perp} (P_1 f)(u, r \theta) d\theta.
\]

3. PROOF OF THE MAIN RESULT

Denote
\[
(3.1) \quad M_\xi(\omega, t) = (P_1 f)(\xi, \omega t), \quad \omega \in S^{n-1} \cap \xi^\perp, \quad t \in \mathbb{R}.
\]
If \( f = \chi_K \) is the characteristic function of \( K \), then (3.1) is a volume of the \( i \)-dimensional plane section of \( K \) in the direction \( \omega \) at distance \( |t| \) from the origin. Our nearest goal is to apply Lemma 2.5 to \( f = \chi_K, K \in \mathcal{K}_c \), and to express \( \psi_u(0) = (P_1 f)(u, 0) = 2\rho_K(u) \) through \( M_\xi(\omega, t) \). Let
\[
(3.2) \quad F_u(t) = (R_i [M_\xi(\omega, t)])(u), \quad t \geq 0,
\]
and suppose that \( K \) has a \( C^\infty \)-boundary. If \( i \) is odd we follow [BFM] and write (2.11) as
\[
\frac{F_u(t)}{\sigma_{i-2}} = \int_0^\infty \psi_u(r)(r^2 - t^2)^{(i-3)/2} dr - t^{i-1} \int_0^1 \psi_u(t s)(s^2 - 1)^{(i-3)/2} s ds.
\]
The first term in the right hand side is a polynomial of degree \( i - 3 \), and simple calculation yields
\[
\frac{F_u(t)}{\sigma_{i-2}} = -\sigma_{i-2}(i - 1)! \psi_u(0) \int_0^1 (s^2 - 1)^{(i-3)/2} s ds
\]
\[
(3.3) \quad = \pi^{i/2-1} 2^i (-1)^{(i-1)/2} \Gamma(i/2) \rho_K(u).
\]
For \( i \) even, the required result can be easily obtained by making use of Marchaud’s fractional derivative which was introduced in 1927 [M], and is well known in fractional calculus. Numerous generalizations and applications of this notion can be found in [R1], [SKM]. We recall some basic facts from the book [R1] containing a detailed account and further development of Marchaud’s method. For a large class of functions \( \varphi \) on a half-line, the Weyl fractional integral

\[
\Phi(t) = (I^\alpha_0 \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \varphi(r)(r - t)^{\alpha - 1} dr, \quad Re \alpha > 0,
\]

can be inverted as \( \varphi = D^\alpha_0 \Phi \) where

\[
(D^\alpha_0 \Phi)(r) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty \left[ \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j \Phi(r + j \ell) \right] \frac{dt}{t^{1+\alpha}},
\]

\[
\kappa_\ell(\alpha) = \int_0^\infty \frac{(1 - e^{-\ell t})^\ell}{\ell^{1+\alpha}} dt, \quad \ell > Re \alpha, \quad \ell \in \mathbb{N}.
\]

The expression (3.5) containing a finite difference of \( \Phi \) of order \( \ell \), is called the Marchaud fractional derivative of \( \Phi \). The integer \( \ell > Re \alpha \) can be arbitrary due to normalization. If \( \Phi \) has \( \ell + 1 \) continuous derivatives in a neighborhood of \( r \) then \( D^\alpha_0 \Phi \) exists as an improper integral. Otherwise it represents a hypersingular integral which is understood in a certain special sense (see Theorem 10.21 in [R1]). For \( r = 0 \), (3.5) yields

\[
(D^\alpha_0 \Phi)(0) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty \left[ \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j \Phi(j \ell) \right] \frac{dt}{t^{1+\alpha}}.
\]

This expression represents analytic continuation (a.c.) of the integral

\[
I(\beta) = \frac{1}{\Gamma(\beta)} \int_0^\infty \Phi(r)r^{\beta - 1} dr \quad \text{in the strip } \mathcal{L} = \{ \beta : -\ell < Re \beta < 0 \} \text{ at the point } -\alpha \in \mathcal{L}.
\]

If \( \Phi^{(2m+1)}(0) = 0 \), \( m = 0, 1, \ldots \), then \( \ell \) can be reduced so that (3.6) holds for all \( \ell > 2[Re \alpha/2] \), \([a]\) being the integral part of \( a \) (see Corollary 10.15 in [R1]).

We note that in applications of fractional calculus, a difference regularization is sometimes more effective than the classical one [GS]

\[
a.c. \int_0^\infty \Phi(r)r^{\beta - 1} dr = \int_0^\infty \left[ \Phi(r) - \sum_{j=0}^{\ell - 1} \frac{r^j}{j!} \Phi^{(j)}(0) \right] r^{\beta - 1} dr,
\]

\[-\ell < Re \beta < -\ell + 1.\]
We shall use both regularizations of \( I(\beta) \). By (2.11) and (3.2), \( F_u(\sqrt{t}) = \pi^{(i-1)/2} (F_u^{(i-1)/2} \psi_u(\sqrt{t}))(t) \). Hence

\[
\psi_u(0) = \frac{\pi^{(i-\alpha)/2}}{\kappa \ell((i-1)/2)} \int_0^\infty \left[ \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j F_u(\sqrt{t}) \right] \frac{dt}{t^{(i+1)/2}}
\]

\[= a.c. \frac{\pi^{(i-\alpha)/2}}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} F_u(\sqrt{t}) dt \bigg|_{\alpha = (1-\alpha)/2} \]

\[= a.c. \frac{2\pi^{(i-\alpha)/2}}{\Gamma(\beta/2)} \int_0^\infty \tau^{\beta-1} F_u(\tau) d\tau \bigg|_{\beta = 1-\alpha}.
\]

Since odd derivatives of \( F_u(t) \) are zeros at \( t = 0 \), and \( \psi_u(0) = 2\rho_K(u) \), according to (3.7) we get

\[(3.8) \quad \rho_K(u) = c_i \int_0^\infty \left[ F_u(\tau) - \sum_{m=0}^{(i-2)/2} \frac{\tau^{2m}}{(2m)!} F_u^{(2m)}(0) \right] \frac{d\tau}{\tau^i}.
\]

\[c_i = \frac{\pi^{(i-\alpha)/2}}{\Gamma((1-\alpha)/2)} = \frac{(-1)^{i/2}(i-1)!!}{(2\pi)^{i/2}}.
\]

This implies the following

**Theorem 3.1.** Let \( \rho_K \in C^\infty(S^{n-1}) \), \( 2 \leq i \leq n-1 \). For \( \xi \in G_{n,i}, \omega \in S^{n-1} \cap \xi^\perp \), and \( t \geq 0 \), we set

\[(3.9) \quad M_\xi(\omega, t) = vol\{K \cap \{\xi + \omega t\}\},
\]

and denote by \( M_\xi^{(j)}(\omega, 0), j = 1, 2, \ldots, \) directional derivatives of \( M_\xi(\omega, \cdot) \) in the direction \( \omega \) at \( t = 0 \). Then \( \rho_K = R^*_h \), where \( h \equiv h(\xi, \omega) \in C^\infty(G_{n,i}) \). The function \( h \) has the form

\[(3.10) \quad h(\xi, \omega) = \frac{(-1)^{(i-1)/2}}{\pi^{i/2-1} 2^{i/2} \Gamma(i/2)} M_\xi^{(i-1)}(\omega, 0)
\]

if \( i \) is odd, and

\[(3.11) \quad h(\xi, \omega) = \frac{(-1)^{i/2}(i-1)!!}{(2\pi)^{i/2}} \int_0^\infty \left[ M_\xi(\omega, \tau) - \sum_{m=0}^{(i-2)/2} \frac{\tau^{2m}}{(2m)!} M_\xi^{(2m)}(\omega, 0) \right] \frac{d\tau}{\tau^i}
\]

if \( i \) is even.

For \( i = n-1 \), Theorem 3.1 was proven in [GKS] and [BFM]. Our proof is essentially simpler and based on different ideas. We recall that the argument in [GKS] relies heavily on the Fourier analysis of
homogeneous distributions in $\mathbb{R}^n$. Derivation of (3.11) in [BFM] (for $i = n - 1$) is more complicated than ours. It is worth noting that for $i < n - 1$, one deals with the family of functions $h$ parameterized by the vector $\omega$ belonging to the $(n - i - 1)$-dimensional unit sphere. In the case $i = n - 1$ there exist only one such function.

Now we can return to the generalized Busemann-Petty problem. We denote by $A(GBP)$ the set of all pairs $(i, n)$, $2 \leq i \leq n - 1$, so that (1.1) $\Rightarrow$ (1.2) for all $K, L \in K_c$, and let $\kappa_K(x)$ be the Gaussian curvature of the boundary $\text{bd}(K)$ at the point $x \in \text{bd}(K)$ [Sch]. Consider the following class of “test bodies”:

$$K^{\infty}_{c+} = \{K : K \in K_c; \quad \rho_K \in C^\infty(S^{n-1}); \quad \kappa_K(x) > 0 \quad \forall x \in \text{bd}(K)\}.$$

**Lemma 3.2.** If $K^{\infty}_{c+} \subset I_{n,i}$ (see (2.6)) then $(i, n) \in A(GBP)$.

**Proof.** Following Gardner’s scheme ([G1], Theorem 3.1), we suppose that for some $K, L \in K_c$ satisfying (1.1), $\text{vol}_n(K) > \text{vol}_n(L)$. There exists an approximating body $K' \in K^{\infty}_{c+}$ so that $K' \subset K$ and $\text{vol}_n(K') > \text{vol}_n(L)$ (see, e.g., [G2], p. 297, and [G1], p. 438). By Lemma 2.1, $K' \not\in I_{n,i}$, and we arrive at contradiction. $\square$

**Proof of Theorem A.** Let $K \in K^{\infty}_{c+} \subset K_c$, be a “test body”. By Lemma 3.2 it suffices to show that

$$\rho_K^{-i}(u) \in R^*_i(L^1_+(G_{n,i})).$$

By Theorem 3.1,

$$h(\xi, \omega) = \frac{1}{2\pi} \int_0^\infty \frac{M_\xi(\omega, 0) - M_\xi(\omega, \tau)}{\tau^2} d\tau$$

if $i = 2$, and

$$h(\xi, \omega) = -\frac{1}{4\pi} M''_\xi(\omega, 0)$$

if $i = 3$. Fix $\omega \in S^{n-1} \cap \xi^\perp$ and consider the $(i + 1)$-dimensional plane $\pi_{i+1}$ spanned by $\xi$ and $\omega$. The $(i+1)$-dimensional body $K \cap \xi_i$ belongs to the class $K^{\infty}_{c+}$ (on $\pi_{i+1}$), and by the Brunn-Minkowski theory [Sch] the function $h$ in (3.13) and (3.14) is positive. Moreover, it is continuous in $\xi \in G_{n,i}$. Let $\tilde{h}(\xi)$ be the average of $h$ over all $\omega \in S^{n-1} \cap \xi^\perp$. By Theorem 3.1, $\rho_K(u) = (R^*_i \tilde{h})(u)$, and owing to Lemma 2.2, $\rho_K(u)$ is represented by the Minkowski-Funk transform of a function $\psi$ which is defined by (2.7) with $\varphi$ replaced by $\tilde{h}$. Since $\tilde{h} \in C_+(G_{n,i})$, then $\psi \in C_+(S^{n-1})$. Next we use Lemma 2.4, which implies (3.12). This completes the proof.
REFERENCES


THE GENERALIZED BUSEMANN-PETTY PROBLEM


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